

Production of Photons in Positronium Decay:

Critique of the Creation-Annihilation Hypothesis

Robert J. Buenker¹

¹*Fachbereich C-Mathematik und Naturwissenschaften, Bergische Universität Wuppertal,*

Gaussstr. 20, D-42097 Wuppertal, Germany

e-mail: rjbuenker@gmail.com

Abstract

The history of the belief in the existence of elements from which all matter is formed begins more than two millennia ago with the writings of the ancient Greeks and Romans. Developments in the late 17th century pioneered by Robert Boyle led to the modern concept of elemental balance, according to which the number and type of various atoms remains exactly the same in the course of a chemical reaction. The concept of the creation and annihilation of matter changed all that. According to Einstein's relativity theory, elements can be converted entirely into energy and therefore no longer exist. The present study examines this theoretical interpretation by looking in detail at the process of positronium decay. It is suggested that the electron and positron constituents of positronium are actually bound so tightly together after decay that they form a massless lower-energy state which can be identified with the photon itself. The Breit-Pauli Hamiltonian is adjusted through the addition of momentum-dependent exponential damping factors so that it possesses eigenfunctions which correspond to such a tightly bound state with $2m_0c^2$ binding energy. The success which the corresponding

Schrödinger-type calculations achieves calls into question the creation-annihilation hypothesis on a completely general basis.

Keywords: Breit-Pauli Hamiltonian, Schrödinger equation, Photon, Positronium, Creation-Annihilation, Einstein $E=mc^2$ mass-energy equivalence relation

I. INTRODUCTION

After the discovery of quantum mechanics, nothing quite so excited the imagination of physicists as Dirac's prediction¹ of the existence of antimatter. Within a few years Anderson had demonstrated² the existence of the positron as the antiparticle of the electron, and within decades the antiproton³ had also been found. Although Dirac's line of reasoning involved first and foremost his treatment¹ of the fine structure in the spectra of hydrogenic atoms, the ultimate theoretical basis for his prediction can be traced back years before to Einstein and the special theory of relativity (STR).⁴ The famous mass-energy equivalence relation, $E=mc^2$, was given a more concrete interpretation by Dirac, namely that two particles of equal rest mass but opposite electric charge could be converted entirely into energy. In the case of e^+ and e^- , the energy appears as electromagnetic radiation or photons. The associated frequency ν is related to the energy E by the Planck relation⁵ $E=h\nu$ (h is Planck's constant: 6.626×10^{-34} Js). A key element in this interpretation is the series of mechanical laws of classical physics, particularly the conservation of energy and each of the components of linear and angular momentum, which were enunciated in the seventeenth century by Newton⁶ and Galileo.⁷

The creation and annihilation of matter, as this phenomenon is generally called, is central to the theory of modern physics, ranging from the sub-microscopic regime of atoms, electrons, photons and more exotic elementary particles to the super-macroscopic world comprising astronomy and cosmology. Indeed, one uses these terms to discuss processes which are commonplace in our everyday experience, such as the absorption and emission of light. Energy released or gained upon electronic transitions in atoms or molecules is said to cause the creation or destruction of a photon, again in accordance with the Planck energy-frequency relation.

As with other fundamental theoretical concepts of natural science, the principle of creation and annihilation of matter forced a rethinking of old themes from the realm of philosophy and metaphysics.

For example, a new interpretation could be given to the famous statement⁸ of Lucretius in the first century B.C. in *de Rerum Natura*: "Nothing can be created from nothing," which is paraphrased by Shakespeare⁹ in *King Lear* as "Nothing will come of nothing." In other words, energy in the form of photons corresponding to a given frequency is to be distinguished from nothing and therefore the production of electrons and positrons from it is thus not inconsistent with Lucretius's rule.

Yet, it is clear that the creation-annihilation concept still represents a revolutionary departure from ancient precepts, particularly certain aspects of the atomic theory of matter dating back to the work of Democritus.¹⁰ According to this traditional view, the elements from which all matter is assumed to be constructed possess very definite characteristics. Again, in the words of Lucretius:¹¹ "Material objects are of two kinds, atoms and compounds of atoms. The atoms themselves cannot be swamped by any force, for they are preserved indefinitely by their absolute solidity." Suffice it to say that the annihilation of an electron and positron to produce pure energy is not consistent with this statement. The present investigation concerns itself with the theory underlying this revolutionary process in which material particles interact with one another with such force that they are claimed to lose the very identity that Lucretius imagined was their inherent property.

II. POSITRONIUM DECAY AND THE CREATION-ANNIHILATION HYPOTHESIS

In a book¹² published in 1661 entitled *The Skeptical Chymist*, Robert Boyle originated the modern concept of chemical elements. Simply stated, he suggested that the elements can be distinguished from all other substances by virtue of the fact that they cannot be split up by chemical reactions into simpler substances. The relationship between his ideas and the atomic theory of Democritus and continuing on to Lucretius is unmistakable. The concepts were made more concrete in Dalton's atomic theory¹³ which was proposed in 1803. It provided a sound basis for interpreting known facts of chemistry. Thus, when one observed two different gases, oxygen and hydrogen, coming together to react explosively to form steam, one could speak of

the process as involving two elements which were simply joined together in different ways before and after the reaction.

Until the advent of Einstein's STR⁴ it was believed that each element had a fixed mass and that the sum of the masses of the reactants was exactly equal to that of the products. Even today one adheres to the underlying principle that the numbers of each type of element are unchanged in the course of chemical reactions, and thus the concept of a balanced equation remains an integral part of the teaching of fundamental chemistry to the present time. Special relativity merely rejects the assumption of fixed masses for the elements and replaces it with the principle that energy absorbed or emitted in the course of a physical transformation must be taken into account by means of the $E=mc^2$ relation in order to predict the combined mass of the products from that of the reactant species.

The concept of elemental balance in chemical reactions is in no way disturbed by this adjustment, and with its help it was possible to carry over the ideas of the early chemists to the interpretation of nuclear reactions involving much larger energy changes. For example, when two deuterium atoms combine to form an alpha particle the energy released is great enough to allow for direct measurement of the corresponding loss of mass of the product system relative to that of the reactants. The discovery of the neutron by Chadwick¹⁴ in 1932 made it possible to specify more precisely what the elements are in this process. The deuteron is thus a compound consisting of a single proton and neutron, whereas the alpha particle is the ⁴He nucleus composed of pairs of each of these elements.

Nonetheless, one of the casualties of the creation-annihilation concept is the principle of elemental balance in all physical transformations. The above example involving a nuclear process is more the exception than the rule in the theory of modern physics, particularly as one makes the transition into the field of elementary particles. If particles can simply be converted into pure energy, there is no longer any basis for demanding that the same number and type of elemental species is present before and after a reaction has occurred. One need only try to imagine how the development of the theory of chemical transformations would have been affected *if the principle of a balanced reaction had not been enunciated* in order to appreciate the consequences of rejecting the idea in other branches of the natural sciences.

A. ELECTRON-POSITRON INTERACTION

With this background it is interesting to analyze in some detail the process which first led to the postulation of the creation and annihilation of matter, namely the interaction of an electron with its antiparticle, the positron. For this purpose it is important to consider the experimental data with reference to familiar theoretical models without accepting their conclusions *a priori*, recalling Newton's prescription¹⁵ that "these laws must be considered as resting on convictions drawn from observation and experiment, not on intuitive perception." Assuming the positron and the electron to be initially at rest, the first observation is that they form a weakly bound complex known as "positronium." After a short lifetime (ca. 10^{-10} s) something much more dramatic happens to the system, however. In the most commonly observed process, the decay of positronium leads to the production of two high-energy photons which fly away in opposite directions to one another. In another branch of this reaction which occurs much less frequently, three photons appear. In the primary decay process the two photons are always found to show opposite polarization, whether circular, plane or elliptical, and have equal energy /frequency.

The existing theory for these various observations can be summarized as follows. First, a typical low-energy phenomenon occurs, corresponding to the binding of the electron and positron together. This process can be described accurately in close analogy to the treatment of the hydrogen atom by means of the non-relativistic Schrödinger equation. An even more thorough description in terms of quantum electrodynamics is also possible.¹⁶ One knows that the ionization potential of the hydrogen atom is 13.605 eV (0.5 hartree) and the Bohr theory¹⁷ of 1912 had shown that the amount of binding is proportional to the reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$ of the proton-electron system. The equality of the rest masses of the electron and positron leads to the conclusion that μ for positronium is only about half that of the hydrogen atom. Hence, the binding energy in this case is one-half as large (6.80 eV). However, *the instability of the positronium complex stands in sharp contrast to the known characteristics of the corresponding H atom*, which is quite stable and appears to exist in this state for indefinite periods in the absence of any outside influence.¹⁸

Were it not for the spontaneous decay of positronium, one would have no difficulty relating these observations to Dalton's atomic theory.¹³ In this case, the elements are one electron and one positron. At the beginning of the reaction they are separated (existing in elemental form, to use Boyle's terminology¹²), whereas afterwards the compound positronium is formed in which the two elements are bound together.

Why then must we give up the atomic theory of elements when it comes to the decay of positronium? The conventional answer to this question is that the elements with which we started, namely an electron and a positron, are no longer present at the conclusion of the process. Instead, one is left with a pair of photons in the most commonly occurring case.

Yet, in a strict sense the creation-annihilation explanation for positronium decay violates Newton's prescription about always basing theory firmly on observation. *How can one truly observe that something disappears?* If natural science is restricted to the domain of observation, how does one fit the phenomenon of creation and destruction into the picture? By definition, these processes involve material transformations either to or from nothing, and therefore can never be observed directly in their entirety. *There is clearly something about nothingness which defies observation.* By the same token, it is impossible to prove that things do not disappear. It only seems prudent to recognize that the inability to observe an object is not an unambiguous sign that it has ceased to exist.

Especially since the natural sciences underwent a long and successful development without having to yield on the ancient view that all material objects are synthesized from impermeable elements, it is important to probe the creation-annihilation hypothesis with utmost scrutiny. To this end, let us follow the tried-and-true principle of mathematics employed whenever a theorem is to be proven, *namely to assume the opposite and examine whether a contradiction can be derived as a result.* In the physical sciences the definitions of initial assumptions cannot always be as clearly drawn as in mathematics, however, so it is difficult to be certain that the list of alternative hypotheses has been exhausted. In the present discussion of particle-antiparticle interactions the finding of an incontrovertible hypothesis which does *not* require that matter is created or destroyed under any circumstances would have its merit, particularly from the point of view of the proponents of the classical atomic theory. To paraphrase another author with less direct involvement with the physical sciences who put these words in the mouth of Sherlock Holmes:¹⁹ "When you have eliminated the impossible, whatever remains, *however improbable*, must be the truth."

B. IS THERE A NON-HYDROGENIC STATE OF POSITRONIUM?

In the first stage of the electron-positron interaction there is a close analogy to what is observed in the hydrogen atom formation. However, the fact that the 1s state of positronium

undergoes spontaneous decay clearly distinguishes it from the proton-electron combination. In a broader sense, however, the positronium decay is similar to emission processes occurring in excited hydrogenic states such as the $^2P_{1/2}$ species, for example. After a short lifetime a decay photon is observed as the hydrogen atom returns to its ground state. Again quantum electrodynamics is able to describe this process with extremely high accuracy. The fact that no subsequent emission has ever been observed from the hydrogen $^2S_{1/2}$ state is why one refers to it as the ground state of this system. *But does this mean that the analogous 1s state of positronium is its ground state?*

The close similarity between the hydrogenic and positronium spectra predicted by quantum mechanical theories ranging from the non-relativistic Schrödinger equation to quantum electrodynamics is the primary justification for answering this question in the affirmative, but the observed positronium decay from its 1s state raises at least the possibility that this conclusion is incorrect. *If there is a state of positronium below that of the 1s species, the observed photon appearance can be explained in a different manner.*

To explore this possibility let us examine the mechanics of the $P_{1/2}-S_{1/2}$ emission process in more detail for the hydrogen atom. The initial system is clearly the atom in one of its excited states, whereas the final system consists of the same atom in its ground state along with a photon associated with a characteristic frequency. With reference to what has been said previously, it can be noted that this theoretical description does not represent a balanced reaction in the traditional sense.

Nonetheless, there are clear similarities between this process and positronium decay from its 1s state. This can be seen by considering the distribution of energy and momentum among the partners of the transition. Because of the relatively large mass of the H atom, conservation of energy and linear momentum requires that the photon carry away most of the energy accompanying the transition, but not all of it. In order that momentum also is conserved in the process, it is necessary that the H atom recoil slightly relative to its initial position. The momentum of the atom is thus given by the de Broglie relation²⁰ as $p = h/\lambda$ (λ is the wavelength of the emitted radiation and h is Planck's constant). Consequently, some of the energy lost in the transition is also carried away by the H atom, specifically an amount equal to $p^2/2M_H$, where M_H is the total mass of the atom, i.e., electron plus proton. For higher-energy transitions, such as gamma decays in nuclear processes, the recoil energy can be so large that the energy of the

emitted photon differs considerably from the internal energy difference of the pair of nuclear levels involved (Mössbauer effect²¹).

