Mass-Energy Equivalence Relation ($E=mc^2$)

by

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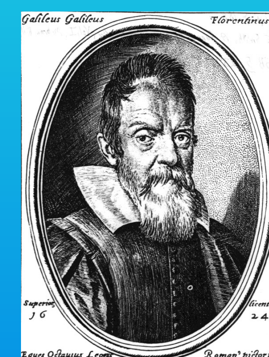

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Perhaps the most well known of Einstein's innovations in the theory of relativity is the $E=mc^2$ formula. It relates the inertial mass m of any object to its total energy content E (c is the speed of light in free space: 299792458 ms^{-1}).

Let us first see how he derived this formula in his seminal 1905 paper (A. Einstein, Zur Elektrodynamik bewegter Körper. *Ann. Physik.* **322 (10)**, 891-921 (1905). He considered an object which radiates energy L/2 in opposite directions. The energy lost in the object's rest frame is therefore L. He then used a result of his theory to compute the amount of energy gained in another rest frame moving with speed v relative to the first Accordingly, this amount is $\gamma L=(1-v^2c^{-2})^{-0.5}L$.

Ironically, he used approximations from non-relativistic theory in his derivation. He assumed that the kinetic energy imparted to the second rest frame is $(\gamma - 1)\ L = 0.5\ v^2c^{-2}\ L$, that is, to first order in v^2

The non-relativistic value of the kinetic energy is $0.5\ mv^2$. Note that comparing these two equations is already an unusual approach. On the one hand, you have an amount of energy L arriving at the second rest frame. On the other hand, kinetic energy is to be applied to an object of mass m in this rest frame. One can imagine that Einstein was at first confused by this, but he forged ahead and simply applied a basic tenet of algebra to go forward, namely to set the two quantities equal.

$$0.5\ v^2 c^{-2}\ L = 0.5\ mv^2.$$

Upon cancellation, one obtains

$$c^{-2}\ L = m.$$

Replacing L with E, the result is $E=mc^2$.

Strictly speaking, the above derivation is only valid for low velocities. To Einstein's credit, he immediately began asking questions about the possible generality of the new relation. It also needs to be understood that there is no way to prove that $E=mc^2$. It is a physical law that needs to be tested with the goal of finding a situation which is not consistent with it. The same holds true for Newton's Laws of Motion, for example, or for the three Laws of Thermodynamics.

Planck later conceived of a different way to derive the formula, however. He proposed another new law, this one concerning inertial masses. He again assumed that the quantity γ (v) is involved. Accordingly, the inertial mass of an object increases with γ (v), the same proportionality Einstein assumed for energies:

$$m(v) = \gamma (v)\, m (0) = \gamma (v)\, \mu.$$

Planck then generalized the non-relativistic definition of momentum p, namely pμv, to the relativistic version:

$$p = mv = \gamma\ (v)\ \mu\ v.$$

The $E=mc^2$ derivation can now proceed using Hamilton's Canonical Equation, $dE=vdp$, which is derived from Newton's Second Law: $F=dp/dt$. The differential of energy dE is equal to the product of F and the differential of the distance x by which the object moved parallel to F).

$$dE= Fdx =(dp/dt)\ dx = dp\ (dx/dt)= vdp.$$

The next step is to use Planck's definition for p:

$$dE= vdp= v\ d\gamma(v)\ (\mu\ v) + v\ \gamma(v)\ \mu\ dv.$$

Note that $d\gamma(v) = (1-v^2c^{-2})^{-3/2}$ $[-1/2\ (-2vc^{-2}dv)] = \gamma^3 vc^{-2}dv$. Thus,

$$dE = vdp = v\ d\gamma(v)(\mu v) + v\gamma(v)\mu dv = \mu v^2(\gamma^3 vc^{-2}dv) + \mu v\gamma dv =$$

$$\mu v\gamma dv\ (\gamma^2 v^2 c^{-2} + 1) = \gamma^3 \mu v dv(v^2c^{-2} + 1 - v^2c^{-2}) = \gamma^3 \mu v dv =$$

$$\mu c^2 d\gamma \quad = d(\mu c^2 \gamma) = dE$$

Upon integration, the result is

$$E = \gamma\mu c^2 = mc^2.$$

In 1909 Planck's assumption about the definition of inertial mass was verified by Bucherer. He measured the inertial mass of electrons that are accelerated in crossed magnetic and electric fields and found that the value varied as $m=\gamma(v)\,\mu$, exactly as Planck had assumed in his definition of relativistic momentum $p = \gamma(v)\,\mu v$.