The decay of positronium can be viewed as similar in nature to the above emission processes, differing from them only in quantitative detail, provided one makes a crucial assumption, as outlined in what follows. It is generally accepted, for example, that the reason at least two photons are always observed after positronium decay is because of the need to conserve energy and momentum in the process. The rest masses of the particles present after positronium decay are equal, however, whereas in the other case the hydrogen atom is far more massive than the emitted photon. Consequently in the two-photon positronium decay the available energy and momenta are equally distributed whereas a much less equal distribution is found among the H-atom emission products. The assumption of a low-energy state of positronium below the 1s entity clearly would give more substance to this analogy, but there still remain difficulties as to how best to correlate the various product and reactant systems in the two processes.

To begin with, it seems straightforward to associate the emission quantum in the H-atom case to *one* of the photons observed in positronium decay. The energies of the H atom and positronium photons are different to be sure, but so as to perfectly satisfy the pertinent conservation laws in each case. In order to make the analogy even closer, however, one might correlate the second photon observed in the positronium decay to the H-atom $1s_{1/2}$ product itself in the other example. The latter association is tantamount to saying that this photon continues to have the same e^+e^- structure as the initial complex, simply existing in a lower-energy state than the 1s species (see Fig. 1). *Is something of this nature not a possibility?*

Pursuing this supposition further, it is important not to forget that both photons accompanying positronium decay (again in the most frequently occurring process) appear to be identical in every way, including with regard to their energies and absolute values of linear and angular momentum; only the directions of the latter two vector quantities are different, i.e., opposing. *There is thus no justification from experiment to attribute a different composition to one of the decay photons than to the other.* Before one can claim that particle balance has been achieved by the above assumption, however, it is necessary to face up to the fact that in this model there is apparently (at least) one more photon (even if its proposed e^+e^- structure is correct) present after the positronium decay than before it. At least there is comfort in realizing

that the same state of affairs exists in the H-atom emission process, and therefore that the analogy under consideration is not weakened on this basis.

In any atomic, molecular, nuclear or other radiative emission process, the conventional view holds that while the initial system consists of a single substance in an excited state, the final system consists of the same substance in a more stable state plus a photon of well-defined energy and momentum. Despite the universality of such processes, however, there is a way to describe them consistently, positronium decay included, without giving up the concept of complete particle balance. It is simply necessary to assume quite generally that the observed emission photons are also present *prior* to such transitions, *but that their energy and momentum are exactly zero in these initial states.*

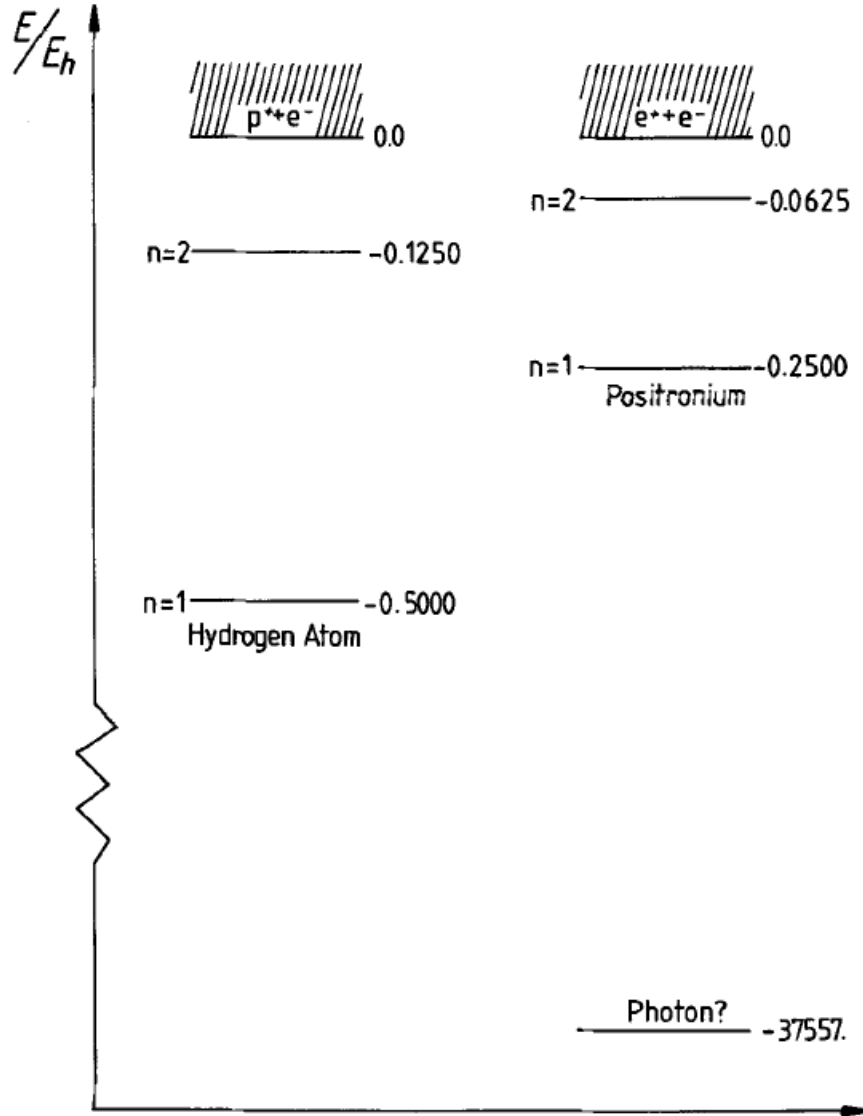


FIG. 1. Energy level diagram comparing the hydrogen atom and positronium. In the standard quantum mechanical theory the lowest (1s) levels of each system occur at -0.5000 and -0.2500 hartree, respectively. Corresponding ionization energies for the excited states of these two systems always differ by a factor of two as well, reflecting the different reduced masses of the electron in the two cases. The hydrogenic 1s state is known to be stable, however, whereas the corresponding e^+e^- state has a short lifetime and decays radiatively. This suggests that the true lowest state of the e^+e^- system actually lies far below the $n = 1$ state of positronium, and as such corresponds to the mass-less state of the photon itself, with an "ionization" energy equal to $2m_0c^2$ or 37557.7306 hartree.

We come then to the crux of the creation-annihilation hypothesis. To deny the creation-annihilation proposition is to insist at the very least that photons with zero energy and

momentum exist in their own right, despite the fact that according to the theory of special relativity they must be in a mass-less state under these conditions. Furthermore, the fact that radiative emission is observed whenever a system populates an excited internal state inevitably forces a new assumption, namely *that such mass-less particles can be found in sufficient numbers anywhere throughout the universe at all times.*

C. RELATIVISTIC CONCLUSIONS ABOUT PHOTONS OF ZERO ENERGY

The concept of photons other than the real variety encountered in everyday experience is by no means foreign to physical theory. So-called virtual photons²² play a key role in relativistic quantum electrodynamics and are invoked to explain details of the interaction of radiation with matter wherever it exists. Care is generally taken to exclude the possibility that such entities have any but a theoretical existence, however. One prefers instead to speak of them as a field quantity, with the non-localized properties of a wave-like substance, rather than simply as particles in the Newtonian sense. Even in classical electrodynamics it has long been known that there is a need to attribute non-zero energy and momentum to an electromagnetic field.²³ It is easy to find situations where failure to do this is tantamount to assuming that neither quantity is conserved in such processes. This line of approach is at least a clear indication that the *properties* of photons must be assumed to be present on a large scale everywhere in the universe, even if it is insisted that the particles as such are non-existent.

Why not actual particles, however? A typical argument is drawn from STR, asserting that once a particle with zero rest mass (as one assumes for a photon in free space) does not move with the speed of light c , it ceases to exist. The justification comes from the law of mass dilation⁴: $m = m_0 (1 - v^2/c^2)^{-1/2} = m_0 \gamma$, where m_0 is the rest mass and m is the relativistic mass of the particle moving relative to the observer with speed v . Accordingly, if $m_0 = 0$ and $v < c$, then γ is finite and $m = 0$. On this basis it is generally concluded that the corresponding particle cannot exist.

Yet examination of this argument shows that the supposed non-existence of the zero-mass photon is really just an assumption. All one learns with certainty from the above formula is that $0 < v < c$ is a condition under which the relativistic mass of such a particle vanishes. *It does not say that the particle itself necessarily ceases to exist as a*

result, however. The equation of non-zero mass with the possibility of a particle's existence cannot be said to be a logical consequence of the theory of special relativity. Instead, it constitutes an additional assumption which has had enormously broad consequences over the length and breadth of modern physical theory. At the very least it seems only prudent to give careful consideration to alternative interpretations of the possible meaning of a mass-less state of a system, as will be done below.

To begin with, it is possible to give a simple continuity argument which makes the existence of zero-mass photons plausible. Combining the mass-energy equivalence with the Planck frequency relation results in $E = h\nu = mc^2$. If $m = 0$, it follows that the corresponding frequency of the radiation is also of vanishing magnitude, whereas the wavelength $\lambda = v/c$ is infinitely large. A photon field with infinite wavelength is inaccessible to experimental detection and thus is not observable in the traditional sense. However, there is a clear distinction to be drawn between non-observation and non-existence. When an infinitesimal amount of energy is added to the same system, the above equation indicates that the photons now correspond to a non-zero frequency and a finite wavelength, which at least in principle can be measured. While there are definite limits as to how long a wavelength or how short a frequency can be measured in practice, it seems arbitrary to insist that beyond this point we dare not think of the particles as continuing to exist. If one can systematically withdraw energy from a single photon, at what point can it be safely assumed that it has been annihilated? The point is surely not that one can prove that a photon with exactly zero energy exists, but rather that *the converse cannot be proven* by this or any other means either. There is even a certain element of logic in dealing with the mass-less photon simply as *the limiting case in the above experiment* as the wavelength of the radiation becomes infinitely large.

Another mathematical point relating to the mass dilation formula should also be considered in the present context. If $m = 0$ for a particle of zero rest mass, must it continue to move at the speed of light? The answer based on the formula alone is clearly negative. Any value of the speed v up to and including c is consistent with zero relativistic mass for a system of zero rest mass. Even $v > c$ is not inconsistent, which fact, if nothing else, demonstrates that the mass dilation formula is really not at all restrictive on this point. The possibility exists in particular that a mass-less photon is *at rest* in a

given rest frame, unlike photons with non-zero relativistic mass. There is no contradiction in this with regard to the postulates of STR, since such photons defy observation in any and all inertial systems. The photon in a mass-less state actually corresponds to the null world-vector of energy, which means that application of any Lorentz transformation with $v < c$ leaves it unchanged. This characteristic thus precludes the possibility that observers in inertial systems moving relative to one another would ever come to different conclusions about whether a given photon's mass is zero or not. A zero-energy photon thus remains undetectable to all observers regardless of their velocities relative to one another.

The same conclusion is reached by consideration of the relativistic Doppler effect,²⁴ which holds that the frequency of light as measured by an observer moving relative to the source with a speed $v < c$ is a finite multiple of the frequency measured in the inertial system of the source itself. With reference to the thought experiment of the last paragraph, a photon with decreasing energy continues to move at the speed of light as long as its mass exceeds null. In the limit of zero energy, however, it becomes free to change its speed over a continuous range, although it must not do so. As a mass-less system, its momentum is unaffected by a change in velocity, so that a gradual reduction to zero velocity in any given inertial system is possible without altering its energy.

D. STATISTICAL MECHANICS OF MASSLESS PARTICLES

In order to obtain a satisfactory explanation for spontaneous radiative emission which avoids the creation-annihilation hypothesis, it is not only necessary to assume that photons can exist with zero energy and momentum, but also that they exist in great numbers everywhere in the universe. One can approach this aspect of the problem on two levels. The first simply relies on the arguments of the last section and takes them a step further, namely if it is not possible to observe a single photon in this state, then it is also not possible to contradict the view that there are great numbers of such systems. However, it is also possible to find more positive indications regarding this point by considering the phenomenon of blackbody radiation.

Quantum mechanics originated with Planck's discovery⁵ that the observed intensity distribution in a perfect absorber can be quantitatively described within the framework of

Maxwell-Boltzmann statistics, provided one assumes that only certain energy values are available for radiation of a given frequency. Specifically, Einstein showed that the mean value of the energy is obtained as

$$\langle E \rangle_v = \frac{\sum_{n=0} n h \nu \exp\left(\frac{-n h \nu}{kT}\right)}{\sum_{n=0} \exp\left(\frac{-n h \nu}{kT}\right)} \quad (\text{II.1})$$

instead of as a ratio of integrals in which n is treated as a continuous variable with non-integer values. The key point of interest in the present context is that *the $n = 0$ term in the above sums must be retained* to provide for an accurate representation of the observed spectral intensity distribution. This term does not alter the sum in the numerator, but it makes a decisive contribution to that in the denominator (partition function).

According to the theory of statistical mechanics, each term in the above sums corresponds to an allowed state for the system, in this case a collection of oscillators or photons with energy $E_n = n h \nu$. The zero-energy ($n=0$) photon is thus an ingredient in Planck's long-accepted solution⁵ to the blackbody problem. Moreover, as the lowest-energy state available to a photon associated with a given frequency, it is also the most frequently populated according to the Boltzmann exponential law, and this at any temperature T . In order to obtain the total intensity distribution it is necessary to integrate over all frequencies from null upwards. It is important to note, however, that zero-energy photon states are present in the distribution for *each value of ν* . This situation is illustrated in Fig. 2, in which the various frequencies are represented by the spokes of a wheel. The allowed states for a given ν can be thought of as being plotted as points along the corresponding spoke at a distance from the center of the wheel which is proportional to their energy. Especially if the Boltzmann populations are taken into account, it is found that by far the largest concentration of photons is at the center of the wheel, i.e. with exactly zero energy and momentum.

Allowed Photon Energies in Blackbody Radiation

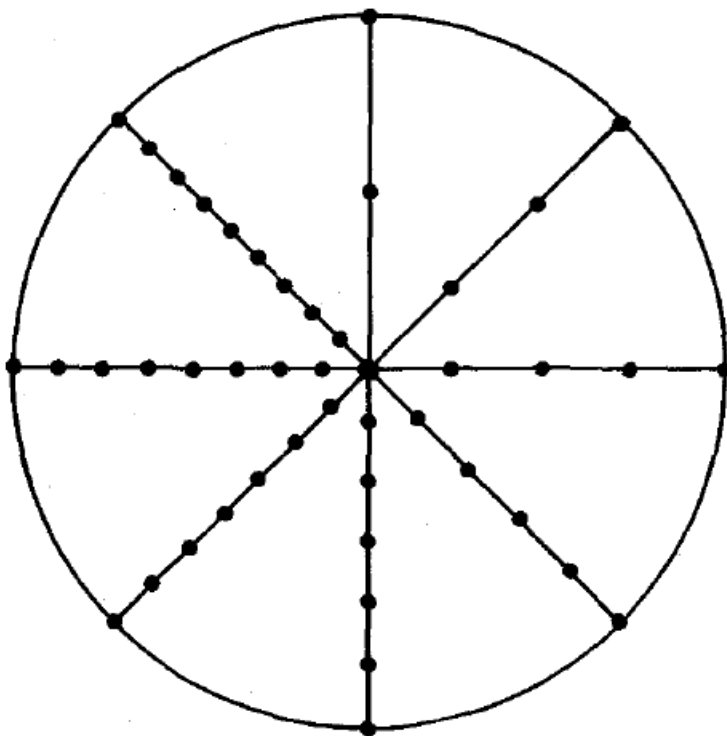


FIG. 2. Schematic diagram representing the distribution of allowed energy levels in the theory of blackbody radiation. Each spoke of the wheel corresponds to a fundamental frequency ν whose energy quantum $E = h\nu$ is always proportional to the distance between adjacent points on such a radius. Each such (equally spaced) point thus corresponds to an allowed energy level, one of which is always found at the hub of the wheel for each spoke, i.e. $E = 0$ is allowed for every value of ν . The magnitude of the energy quantum is shown to decrease monotonically as one proceeds in a clockwise fashion from the twelve o'clock position. The partition function used in Einstein's explanation of the blackbody radiation phenomenon must include the $E = 0$ levels explicitly for each fundamental frequency in order to obtain results which agree with experiment. It is thus clear from the diagram that the highest concentration of allowed energy levels by far is located at the hub of the wheel, and the form of the Boltzmann exponential factors $\exp(-E/kT)$ insures that the highest photon population always occurs for this energy value.

Since a blackbody of a given temperature displays the same intensity distribution regardless of its location, it must be assumed that this state of affairs exists everywhere. The relatively high density of zero-energy photons is a theoretical assumption apparently needed to explain observed phenomena. This circumstance does not constitute a proof of the hypothesis in the mathematical sense, but at least it can be said that the idea does not lead to

a contradiction either. Put the other way around, it would be a very damaging piece of evidence to the mass-less photon concept if states of zero energy had to be *excluded* from the partition function in order to achieve a satisfactory representation of the experimental observations. Quite to the contrary, Einstein made the opposite assumption of a high density of zero-energy photons. Seemingly the most natural interpretation for this theoretical approach is to conclude that the populations of *all* the various photon states are given correctly by the Boltzmann exponential factors in eq. (II.1), not only for those corresponding to non-zero quantum numbers. More details on this general subject may be found elsewhere.²⁵

A different, and theoretically superior approach, was introduced by Bose,²⁶ however. He concluded that Maxwell-Boltzmann statistics do not apply for photons. Specifically, the population index for photons needs to be changed from $\exp(-E/kT)$ to $(\exp(E/kT)-1)^{-1}$. This change has the effect of *greatly increasing the population of low-energy photons relative to the prediction of Maxwell-Boltzmann statistics*. In the present context, the key conclusion is that photons of *zero energy must be ascribed infinite population*. This result is thus the same as is reached using Einstein's quantization assumption discussed above, but the manner in which it is reached, namely on the basis of Bose-Einstein statistics, is completely in line with applications for atoms and molecules with non-zero rest mass.

E. SEARCHING FOR A QUANTUM MECHANICAL REPRESENTATION OF THE ZERO-ENERGY STATE OF THE PHOTON

It is now of interest to shift the discussion from the general topic of radiative emission back to the original subject of positronium decay. By comparison with details of the hydrogen atom emission process, it was concluded that the appearance of only photons after positronium decay does not rule out the possibility that the internal structure of both the initial and final participants in this reaction is exactly the same. At least, the creation-annihilation hypothesis can be avoided by means of such an alternative assumption. In other words, it is proposed that the photon may also have an e^+e^- elemental composition, simply existing in a state of lower energy than the $1s$ species associated with positronium itself. The observed frequencies of the decay photons dictate that the energy of such an e^+e^- state must be $-2m_e c^2$ or -37557 hartree relative to that of the separated electron and positron, i.e. the

negative of what one conventionally refers to as the annihilation energy. By comparison the energy of the positronium 1s state is -0.25 hartree (Fig. 1).

Qualitatively one can imagine an attractive potential *which binds the electron and positron so tightly together* that there is a mass reduction similar to that known to occur in nuclear reactions. The difference in this case is that there is a total loss of mass, and not just a few parts in a thousand.

Moreover, such a tightly bound e^+e^- state can have no counterpart in the hydrogen atom spectrum. It is clear that quantum electrodynamics provides no such attractive potential or corresponding internal state, but one also knows that the range of validity for this theory is limited to electromagnetic interactions. Nonetheless, any conceivable extension of this theory to include a tight-binding e^+e^- state of the type required must remain consistent with the latter theory in its description of conventional electromagnetic phenomena. Before looking for a more quantitative model to describe such a positronium state, however, it is well to remain on the phenomenological level in considering the consequences that avoiding the creation-annihilation hypothesis have upon the interpretation of other experimental observations in modern physics.

III. A SURVEY OF OTHER EXPERIMENTS INVOLVING PHOTONS

The discussion in the preceding chapter demonstrates first and foremost that there is no compelling proof that particles pass to and from existence in the decay of positronium. It is impossible to distinguish between objects which have gone out of existence from those which simply cannot be detected experimentally. The alternative assumption to the creation and annihilation of matter is thus that particles can exist in great abundance in a mass-less (zero-energy) state without being directly observable. Put more descriptively, this amounts to saying: "*We live in an infinite sea of mass-less photons.*" The question to be explored in the present chapter is how these concepts can be used to explain other fundamental observations in modern physics.

A. PROPERTIES OF THE PHOTON

The interpretation of positronium decay as an emission process involving different states of the same physical system has been shown to suggest that the photon itself is a

compound of a single electron and positron. It is therefore interesting to compare the properties expected for such an e^+e^- structure with those known experimentally for the photon. To begin with, it can be noted that a system containing two fermions in a highly bound state would be expected to obey the Bose-Einstein statistics observed for photons. The spin of the combined system must be integral, just as for positronium in any of its hydrogenic states. Whether a system consisting of an even number of fermions behaves as a boson or not is known to depend on the strength of the interactions holding the individual particles together.²⁷ The ^3He isotope, for example, is fermionic and non-superconducting, but combining it with another fermion (the neutron) produces ^4He , which behaves as a boson.

Otherwise, what we know of photons is that they have zero rest mass and no charge, the latter property being clearly consistent with an electron-positron composition. The fact that photons of a given energy are characterized by a definite frequency and wavelength does not distinguish them from other particles, as emphasized by the de Broglie relation²⁰ $p = h/\lambda$, and the Planck frequency law⁵ $E = h\nu$, and demonstrated explicitly for electrons by Davisson and Germer.²⁸ For photons there is the additional feature of oscillating electric and magnetic fields being involved explicitly in the wave motion. However, especially for optical photons, the frequency of the oscillations is too large to enable a direct measurement of the individual electric or magnetic fields.²⁹ The oscillating properties of photons/light are actually deduced from theoretical considerations, namely the solution of Maxwell's classical equations of electromagnetism.³⁰ In quantum mechanics photons have traditionally been treated as oscillators, without giving a detailed description of the internal structure which is ultimately responsible for such characteristics. All that can be said in the present context is that an e^+e^- composition for the photon is at least consistent with electromagnetic phenomena.

The dipolar nature of such a binary system meshes qualitatively with the photon's capacity for interacting with charged particles, especially when the photon is in relative motion to the latter. One would have to have much more detailed information concerning the wave function of the e^+e^- system in a given state of translation to make more specific comparisons with real photons. Similarly, since the speed of the photons is a consequence of their zero rest mass, this is again a conceivable property for a system with such a dipolar

composition, one whose verification would require a more quantitative theoretical treatment.

The polarization of light has been one of its most intriguing properties. It has been interpreted by Wigner³¹ to result from the fact that the photon possesses non-zero angular momentum \mathbf{J} . The “twoness” of the photon's polarization is thereby explained as a relativistic requirement according to which a particle moving with the speed of light must have \mathbf{J} oriented either parallel or anti-parallel to its line of motion. Quantum mechanically this means that only $M_J = \pm 1$ is allowed for photons, despite the requirement of symmetry that components with $M_J = 0$ also must exist. Circularly polarized light corresponds to an eigen-function of J_z , while plane-polarized implies a 50-50 mixture of both allowed M_J values and elliptically polarized light is any combination in between, all of which is consistent with the existence of an effective two-fold degeneracy. Careful experiments^{32,33} have demonstrated that the magnitude of a circularly polarized photon's spin component is \hbar , corresponding to $|J| = 1$, which is consistent with the Wigner interpretation,³¹ but also with a possible e^+e^- constitution for the photon itself.

Altogether it should be recalled that despite intense investigation over centuries, going back at least to the work of Newton³⁴ and Huygens,³⁵ there is very little consensus about the structure of the photon itself, or indeed whether it has any internal structure at all. Einstein remarked³⁶ in 1951 that, despite his efforts of the preceding half-century, he did not feel that he had come any closer to answering the question of what a light quantum is. He went on to say that apparently many people³⁷ did think they understood the matter, but that they were only deceiving themselves. At the very least his comments would seem to allow considerable latitude for further research into this question.

B. PRODUCTION OF PARTICLE-ANTIPARTICLE PAIRS FROM PHOTON COLLISIONS

The reverse process to positronium decay, in which an electron and positron are produced with the aid of high-energy photons, also needs to be considered in the present context. The assumption of an e^+e^- structure for each photon is obviously consistent with this result, but a few details require special attention. When a photon with energy equal to $2m_0c^2$ collides with a massless photon, no electrons are produced unless a heavy nucleus is also present. By contrast, if two photons collide head-on, and each has m_0c^2 energy,

electron production is possible in free space. The distinction can be understood from relativity theory.

A collision between such a mass-less photon and one with $E = 2m_e c^2$ is characterized by a total momentum of $p = E/c = 2m_e c$. If one of the photons were to dissociate into its elements e^+ and e^- , all the available energy would be used up for this purpose, so that the translational energy of the two electrons produced would have to be null. The latter condition makes conservation of linear momentum in such a process impossible, however. By contrast, if both photons have $E = m_e c^2$ and collide head-on so that the momentum sum $\Sigma p_i = 0$, it follows that the electron and positron can be set free, but must remain at rest in the original inertial system. The latter process is seen to be simply the reverse of the positronium decay process, or more precisely the reverse of the interaction of a free electron and positron which are initially at rest in a given rest frame.

More generally, it needs to be recognized that for a given energy E , the momentum of the photon (E/c) is always greater than for any particle with rest mass $m_{0A} > 0$, for which $p_A = (E^2/c^2 - m_{0A}^2 c^2)^{1/2}$. This fact prevents a single photon of any energy from causing a zero-energy photon to dissociate, because no matter how much energy is transferred, there is a disparity in the corresponding photon momentum lost and that which could be theoretically given to each of the electron products. The presence of a third body can remove this restriction, as is well known, but the point to emphasize in the present discussion is that the same result is found whether free space is thought to be involved, as foreseen in the creation-annihilation hypothesis, or if a mass-less *but existing* photon of e^+e^- structure is assumed instead.

To make this point more clearly, it is interesting to consider the effect of relative motion of the observer on the outcome of such experiments. The relativistic Doppler effect²³ tells us that the energy (frequency) of the photons in the above examples is dependent on the relative speed of the inertial system from which these quantities are measured. There is a clear exception to this rule, however, namely if the energy of the photon is zero in one inertial system, it must remain zero in any other. Thus, it is not possible to make the transition between the above two cases simply by changing the relative speed of the observer. As noted in Sect. II.C, a mass-less photon corresponds to a null vector in Minkowski space,³⁸ and as such is unaffected by any Lorentz transformation. At the same time, a photon with non-zero

mass can have its energy changed to any conceivable value other than zero by virtue of such a transformation. The consequences of these relationships are crucial in the present case, with electron-positron production in "free space" occurring only if both photons have non-zero energy, just as is observed experimentally.