It is important to realize that there are a number of such proportionality relations, including for energy, distance D and time T:

$$E(v) = \gamma(v)\; E(0)$$
$$D(v) = \gamma(v)\; D(0)$$
$$T(v) = \gamma(v)\; T(0)$$

The fact that both E and m increase by the same factor γ (v) is clearly consistent with $E=mc^2$ since one expects c to be the same in all rest frames. This in turn demands that the ratio of distance D to elapsed time T must also be the same in each rest frame, independent of the orientation of the the object of measurement, hence the proportionality relation for D (v). Note that this means that distances increase isotropically with time dilation, in stark contrast to Einstein's belief in the FitzGerald-Lorentz length contraction effect according to which distances decrease by varying amounts when objects are accelerated.

This leads to the more general law of Uniform Scaling. Accordingly, there must be a similar proportionality relation for all physical properties. This is because all properties are products of the three fundamental properties of inertial mass, distance and time. For example, angular momentum L is a product of mass, distance^2 and inverse time, so $L(v) = \gamma^2(v)\ L(0)$.

Moreover, the fact that the speed of light is the same for all observers requires that the same hold true for all relative speeds between two rest frames. Otherwise, the ratios of v/c would vary from one rest frame to another, in contradiction to the Relativity Principle. The proportionality relation for momentum p is also to be understood in this way.

Hamilton's Canonical Equations can also be used to obtain an energy-momentum transformation similar to space-time transformations:

$$E = \gamma\,(E' + v p_x')$$
$$p_x = \gamma\,(p_x' + c^{-2} v E').$$
$$p_y = p_y'$$
$$p_z = p_z'.$$

If the primed variables correspond to the system at rest, then p_x'=0, and E' is the rest energy of the object. Thereupon, we obtain the same equation as before for the effect of acceleration on energy, i.e. E= γ (v) E'. The inverse of the equation for p_x is then:

$$p_x' = \gamma (p_x - c^{-2} v E).$$

By definition, p_x = mv, so when p_x'=0,

$$m = c^{-2} E.$$

While everything has been consistent so far, there remains a question about the proportionality relations. In the inertial mass study, all measurements were made relative to the laboratory. The mass is moving with speed v relative to the laboratory Suppose the observer is also moving with respect to the laboratory with speed v'? His unit of mass is therefore γ (v')μ, not μ as for the laboratory observer. Therefore, when he measures the mass of the object, his value will differ by a factor of $\gamma (v')/\gamma (v) < 1$ because the measured value is inversely proportional to the size of the unit used.

This means that the previous formula is not general enough. Instead, we need to define a conversion factor which allows us to convert the value of the mass in its own rest frame to that of an observer moving relative to it with speed v. The general formula is thus:

$$m(v) = Q\, m(0) = Q\, \mu$$

If the speed of the object relative to the laboratory is v' while the speed of the observer relative to the same laboratory is v, this means that

$$Q = \gamma (v')/\gamma (v).$$

Note that if the observer is in the rest frame of the laboratory, v=0 and $Q = \gamma (v')$, so the above formula reverts back to the original in this case: $m (v') = \gamma (v') m(0) = \gamma (v') \mu$.

The same argument holds for every physical variable, hence:

$$E (v) = Q \; E(0)$$
$$D (v) = Q \; D(0)$$
$$T (v) = Q \; T(0)$$

Since both E and m scale with the same factor q, it follows that $E=mc^2$ in all rest frames because c is the same in every rest frame.

Moreover, by construction all physical laws remain the same in each rest frame, regardless of the value of the conversion factor Q. For example, Planck's Law $E=h/T$ is multiplied on the left by Q and on the right by $Q^2/Q = Q$, so it remains intact after the scaling is applied. The scale factor for h is the same (Q^2) for angular momentum (mass x speed x distance).

Although there are many plausible derivations of $E=mc^2$, it must be remembered that these are not proofs in any strict mathematical sense. Just as is the case for Newton's Laws of Kinetics, Einstein's mass-equivalence relation is is to be used to deduce other facts of Nature which can be tested by experiment.

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i