With much higher energies it is also possible to generate proton-antiproton pairs,³ again as predicted by the Dirac theory.¹ It is clear that this result cannot be entirely explained by assuming an e^+e^- structure for the photon. Nonetheless, it cannot be said that such observations are inconsistent with what has been assumed so far. Rather, they force an additional assumption, namely that other types of mass-less particle-antiparticle binaries exist as well. There is, of course, a natural tendency to avoid introducing new types of particles into any theoretical framework, however. At the very least one hopes to keep their number to an absolute minimum.

As long as its rest mass is exactly zero, the mechanical properties already mentioned for e^+e^- , such as $v=0$, $\lambda = \infty$ and the like, could also apply to p^+p^- or related entities. One can only speculate that a p^+p^- system of zero rest mass will exhibit different properties under translation than do the corresponding e^+e^- species. Clearly, the dissociation energy of p^+p^- must be 1836 times greater than for e^+e^- , which condition already constitutes a major distinction. By the same token, the fact that neutron decay produces neutrinos,³⁹ whose rest mass is already close to or equal to zero, implies that there must be $\nu\bar{\nu}$ binaries as well, with extremely small to vanishing dissociation energies. The real challenge presented by these observations is to construct a quantitative theory, requiring as input at most such quantities as the rest mass, charge and perhaps magnetic moment of the interacting species, *which leads to binding energies of the above particle-antiparticle pairs which are equal to $2c^2$ times the rest mass of each of the respective constituents.*

Since the charge-to-mass ratio is much smaller for the proton than the electron, it seems clear that a p^+p^- binary would show much weaker electromagnetic effects than its e^+e^- counterpart. On this basis, it seems plausible that the traditional properties of a photon, i.e. oscillating electromagnetic field which is involved even in low-energy emission and absorption processes, are exhibited exclusively by the electron-positron mass-less binary systems. The statistical arguments given above, in conjunction with the discussion of blackbody radiation (Sect. II. D), are equally consistent with a high density of other systems of zero rest mass. At

least one knows that protons and antiprotons can be produced together wherever the appropriate energy and momentum conditions are fulfilled.

C. QUANTUM CONDITIONS OF PHOTON INTERACTIONS

The quantum jumps associated with photon interactions provided an important clue regarding the particle nature of light. In his explanation of the photoelectric effect,⁴⁰ Einstein reversed a trend away from the Newtonian view³⁴ of light as "corpuscles". He showed that surface ionization of metals could be most consistently explained by assuming that a single quantum of light gives up all its energy to a single electron. He used the word "heuristic" in describing his ideas because the (exclusively) wave theory of electromagnetic radiation was widely accepted by the physics community at that time.

While there can be general agreement that the photoelectric effect is inconsistent with a totally wave-like nature for light, it still must be regarded as extraordinary that any particle would transmit *all* its translational energy to a single electron in a given interaction. Such a property of photons is consistent with the concept of annihilation, because it is reasonable to assume that a particle which has gone out of existence does so by leaving behind all its energy and momentum. However, if it is assumed instead that the photon retains its existence after photo-ionization has occurred, but simply assumes a mass-less state which defies direct experimental observation, it is necessary to look more closely at the dynamics of this process to better understand the nature of the quantization phenomenon.

To this end, it is instructive to apply the laws of energy and momentum conservation to the absorption process, as depicted in Fig. 3. If the photon γ were to give off an arbitrary amount ΔE of its energy to an atom A with mass M_A , its momentum would decrease by $\Delta p_\gamma = \Delta E/c$. If the atom were to remain in the same internal state, this amount would appear in the form of translational energy, which means that the momentum of the atom would change by $\Delta p_A = (2M_A\Delta E)^{1/2}$. Conservation of momentum requires that Δp_A and Δp_γ be equal. For small ΔE this can never be the case, however, in view of the large mass of A. Setting Δp_A equal to Δp_γ shows that ΔE would have to be equal to twice the rest energy of A or $2M_Ac^2$, which corresponds to the GeV range.⁴¹

There is a solution to this dilemma, however, namely to have a part of the photon's energy be added to the internal energy of the atom, *i.e.* that another electronic state of the more

massive system be reached. If the excited electronic state differs by $h\nu$ in energy from that of the initial state, conservation of momentum requires that

$$\Delta p_\gamma = \Delta \frac{E}{c} = \Delta p_A = (2M_A)^{\frac{1}{2}} (\Delta E - h\nu)^{\frac{1}{2}}. \quad \text{III.1}$$

This is possible provided $h\nu$ is only slightly smaller than ΔE , again by virtue of the relatively large mass of A as well as the magnitude of c .

It is important to distinguish between two aspects of the absorption process in the foregoing discussion. First, the quantized nature of the atomic spectrum is seen to be directly connected with the large disparity between the respective masses of the atom and the photon. When one considers the translational motion of the atom, it is recognized that the energy levels available to it are actually continuous. It is the requirement of momentum conservation which restricts the possible transitions between different states of the same atom and thereby produces the quantization phenomenon. On the other hand, on the basis of these arguments by themselves there is no restriction put on the magnitude ΔE of the energy lost by a photon in the absorption process, save that it be less than its total energy $E = m_\gamma c^2$. Indeed, the analogous excitation brought about by electron impact is well known.⁴² One is thus still left with the conclusion that there is something special about a zero-energy, zero-momentum state of the photon, even though many aspects of the absorption phenomenon can be understood by just assuming that the photon is a particle of relatively small mass compared to the system with which it interacts.

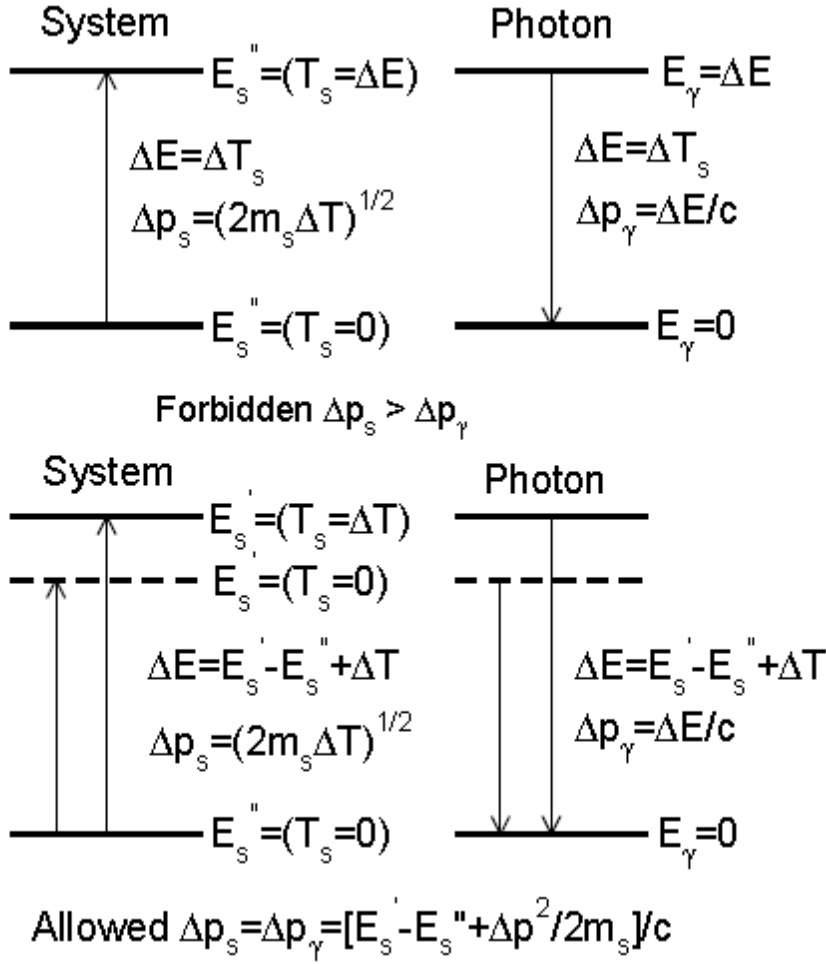


FIG. 3. Energy level diagram detailing the role of conservation laws in determining whether a given radiation absorption process is allowed or not. At the top of the diagram, the system is to retain the same internal energy E_s'' in the transition, *i.e.* the two levels shown differ only in translational energy $\Delta T_s = \Delta p_s^2 / 2m_s$ (non-relativistic theory), where m_s is the inertial mass of the system and Δp_s is the corresponding change in the momentum of its center of mass. Such a radiation process is always forbidden by the law of conservation of linear momentum because the rest mass of the photon is so much smaller than that of the system ($\Delta p_s > \Delta p_\gamma$). The only way for radiation absorption to occur is if the system changes its internal energy (from E_s'' to E_s') as well as its translational energy, as depicted in the lower part of the diagram. Under these circumstances the momentum conservation law can be satisfied for a particular value of Δp_s , namely one that is equal to $(E_s' - E_s'' + \Delta p_s^2 / 2m_s) / c$, where c is the speed of light. This condition rules out the occurrence of a radiation absorption process in which the system's translational energy does not change at all, also as indicated. Thus the "quantized" nature of radiation transitions is seen to be intimately connected with the photon's vanishing rest mass.

The fact that the energy transferred in the above process is exactly equal to $E_\gamma = m_\gamma c^2$ is thus seen to be a separate issue from the photo-ionization phenomenon itself. In other words, why doesn't the photon give off only part of its energy in inducing a transition in another system? Dirac used time-dependent perturbation theory⁴³ to answer this question, arguing that the incident radiation introduces a frequency-dependent term in the Hamiltonian of the atomic system. A resonance condition results according to which the energy of the most probable atomic transition, $h\nu = E_i - E_f$, must be the same as the energy of the incident photon, $E_\gamma = m_\gamma c^2$.

The prospect of a mass-less photon being formed as a result of this energy exchange (rather than that the original photon is annihilated in the process) suggests a somewhat different interpretation for this phenomenon, however, one which does not rely on the *ad hoc* assumption of wavelike properties for the incident radiation. If one simply looks upon the process as a collision between an atom and a photon moving with speed c , it seems plausible to demand that the observed energy exchange take place over a relatively small but finite period of time. As a consequence, the temporal requirements of the interaction are more readily fulfilled by an outgoing system whose velocity has been considerably reduced below the speed of light in a vacuum. As long as the departing photon possesses a non-zero amount of energy, this condition can never be fulfilled, but as has been pointed out in Sect. II.C, a *mass-less* photon is free of any such restriction, and thus can move at any speed less than c , including zero. In this view, the only practical means available to a photon to reduce its energy by virtue of an atomic collision is to assume a mass-less state, so that its relative speed compared to the system with which it interacts can be made as close to zero as possible. Accordingly, this interaction mode represents the only inelastic collision process available to a system of zero rest mass, since it is otherwise forced to move with the speed of light as long as it possesses any non-zero amount of translational energy.

By combining this result with the conservation of energy and momentum arguments first discussed, it is seen that the quantum characteristic associated with radiation absorption (and emission⁴⁴) can be deduced exclusively on the basis of the rest mass values of the photon and the interacting system, respectively. There is no need to postulate any wave characteristics for the field inducing the transition. Instead, one is led to conclude from knowledge of the internal energy states of the interacting system and the magnitude of its rest mass exactly which photon energy is required to induce maximum transition probability. The magnitude of this transition probability itself cannot be determined quantitatively on the basis of the above information

alone, and thus for this purpose one does have to introduce some additional information about the nature of the perturbing Hamiltonian, which itself is ultimately based on other experimental observations. This state of affairs does not affect the main conclusion in the present discussion, however, namely that the properties expected on the basis of relativity theory for a mass-less but nonetheless existent system are sufficient in themselves to allow for a suitable explanation of the observed tendency of photons to give up all their translational energy upon interacting with other particles.

These observations are also relevant to the positronium decay process discussed in Sect. II. In order for the de-excitation process to occur from the positronium $1s$ state to the proposed tightly-bound e^+e^- photon state (as depicted in Fig. 1), it again seems highly desirable that there be a minimum of relative motion between its initial and final systems. This condition cannot be said to be satisfactorily fulfilled when the product photon carries translational energy, because it must then move away from the original point of interaction with the speed of light. That would be something akin to a business transaction carried out between two people, one of which is riding on a speeding train while the other is standing on the station platform. In its mass-less state the photon can move with exactly the same velocity as the initial positronium system, thereby greatly improving the chances for such a transition. In this way, momentum can be conserved in the process, but the energy lost by the positronium complex still has to be carried away.

As shown in Fig. 4, the simplest way to accomplish this objective is to have the released energy divided up equally between two other photons which are in the neighborhood of the interaction locale, which again means they must initially possess zero translational energy. The conservation laws can then be satisfied by dividing the emitted energy equally among the two departing photons and having them move with exactly opposed momentum.²⁵ There are also angular momentum conditions to be satisfied, which is why the number of emitted photons is different depending on the multiplicity of the positronium state prior to its decay. This point will be considered more closely in Sect V.

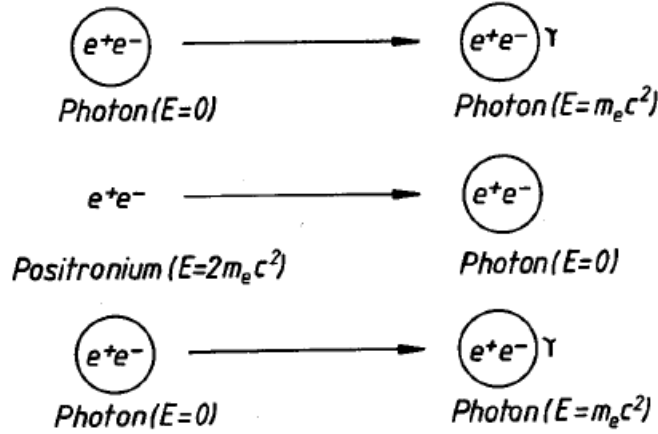


FIG. 4. Schematic diagram for the two-photon decay of (singlet) positronium. By assuming that the photon also has an e^+e^- composition, it is possible to describe this transition without assuming that particles are either created or annihilated in the process. In this model three e^+e^- binaries are involved, two of which are mass-less photons at the start of the process. They share the energy released by the positronium decay, and are observed as γ photons of equal energy at its conclusion. The final state of the original positronium system is another mass-less photon which, as its two counterparts at the start of the transition, escapes detection by virtue of its lack of energy.

To summarize, it is possible in this way to look upon the most commonly occurring positronium decay process (Fig. 4) as involving three distinct photons, each of which exists in its mass-less state at some point in the interaction.²⁵ One of them is formed as a result of the de-excitation of the (singlet) positronium 1s state (Figs. 1 and 4), thus eluding detection by virtue of its null frequency. The other two are already present at the start of the reaction and are also unobservable as a consequence of their lack of inertial mass. Upon taking up their share of the energy released in the decay process they are detected, however, giving rise to the "two-photon" classification commonly ascribed to this interaction.

Another important aspect of this topic arises in the treatment of the blackbody radiation phenomenon. As discussed in the preceding Section, Einstein's quantum assumption assigns allowed states to a given photon with only integral multiples of its frequency ν . Since a blackbody is a perfect absorber, each frequency is present, at least in principle, and one can think of such a system as a collection of atoms of the type just discussed. The success of Planck's assumption⁵ indicates that each frequency can be treated independently of the other, however. It also is important to recall that equilibrium is present which is conventionally thought of as arising from a series of collisions between the participating systems over a long period of time.

This suggests that photons of energy $h\nu$ only interact readily with each other or with photons possessing a multiple of this energy. The same conclusion can be inferred from the occurrence of the coherence phenomenon⁴⁵ in electromagnetic radiation.

It seems at least possible that the stability of the photon in its mass-less state is at the root of the observed quantum characteristics of the blackbody intensity distribution. The special conditions of velocity seen to be permitted in this state, namely $0 \leq v \leq c$, are at least suggestive in this regard. The presumed high density of zero-energy photons based on the results of the blackbody experiment clearly derives from the fact that this energy value is the minimal amount available to photons in general. The key point which distinguishes the blackbody radiation phenomenon from the other processes discussed previously in this chapter is clearly that a large ensemble of photons is required to describe the effect in a meaningful way. It is not surprising as a result that it is quite difficult to analyze this particular experiment in the kind of microscopic detail needed to deal with the general question of whether individual mass-less photons can exist or not.

D. COMPTON AND RAMAN EFFECTS AND BREMSSTRAHLUNG

The Compton effect⁴⁶ involves collisions between x-ray photons and weakly bound electrons and can also be interpreted in a very straightforward manner using conventional energy and momentum conservation arguments in conjunction with the Planck frequency and de Broglie relations. In this experiment a photon with a given energy is scattered off an (essentially free) electron and another photon is observed after the collision with lower energy and momentum than the first. It might be argued that the same (x-ray) photon is involved before and after the collision, but in view of experience with the absorption and emission of photons it is generally assumed that the first photon gives up all its energy initially and that afterwards this is distributed between the electron and a second photon. Again one conventionally speaks of annihilation of the first photon and creation of the second in the process, but one can just as well imagine that the first photon simply assumes a mass-less state upon collision, while another mass-less photon takes up the energy left over from the electron collision and appears as an x-ray photon, generally moving in a different direction than the first.

The Raman effect^{47,48} is closely related to the Compton effect, and involves inelastic scattering of visible light off molecular systems. If the initial frequency is ν , it is found that

photons emerge at right angles to the incident radiation with frequencies $\nu \pm \nu'$, where ν' is a characteristic infrared frequency which is small compared to ν . Again there might be a tendency to interpret incoming and outgoing photons as one and the same, only with changed energy, but the problem with this view are clearly the same in this regard as for the Compton effect. In both cases it is clear that a particle interpretation for the electromagnetic radiation allows for a quantitative description of these phenomena. There is no particular difficulty interpreting these effects in terms of mass-less photons located in the neighborhood of the pertinent collision processes.

Finally, it is pertinent in this connection to consider the *Bremstrahlung* phenomenon as well. In this case an x-ray photon is produced with an energy which is essentially equal to the decrease in an electron's kinetic energy caused by its collision with a heavy nucleus. The process can thus also be thought of as an interaction in which a mass-less photon picks up energy, similarly as in the emission processes discussed earlier. The fact that a third (heavy) body is required is again related to the energy-momentum conservation laws, especially the fact that a given energy value always corresponds to a greater momentum for a photon than for a neighboring electron by virtue of the disparity in their respective rest masses.

IV. VARIATIONAL THEORY FOR THE DECAY OF POSITRONIUM

In the preceding sections it has been suggested that there is merit in considering the decay of positronium as an interaction in which an electron and positron are so strongly attracted to one another that the resulting binding energy is exactly equal to the sum of their rest masses, i.e. $2m_{oe}c^2$ or 1.02 MeV. In the present section, we will focus on the goal of finding a suitable potential which is capable of producing such a relatively large binding energy, while at the same time giving consideration to the possibility that the solution to this problem may have relevance for other types of interactions, particularly those involved in the study of nuclear physics.

A. A SHORT-RANGE POTENTIAL

The natural point at which to begin this investigation is with the nature of the potential which might be capable of bringing about such a strong attraction between an electron and a positron, although careful consideration must later be given to the manner in which the kinetic

energy is treated as well. It is clear from the outset that this must be a distinctly relativistic problem, because the rest mass of the combined e^+e^- system is assumed to be much lower than the sum of those of the separated particles. Both the Schrödinger non-relativistic⁴⁹ and Dirac relativistic⁵⁰ treatments of positronium tell us that the lowest possible state for this system is analogous to the 1s state of the hydrogen atom. In this case the primary interaction is Coulombic, exclusively so in the non-relativistic treatment and almost exclusively in the relativistic.

A two-component reduction of the Dirac equation leads to the characterization of a number of perturbative terms which are basically magnetic in nature. The most commonly employed such approximation is that of Breit-Pauli theory,⁵¹⁻⁵³ including the Breit interaction. The perturbations are on the order of $\alpha^2/2 \cong 10^{-5}$ hartree ($\alpha = e^2/2\epsilon_0\hbar c = 0.007297 = 137.036^{-1}$, the fine structure constant), and include the spin-orbit (same- and other-orbit), spin-spin, orbit-orbit and Darwin terms, as well as the mass-velocity correction to the non-relativistic kinetic energy.⁵³ These terms increase as Z^3 or Z^4 (spin-same-orbit) for atoms with nuclear charge Z . For positronium as well as the hydrogen atom they remain quite small, however, and their effects are observed only as fine structure in spectroscopic studies. The potential terms all vary as r^{-3} and thus have relatively short ranges compared to the Coulomb interaction. This point bears further consideration, however, since binding energies of 1.0 MeV and higher are otherwise known only for nuclei, in which case there is clear evidence that short-range forces are involved to a high degree.

We can represent the presumed interaction schematically by plotting the total energy as a function of the average distance r between an electron and positron (Fig. 5). The 1s state of positronium can be thought of as corresponding to a minimum in total energy occurring at $r = 2.0$ bohr $= 1.016 \text{ \AA}$, i.e. roughly double the corresponding value for the H atom by virtue of the smaller reduced mass of the e^+e^- system. Toward larger separations the energy gradually increases to zero, i.e. the energy of the separated particles. The attractive Coulomb potential varies as r^{-1} , while the kinetic energy varies as $p^2 \approx r^{-2}$, from which it follows that the total energy itself at first *decreases* as the particles approach one another from a large distance. At the location of the energy minimum the shorter range of the kinetic energy term becomes the dominant factor, which explains why the total energy thereafter increases rapidly toward still shorter distances. It can be seen that these arguments are very close to those used by Bohr¹⁷ in arriving at his theory of hydrogenic atoms in 1913.

The binding energy at the latter e^+e^- minimum is only 6.8 eV, which is quite small compared to the 1.02 MeV given off when positronium decays from the corresponding (1s) state. The possibility we wish to explore in this work is *whether a second energy minimum does not occur at a much smaller electron-positron separation*. It can be imagined, for example, that at some point the total energy stops increasing toward shorter distances because an attractive short-range potential term begins to overcome the effects of the increasing kinetic energy in this region. Such a potential term would have to vary at a higher inverse power of r than either of the other two terms in the non-relativistic electrostatic Hamiltonian, and would have to be relatively unimportant in the region of the first hydrogenic energy minimum. At the same time, it is evident that some effect with an even shorter range eventually must take over and cause the energy to increase (Fig. 5) once more toward even smaller distances after the assumed 1.02 MeV absolute minimum value is attained. It is also clear that such a second energy minimum must be *totally absent in the corresponding hydrogen atom treatment*. Finally, it is only consistent to assume that an analogous double-minimum curve exists for the proton-antiproton system, but with a binding energy which is 1836 times larger, i.e. 1.85 GeV for its short-range minimum. At least one knows that this much energy is given off when the proton and antiproton interact, whereas no comparable loss of energy is observed for the combination of a proton and an electron.

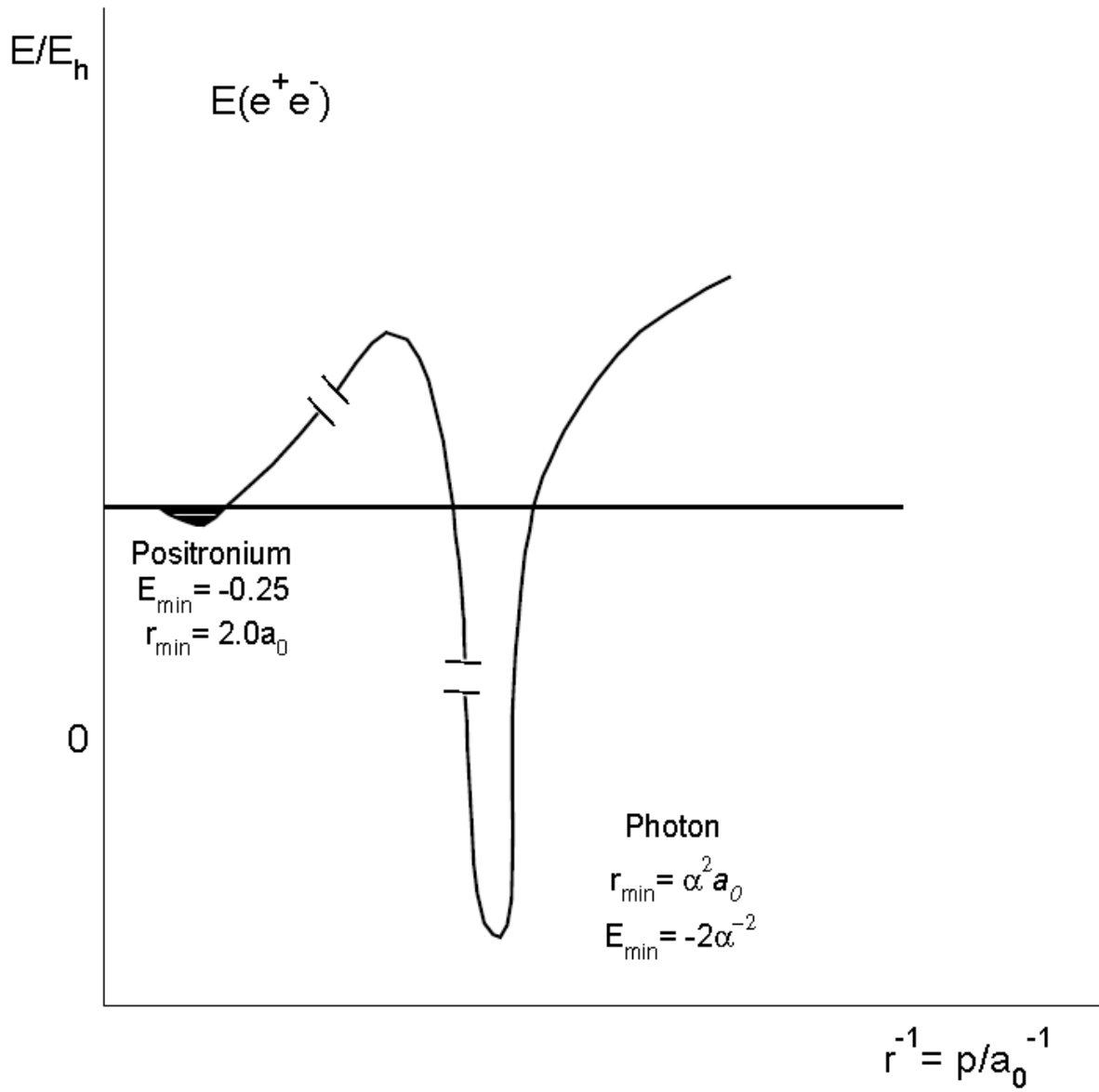


FIG. 5. Schematic diagram showing the variation of the internal energy of the e^+e^- system as a function of the reciprocal of the distance between the two constituent particles. At large separations the Coulomb attractive interaction dominates because of its long-range (r^{-1}) character. A minimum of energy of only -0.25 hartree is eventually reached, corresponding to the familiar hydrogenic 1s state of positronium, after which the total energy begins to rise because the shorter-range kinetic energy term begins to dominate. In the present model this trend is eventually reversed again at a much smaller e^+e^- separation, at which point attractive forces of even shorter range ($\approx r^{-3}$) begin to change more rapidly than the kinetic energy. Finally, a second potential minimum, much deeper than the first, is reached which corresponds to a binding energy exactly equal to the sum of the rest energies of the electron and positron. At still smaller inter-particle distances the total energy rises again, reflecting the effect of some momentum-dependent damping of the short-range potential in this region. A form for the potential is sought for which only the Coulomb energy minimum survives when the positron is replaced by a proton

Concentrating on the comparison of e^+e^- with p^+e^- , the obvious question is how can the differences in the properties of the proton and positron lead to such an enormous distinction in their respective attractions to the electron. The traditional view embodied in the Schrödinger⁴⁹ and Dirac⁵⁰ equations for one-electron atoms holds that the large difference in mass of the two positively charged particles only plays a minor role in this connection, simply affecting the reduced mass of the electron. The magnetic moments (which have the charge-to-mass ratios as a factor) of e^+ and p^+ differ by a far greater amount because of the difference in rest masses, but this distinction is found to be of only minor importance in the Dirac equation treatment, in which effects such as the spin-orbit and spin-spin interactions depending on this quantity are accounted for explicitly.

Yet one knows from the outset that if there is indeed a much lower-lying state of the e^+e^- system than the familiar $1s$ species, it cannot be found among the solutions of the hydrogen atom Dirac or Schrödinger equations. *To progress further in this regard it is necessary to do something differently.*⁵⁴ Especially since the effect that might cause such a novel tight-binding e^+e^- state seems almost certainly short-range in nature (Fig. 5), there is reason to give closer consideration to the above magnetic-type interactions. As noted previously, there are numerous Breit-Pauli terms which fall in this category, varying as the inverse cube of the distance between interacting particles. They are all of order $\alpha^2/2 \cong 10^{-5}$ hartree for typical *atomic* electron-nucleus separations, so this characteristic fulfills another requirement from Fig. 5, namely that such a short-range effect be relatively insignificant at these distances.

Most importantly, however, all these Breit-Pauli terms⁵³ depend on the product of the magnetic moments (or charge-to-mass ratios) of the interacting particles. For the spin-other-orbit, spin-spin and orbit-orbit terms each of these quantities appears once in the corresponding product. Thus these terms are weighted by a factor of 1836 (the ratio of the rest masses of proton and positron) larger for e^+e^- than for p^+e^- . For the spin-same-orbit and Darwin terms the distinction is less important because in these cases the square of the mass of one of the constituent particles is involved rather than the product of both. As a result, there is an extra factor of two for these interactions for e^+e^- than for the hydrogen atom, by virtue of the fact that the square of the charge-to-mass ratio of the proton is negligible compared to that of the positron. At atomic distances these distinctions are still relatively unimportant because in this range the

Coulomb interaction dominates, but it is not difficult to imagine the situation could be far different at shorter range.

Before considering this possibility further, however, it is well to note that the Breit-Pauli Hamiltonian terms as such also have some undesirable properties which make them unsuitable for a variational calculation, that is, one in which the charge distributions of the particles involved are allowed to assume optimal forms so as to minimize the total energy. Since these terms vary as r^{-3} , there is nothing to keep the total energy from decreasing beyond any limit. They do not therefore provide a possibility of a second energy minimum of the type indicated in Fig. 5, rather only the attractive branch to the long-distance side of it. This fact has long deterred giving serious consideration to the Breit-Pauli terms as having any truly dominant role to play in quantum mechanical calculations. Nonetheless, there are other higher-order terms in a power-series expansion of the Dirac equation which need to be considered to properly understand their role in determining atomic fine structure. Specifically, the next terms in such expansions are of the order of $\alpha^4 r^{-4}$, and these higher-order effects prevent variational collapse in Dirac-equation solutions that would otherwise occur if only the $\alpha^2 r^{-3}$ terms were included.

A similar situation exists for the Lorentz force in classical electromagnetic theory. There one has a term of the form $|\mathbf{p} + e \mathbf{A}/c|^2$, where \mathbf{A} is the vector potential. The cross term involving $\mathbf{p} \cdot \mathbf{A}$ is typically^{54,55} also of the form $\alpha^2 r^{-3}$ ($c = \alpha^{-1}$ in atomic units) for the interaction of a charged particle with an electromagnetic field, but it is damped at short range by the $|\mathbf{A}|^2$ term, which varies as $\alpha^4 r^{-4}$. An interesting possibility is nonetheless opened up by the fact that such repulsive terms *are of even shorter range and higher order in α^2 than their Breit-Pauli counterparts*. Including such terms might not keep the total energy in Fig. 5 from turning downward at short distances because of the attractive $\alpha^2 r^{-3}$ interactions, but they would insure that this trend not continue indefinitely toward still smaller inter-particle separations, *with the result that the proposed non-hydrogenic second minimum of energy could be formed*.

If one simply assumes a potential which is the difference of these short-range terms, $-\alpha^2 r^{-3} + \alpha^4 r^{-4}$, it is easily shown that it possesses a minimum near $r \cong 4\alpha^2/3$. Such a distance corresponds to roughly 10^{-5} bohr, which is a typical separation for bound nucleons ($r = \alpha^2/2$ is normally given for the range of the nuclear force, for example). The possible connection between an e^+e^- tight-binding state and the nuclear force is thus reinforced by these considerations as well.

There are basically three arguments for pursuing this line of reasoning further. First, the theory of quantum electrodynamics has a very precisely defined range of applicability. Despite its ability to make extremely accurate predictions for interactions involving electrons and photons in an atomic environment, it is generally accepted that the theory in its established form is not capable⁵⁶ of describing high-energy interactions such as those involved in nuclear bonding. Any indication as to how *the framework of quantum electrodynamics could be adapted so as to become relevant to the description of short-range forces* therefore merits serious consideration.

Secondly, the analysis of the positronium decay process has provided a basis for associating the assumed e^+e^- tight-binding state with the photon itself. Given the prominent role of the photon in quantum electrodynamics, it seems likely that its internal structure would also have an important relationship to the electromagnetic force.

Finally, examination of the multiplet structures and other properties exhibited by nuclei has already led to the conclusion^{57,58} in the nuclear-shell model that spin-orbit or related terms⁵⁹ are almost certainly involved in this type of high-energy interaction. Taken together these observations suggest that a solution to the proposed problem may lie in an adaptation of the Dirac equation which does not detract from the reliability of the original theory's predictions for quantum-electrodynamics phenomena, but which at the same time enables an accurate treatment of interactions of much shorter range and higher binding energy. This eventuality would amount to an extension of the Bohr correspondence principle,⁶⁰ which proved so effective in making the transition from classical to quantum mechanics at the beginning of this century.

B. KINETIC ENERGY CONSIDERATIONS

As remarked above, it is not satisfactory to use a non-relativistic form for the kinetic energy of the positron and electron if a large binding energy is assumed. In this respect the problem is significantly different than in the conventional treatment of nuclear binding, because there the kinetic energies of the nucleons are still relatively small compared to the energy equivalent of their rest masses.

As Einstein showed⁴ on the basis of his special theory of relativity, the non-relativistic $p^2/2m_0$ term is actually just an element in the power series of the square-root quantity $(p^2c^2 + m_0^2c^4)^{1/2} - m_0c^2$. The Breit-Pauli approximation includes⁵³ a term of order p^4 to account for relativistic kinetic energy contributions, but just as for the corresponding potential terms, it is

known that this correction leads to variational collapse when the electronic charge distribution is allowed to vary freely so as to minimize the total energy. It is possible to circumvent this difficulty,⁶¹ however, by employing the Einstein square-root expression directly in the Hamiltonian instead of relying on a truncated power-series expansion for it. The inconvenience of employing a square-root operator can be dealt with for general atomic and molecular systems by means of a standard matrix procedure.⁶¹ Each particle has its own kinetic energy and thus the square-root terms are treated as one-electron operators, exactly as their non-relativistic $p_i^2/2m_i$ counterparts in conventional quantum mechanical treatments.

This procedure brings with it another difficulty, however, namely how to separate the total kinetic energy into its internal and translational (center-of-mass) components. In the non-relativistic case it is well known that the $p_1^2/2m_1 + p_2^2/2m_2$ term can be replaced exactly⁶² by $p^2/2\mu + P^2/2M$ by employing a linear coordinate transformation (p is the internal and P , the center-of-mass, momentum respectively, and $M = m_1 + m_2$ is the sum of the individual particle masses and μ is the reduced mass). The new coordinates are defined as $x = x_1 - x_2$ and $X = (m_1x_1 + m_2x_2)/M$, where x_1 and x_2 are Cartesian coordinates of the two particles, with analogous expressions for the y and z directions. Since the Coulomb potential only depends on the internal coordinates, it is thus possible to effect a separation of coordinates which leads to the familiar situation that the solutions of the corresponding Schrödinger equation can always be formed as simple products of the type $\Psi(r) \chi(R)$, i.e. as a product of a function of the internal coordinates with one involving only those of the center of mass.

This procedure can be generalized for any number of particles, but there is an important assumption to be noted. The derivation generally used⁶³ comes from classical mechanics and relies on the fact that $\mathbf{p} \cdot \mathbf{P}$ cross-terms cancel out as a result of the coordinate transformation. The cancellation is perfect for the non-relativistic kinetic energy, but *use of the relativistic one-particle form*, in which p_1^2 and p_2^2 appear in separate square-root expressions, *does not lend itself to the same simplification*. If one of the particle's masses is much larger than the other, it is not difficult to find approximations from which the desired coordinate separation effectively results even in the relativistic case. Since the nuclear masses are so much larger than the electron's, it is therefore easy to justify this approach for atomic calculations, both for the Dirac equation itself and in the Breit-Pauli approximation.

The difficulty is not so easily circumvented when both masses are equal, however, as is the case for the present e^+e^- interaction. It might be argued that it is intuitively obvious that the center-of-mass motion can always be separated out, but for quantitative work, one would like to have a more solid basis for making this simplifying assumption. In general this question seems to have received relatively little attention in the literature, but it has been given more careful consideration elsewhere.⁶⁴

The most straightforward method under the circumstances is to simply work with the original Cartesian coordinates of each particle. The quantum-electrodynamics treatment of the positron uses a similar approach^{16,65} in evaluating higher-order effects, including transition probabilities for the decay into photons out of various positronium electronic states, except that it imposes an additional condition of $\mathbf{P} = \sum \mathbf{p}_i = 0$. In such a two-particle application this means that $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$. No comparable transformation is employed for the spin coordinates of both particles in this approach, however. Radiation corrections to the Dirac-equation results can also be computed by employing the same condition for the center-of-mass momentum.^{16,65}

When more than two particles are involved, complications arise in attempting to generalize this procedure, however, particularly in the relativistic treatment of the kinetic energy. In the transformed coordinate system, one of the particles is effectively singled out as a reference for the internal coordinates, so that $\mathbf{x}_i' = \mathbf{x}_i - \mathbf{x}_1$ is now employed instead of \mathbf{x}_i , for example (with \mathbf{x}_1 itself being retained in the new basis). As a result the expression for the conjugate momentum of x_1 is $\mathbf{p}_1 = -\sum_{i \neq 1} \mathbf{p}_i'$, i.e. a sum of internal momentum values rather than a single such quantity as in the two-particle case.

In order to develop a computational scheme which can be conveniently applied to systems containing more than two particles, it therefore seemed advisable to avoid making any type of transformation to internal coordinates, and hence to simply employ the original Cartesian coordinates of each constituent particle directly in constructing the corresponding quantum mechanical Hamiltonian operator governing the interactions of a given system. Use of such a coordinate system at all stages of the theoretical treatment carries with it the complicating feature that internal and translational characteristics are mixed together in the resulting wave-functions.

In this connection, it is worth recalling that $\Sigma \mathbf{p}_i$ commutes with each \mathbf{p}_i and \mathbf{r}_{ij} (particle separation) quantity, so that the translational energy operator itself must have a common (complete) set of eigenfunctions⁶⁶ with any Hamiltonian containing exclusively these kinds of variables. Hence, any exact solution of a corresponding Schrödinger *equation must always be characterized by a definite value of the translational energy*. This observation underscores another assumption in the usual separation-of-variables argument for internal and center-of-mass coordinates, however. The Breit-Pauli terms mentioned in the last section contain momentum factors as well as internal distances. Consequently, even when the non-relativistic kinetic energy is employed, the desired separation is not complete for a Hamiltonian containing these types of interactions. Again, this presents no real problem for calculations of one-electron atoms, in which the masses of the constituent particles differ greatly, but for a system consisting of only e^+ and e^- , there is need for more careful consideration.

Finally, it should be mentioned that there is a covariant two-electron equation⁶⁷ which does employ a separation of internal and translational coordinates just as for the Dirac equation. It is characterized by two separate time coordinates, whereas in what has been described above it is always assumed that there is only one such variable, and that it can be separated from its spatial counterparts in the usual way by virtue of the time-independent nature of the corresponding Hamiltonian operator. Since the stated *modus operandi* in the present study is to depart from the purely quantum-electrodynamics treatment of the e^+e^- system, and especially since no other heavy system is present, it is preferable to work with a relatively simple Schrödinger-equation formulation of this problem which does not make any assumptions regarding the way in which the internal motion is separated from that of the center of mass.

C. SUGGESTED DAMPED FORM FOR THE BREIT-PAULI TERMS

A prerequisite for constructing a Schrödinger equation to investigate high-energy processes is the use of a potential which is suitably bounded. The assumed tight-binding e^+e^- state most likely is the result of a short-range interaction which is strongly attractive over a given region of inter-particle separation but even more strongly repulsive at still smaller distances (see Sect. IV. A). The Breit-Pauli approximation employs attractive terms fitting this description but lacks corresponding shorter-range effects which would produce the desired second minimum in the e^+e^- total energy curve sketched in Fig. 5. We have seen how the Breit-Pauli kinetic energy term

(including the mass-velocity correction) can be replaced by the Einstein free-particle square-root expression to avoid variational collapse, without giving up the advantages of having a reliable approximation to the Dirac equation results for hydrogenic atoms. It remains to find a similar means of dealing with the Breit-Pauli potential terms.

If we look upon the spin-orbit and related interactions as terms in a power series,⁵⁴ it seems reasonable to look for a closed expression which approaches this result in the low-energy relativistic regime encountered in calculations of atoms with moderately heavy nuclei, something analogous to the Einstein relativistic kinetic energy, in other words. At least one knows from the form of the Lorentz electromagnetic force that the next higher-order terms after those of $\alpha^2 r^{-3}$ spin-orbit type vary as $\alpha^4 r^{-4}$.

Simply adding such terms to the Hamiltonian has several disadvantages, however. It falls short of the goal of replacing the Breit-Pauli interactions with closed expressions which themselves correspond to infinite-order power series. In addition, it is difficult to carry out computations with an operator varying as r^{-4} since not all integrals which would be required in a variational treatment possess finite values. For example, those involving only s-type basis functions do not fall in this category.

The form of the desired potential (Fig. 5) is reminiscent of that observed in nuclear scattering, and this suggests the following alternative. The r^{-3} and r^{-4} terms already discussed can be grouped together as $\alpha^2 r^{-3} (1 - \alpha^2 r^{-1} + \dots)$. The terms in parentheses are the beginning of a power series expansion of the exponential function $\exp(-\alpha^2 r^{-1})$, which in turn is quite similar to that appearing in the Yukawa potential^{68,69} in nuclear physics, thus making explicit the connection with the nuclear force description. By multiplying the Breit-Pauli α^2 terms with such an exponential function, we have a potential which is capable of producing the second minimum for e^+e^- in Fig. 5 while at the same time retaining the correct behavior needed at relatively large inter-particle separations to properly describe the conventional positronium (hydrogenic atom) states.⁵⁴

Since the damping effects produced by the exponential factor are relativistic in nature, it seems somewhat more likely that the corresponding argument is a function of the momentum of the particles rather than the distance between them, i.e. of the form $\exp(1 - \alpha^2 p)$, with $r^{-1} \sim p = |p|$. This choice has computational advantages as well, because it means working with individual

quantum mechanical operators which depend on the coordinates of a single particle rather than two. From the Lorentz classical Hamiltonian we can also anticipate that a given particle's momentum p is multiplied by its charge-to-mass ratio q/m . Finally, to obtain the desired binding energy for a given particle-anti-particle system it is convenient to introduce a free parameter A as an additional factor in the exponential argument.

The negative of the resulting potential is plotted in Fig. 6 (atomic units are employed throughout) as a function of the reciprocal inter-particle distance. For this purpose we use the approximate representation $V(r) = -\alpha^2 r^{-3} \exp(-\alpha^2 r^{-1})$. For high particle velocities it can be assumed that the kinetic energy varies as pc in the range of interest, which can therefore be represented in an analogous manner as $r^{-1} \propto^{-1}$ in atomic units, i.e. as a straight line. This diagram⁵⁴ is useful in analyzing how binding can be achieved with such an exponentially damped potential over a very narrow range, consistent with the total energy curve shown in Fig. 5. For small momentum values typical of electrons in atoms, the kinetic energy far outweighs the short-range potential because of the factor of α^2 in the latter expression. Coulomb effects are omitted from consideration for the time being.

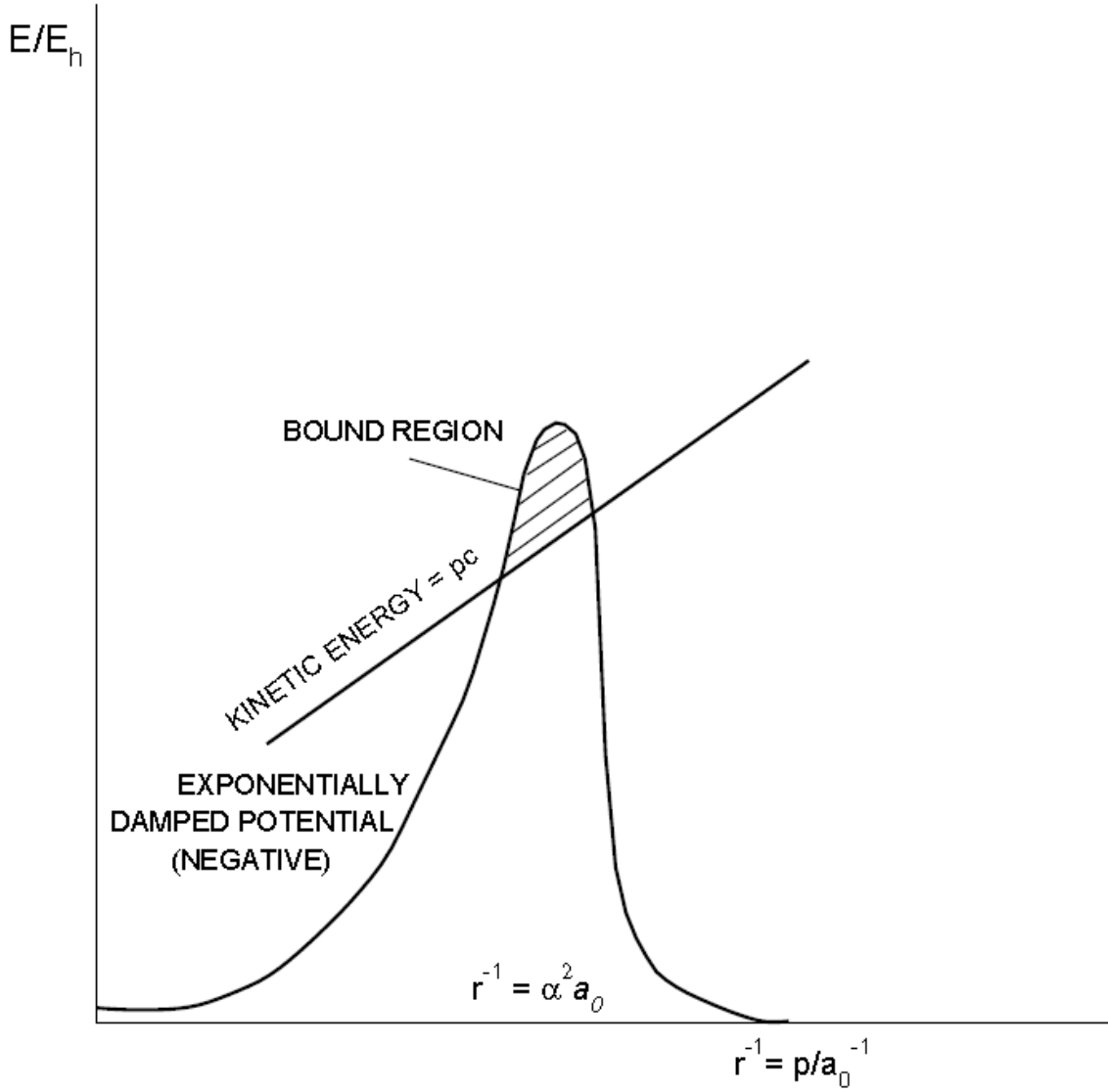


FIG. 6. Schematic diagram exploring the nature of a short-range potential required to produce the type of e^+e^- total energy curve depicted in Fig. 5. At relatively small inter-particle distances ($r \approx \alpha^2$) the relativistic kinetic energy varies nearly linearly with momentum $p \approx r^{-1}$. In order to obtain strong binding within a very narrow range of the $e^+ - e^-$ distance, the negative of the attractive potential term must reach a maximum shortly after it crosses the kinetic energy line from the long-distance side of the diagram, and then drop off again very sharply. Such an extreme cancellation effect requires a potential which fulfills at least three conditions: a) a small coupling constant (order α^2), b) a shorter range ($\sim r^{-3}$) than the kinetic energy and c) a momentum-dependent damping which is exponential in nature.

The absolute value of the potential increases as the cube of the momentum (or inverse distance) while the kinetic energy changes in a nearly linear manner, so it can be imagined that the two quantities eventually become equal at some point and binding becomes possible. The exponential damping becomes noticeable in the same region, however, and thus the above term does not increase as quickly as before and finally reaches a maximum. At the same time, the kinetic energy continues to increase linearly with p and eventually a second crossing with the negative of the potential occurs in Fig. 6. The area in which the negative of the potential exceeds the kinetic energy corresponds to a very small range of r , but the amount of binding with which it is associated can still be quite large. For example, at $r = \alpha^2$ the un-damped Breit-Pauli potential is of the order α^{-4} hartree, compared to the kinetic energy's order of α^{-3} hartree. Since the assumed binding energy for the e^+e^- system is 1.02 MeV or $2\alpha^{-2}$ hartree, it is clear that an enormous cancellation must occur because of the damping of the potential to obtain physically acceptable results.

This state of affairs is probably the strongest argument for employing an exponential damping to produce such a large degree of binding over a narrow range of interparticle separation. The fact that the Coulomb energy is also of order α^{-2} hartree for $r = \alpha^2$ suggests that it is not possible to ignore this effect either, however, despite its relatively long-range character. Nonetheless, the predominant feature in the tight-binding scenario given above is clearly the delicate cancellation at small interparticle separations between the exponentially damped Breit-Pauli terms and the relativistic kinetic energy.

D. SCALING PROPERTIES OF THE BREIT-PAULI HAMILTONIAN

One of the key requirements for the Schrödinger equation under discussion is that it leads to maximum binding energies for particle-antiparticle systems of $2Mc^2$, consistent with the Einstein mass-energy equivalence relation. One postulates that a Hamiltonian exists which has the required energy as its *minimal* eigenvalue instead of assuming that annihilation occurs and the total mass of the particles simply appears as the equivalent amount of energy. It is well known that the Schrödinger and Dirac equations for purely electrostatic potentials both have the property that their binding energies are proportional to the reduced mass of the electron in hydrogenic atoms, and this result is easily generalized for systems containing other charged particles such as muons or antiprotons. More interesting in the present context is the fact that the

proportionality between energy and mass also holds when the various Breit-Pauli relativistic corrections are added to the Hamiltonian.

To show this, let us assume that a solution to the Schrödinger equation is known for a particle-antiparticle pair with charge $\pm q$ and rest mass m_0 .⁵⁴ Furthermore, its lowest energy eigenvalue is taken to be $-2m_0c^2 = -2m_0\alpha^{-2}$ (in atomic units), corresponding to the eigenfunction $\Psi(r)$. The Hamiltonian itself consists of a series of kinetic and potential energy operators of the type discussed earlier, including exponential damping factors $F(p, q, m_0)$:

$$H(p, r, q, m_0) = (p^2\alpha^{-2} + m_0^2\alpha^{-4})^{1/2} - m_0\alpha^{-2} - q^2r^{-1} - q^2m_0^{-2}\alpha^2r^{-3}F(p, q, m_0) \quad \text{IV.1}$$

If the coordinates are scaled so that

$$p' = M_0m_0^{-1}p \text{ and } r' = M_0^{-1}m_0r \quad \text{IV.2}$$

the original Hamiltonian becomes:

$$\begin{aligned} H(p, r, q, m_0) &= M_0^{-1}m_0\{(p'^2\alpha^{-2} + M_0^2\alpha^{-4})^{1/2} - M_0\alpha^{-2} \\ &\quad - q^2r'^{-1} - q^2M_0^{-2}\alpha^2r'^{-3}F(p', q, M_0)\} \\ &= M_0^{-1}m_0H(p', r', q, M_0), \end{aligned} \quad \text{IV.3}$$

provided $F(p, q, m_0) = F(p', q, M_0)$. The corresponding Schrödinger equation in the scaled coordinate system thus becomes:

$$H(p', r', q, M_0)\Psi(r') = -2M_0\alpha^{-2}\Psi(r') \quad \text{IV.4}$$

i.e. by multiplying both sides of the Schrödinger equation for the original Hamiltonian by $M_0m_0^{-1}$. As a result, it is seen that $\Psi(r)$, or the function $\Psi'(r')$ obtained by making the corresponding coordinate substitution for it, is an eigenfunction of the analogous Hamiltonian for a particle-antiparticle system of the same charge q as before, but with rest mass M_0 instead of m_0 . Its energy eigenvalue is $-2M_0\alpha^{-2}$, exactly as required by the mass-energy equivalence relation.⁵⁴

Moreover, this result is quite general, since it is easily seen that the above scaling procedure has the effect of producing an entire spectrum of Schrödinger equation eigenvalues which differ by a factor of $M_0m_0^{-1}$ from those obtained for the original particle-antiparticle

system. Furthermore, by choosing the argument of $F(p, q, m_o)$ to contain the ratio p/m_o , as suggested by the form of the Lorentz electromagnetic force Hamiltonian discussed in Sect. IV.A., the requirement that this damping factor be unaffected by such a coordinate scaling is fulfilled.

The Breit-Pauli interactions also contain angular orbital momentum terms not included explicitly in the above Hamiltonian, but these are clearly unaffected by the above scaling procedure because they either involve only products of r and p , or in the case of the spin interactions, are completely independent of spatial coordinates. It is thus shown that the desired proportionality between binding energy and rest mass of the constituents of a particle-antiparticle binary system holds for the Breit-Pauli interaction as long as the charge q of the individual particles remains the same. This is clearly the case in comparing the e^+e^- system to p^+p^- , so the original objective sought at the beginning of this section is guaranteed by the use of such a Hamiltonian.

This result also tells us that the proportionality constant A in the exponential damping factor $F(p,q,m_o) = \exp [-A\alpha^2(q/m_o)p]$ must be the same for proton-antiproton interactions as between electron and positron. Alternatively, one might have assumed a different constant for the electron than for the proton, in which case one would have had to adjust the inverse mass dependence of the present exponential argument in order to obtain the desired scaling property. From the point of view of economy of assumptions and relation to established theoretical models, the former arrangement is clearly superior.

The units for the constant A remain to be discussed. The exponential argument as a whole must be dimensionless. Examination of the damping factor F shows that A must have units of the product of inverse charge and velocity (p/m) in order to ensure that the argument as a whole is dimensionless. The units of A are hidden when one uses atomic units directly in the corresponding calculations, as has been done in the preceding discussion. If mks units are used instead, A must have a unit of $(e\alpha)^{-1} \text{ s/Coul m} = 2.8530 \times 10^{12} \text{ s/Coul m}$. The actual quantity in the exponential is $A\alpha^2$, however, which has the value of 1.51926×10^8 .

E. EXPLICIT REPRESENTATION OF THE EXPONENTIALLY DAMPED HAMILTONIAN

The explicit form of the Hamiltonian discussed above is given in Table I.⁵⁴ It will be referred to subsequently as the XBPS Hamiltonian, the abbreviation standing for "Exponentially-damped Breit-Pauli Schrödinger Equation." As usual, there is a one-particle interaction for each constituent in the system, and a set of two-particle operators for each pair of such species. The Hamiltonian of Table I is given explicitly for only two (representative) particles with respective charges and rest masses q_i (q_j) and m_{oi} (m_{oj}), but in view of the above discussion it is easily generalized for the description of any number and combination of different particle types.

The only one-particle term in the Hamiltonian is the relativistic kinetic energy. Other common one-electron interactions such as the Coulomb nuclear attraction for an electron are to be found among the two-particle interactions. The rest energy $m_{oi}c^2 = m_{oi}\alpha^{-2}$ (atomic units are used throughout) is subtracted from the Einstein square-root operator in the usual way to represent the kinetic energy. This procedure effectively defines the zero of energy as that of the (infinitely) separated particles at rest.

The first two-particle interaction listed in Table I is the Coulomb term and it is the only one which is unchanged relative to the standard Breit-Pauli treatment.⁵³ The remaining terms are all exponentially damped, but fall into distinct categories, depending on whether they are multiplied with $q_1q_2/m_{o1}m_{o2}$ or q_1^2/m_{o1}^2 (or q_2^2/m_{o2}^2). In the first group are the spin-other-orbit, orbit-orbit and spin-spin terms, the latter including a δ -function component. The spin-same-orbit and Darwin interactions comprise the second category. Since all particles are to be treated equivalently, it is essential that each two-particle interaction be symmetric with respect to interchange of the corresponding indices, as provided for in the original Breit-Pauli Hamiltonian. The Coulomb term obviously satisfies this condition, since r_{12} is a scalar quantity denoting the magnitude of the distance separating the two particles.

The most significant change in the operators of Table I compared to those in the conventional Breit-Pauli Hamiltonian is their multiplication with exponential damping factors. For the terms of orbit-orbit type, it is assumed that two such factors are needed:

$\exp(-A\alpha^2|(q_i/m_{oi})\mathbf{p}_i|)$, for $i=1$ and 2 . To insure that the Hamiltonian be Hermitian, it is necessary that the same pair of factors appear on both sides of the original Breit-Pauli terms, since \mathbf{p}_i and \mathbf{r}_{ij} do not commute with one another. The arguments of all the exponential terms are defined to be negative-definite, hence the absolute value sign in these expressions. The damping factors for the spin-same-orbit and Darwin terms have a somewhat more complicated form, however. Since

the q/m_o factors appear squared for these terms in the original Breit-Pauli Hamiltonian,⁵³ it seems consistent to also use squares of the corresponding damping factors for the same particle in this instance instead of products of two different kinds as before.

The sign of the interaction in the latter class of operators requires additional comment as well. Since the pre-multiplying factor is either q_1^2/m_{o1}^2 or q_2^2/m_{o2}^2 , it is not possible on this basis alone to specify whether the interaction is attractive or repulsive. This information is contained in the product of the charges of the two particles, which appears explicitly in the orbit-orbit-type terms but not for the Darwin or spin-same-orbit operators. It is thus necessary to define a sign convention based on the product $(q_1/m_{o1}) (q_2/m_{o2}) \equiv G(1,2)$ according to which a positive result corresponds to a negative sign for both terms, while the opposite choice is made if the result is negative. Similarly as in the other case, the $\exp(-2A\alpha^2|(q_i/m_{oi}) \mathbf{p}_i|)$ factor must appear on both sides of the corresponding Breit-Pauli terms in order to preserve the required Hermitian character. The absolute value sign guarantees that the argument of the exponential is negative, i.e. the factor always reduces the absolute magnitude of the original Breit-Pauli interaction for the same charge distributions.

Designation	Operator	
Relativistic Kinetic Energy (one-particle only)	KE	$(\mathbf{p}_i^2 \alpha^{-2} + m_{oi}^2 \alpha^{-4})^{\frac{1}{2}} - m_{oi} \alpha^{-2}$
Coulomb	C	$q_i q_j r_{ij}^{-1}$
Spin-same-orbit (exponentially damped)	SsO	$-\frac{\alpha^2}{2} G(i, j)$ $\times \left\{ \left(\frac{q_i}{m_{oi}} \right)^2 \exp \left(-2A\alpha^2 \left \frac{q_i}{m_{oi}} \mathbf{p}_i \right \right) \times (\mathbf{r}_{ij} \times \mathbf{p}_i \cdot \mathbf{s}_i) r_{ij}^{-3} \right.$ $\left. + \left(\frac{q_j}{m_{oj}} \right)^2 \exp \left(-2A\alpha^2 \left \frac{q_j}{m_{oj}} \mathbf{p}_j \right \right) \times (\mathbf{r}_{ji} \times \mathbf{p}_j \cdot \mathbf{s}_j) r_{ij}^{-3} \right\},$

$$\text{where } G(i, j) = \begin{cases} 1, & \text{if } \frac{\mathbf{q}_i \mathbf{q}_j}{m_{0i} m_{0j}} > 1 \\ -1, & \text{if } \frac{\mathbf{q}_i \mathbf{q}_j}{m_{0i} m_{0j}} < 1 \end{cases}$$

Spin-other-orbit
(exponentially damped)

SoO

$$\begin{aligned} & -\alpha^2 \left(\frac{\mathbf{q}_i}{m_{0i}} \right) \left(\frac{\mathbf{q}_j}{m_{0j}} \right) \\ & \times \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_i}{m_{0i}} \mathbf{p}_i \right| \right) \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_j}{m_{0j}} \mathbf{p}_j \right| \right) \\ & \times \left(\mathbf{r}_{ji} \times \mathbf{p}_j \cdot \mathbf{s}_i + \mathbf{r}_{ij} \times \mathbf{p}_i \cdot \mathbf{s}_j \right) r_{ij}^{-3} \end{aligned}$$

Darwin Term
(exponentially damped)

D

$$\begin{aligned} & -\pi \frac{\alpha^2}{2} G(i, j) \delta(r_{ij}) \\ & \times \left\{ \left(\frac{\mathbf{q}_i}{m_{0i}} \right)^2 \exp \left(-2A\alpha^2 \left| \frac{\mathbf{q}_i}{m_{0i}} \mathbf{p}_i \right| \right) \right. \\ & \left. + \left(\frac{\mathbf{q}_j}{m_{0j}} \right)^2 \exp \left(-2A\alpha^2 \left| \frac{\mathbf{q}_j}{m_{0j}} \mathbf{p}_j \right| \right) \right\}, \end{aligned}$$

Orbit-orbit
(exponentially damped)

OO

$$\begin{aligned} & -\frac{\alpha^2}{2} \left(\frac{\mathbf{q}_i}{m_{0i}} \right) \left(\frac{\mathbf{q}_j}{m_{0j}} \right) \\ & \times \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_i}{m_{0i}} \mathbf{p}_i \right| \right) \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_j}{m_{0j}} \mathbf{p}_j \right| \right) \\ & \times \left[(\mathbf{p}_i \cdot \mathbf{p}_j) r_{ij}^{-1} + (\mathbf{r}_{ij} \cdot (\mathbf{r}_{ij} \times \mathbf{p}_i)) \mathbf{p}_j r_{ij}^{-3} \right] \end{aligned}$$

Spin-spin
(exponentially damped)

SS

$$\begin{aligned} & -\alpha^2 \left(\frac{\mathbf{q}_i}{m_{0i}} \right) \left(\frac{\mathbf{q}_j}{m_{0j}} \right) \\ & \times \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_i}{m_{0i}} \mathbf{p}_i \right| \right) \exp \left(-A\alpha^2 \left| \frac{\mathbf{q}_j}{m_{0j}} \mathbf{p}_j \right| \right) \end{aligned}$$

		$\times \left[(\mathbf{s}_i \cdot \mathbf{s}_j) r_{ij}^{-3} + 3(\mathbf{r}_{ij} \cdot \mathbf{s}_i)(\mathbf{r}_{ij} \cdot \mathbf{s}_j) r_{ij}^{-5} \right]$
Spin-spin δ	SS δ	$-\frac{8\pi\alpha^2}{3} \left(\frac{q_i}{m_{0i}} \right) \left(\frac{q_j}{m_{0j}} \right)$
(exponentially damped)		$\times \exp \left(-A\alpha^2 \left \frac{q_i}{m_{0i}} \mathbf{p}_i \right \right) \exp \left(-A\alpha^2 \left \frac{q_j}{m_{0j}} \mathbf{p}_j \right \right)$
		$\times \mathbf{s}_i \cdot \mathbf{s}_j \delta(\mathbf{r}_{ij})$

Table I. Definition of quantum mechanical operators present in the exponentially damped Breit-Pauli Hamiltonian employed throughout the present study (α is the fine-structure constant; atomic units employed throughout).⁵⁴ The indices i and j are used generically to represent two interacting particles; the quantities q_i (q_j) and m_{0i} (m_{0j}) are the electric charges and rest masses of the i^{th} (j^{th}) particle, A is the exponential damping constant (see text for units), and \mathbf{p}_i (\mathbf{p}_j), \mathbf{s}_i (\mathbf{s}_j) and \mathbf{r}_{ij} are the standard vectorial symbols for the linear and spin angular momenta of a single particle and the distance between the i^{th} and j^{th} particles, respectively. throughout the present study (α is the fine-structure constant; atomic units employed throughout).⁵⁴ The indices i and j are used generically to represent two interacting particles; the quantities q_i (q_j) and m_{0i} (m_{0j}) are the electric charges and rest masses of the i^{th} (j^{th}) particle, A is the exponential damping constant (units of e^{-1} ; see text), and \mathbf{p}_i (\mathbf{p}_j), \mathbf{s}_i (\mathbf{s}_j) and \mathbf{r}_{ij} are the standard vectorial symbols for the linear and spin angular momenta of a single particle and the distance between the i^{th} and j^{th} particles, respectively. Note that there is an error in the original version of this table,⁵⁴ namely the exponential factors appear more often than above.

It might be argued that the above sign convention for the Darwin and spin-same-orbit terms can be satisfied more simply by replacing q_1^2/m_{01}^2 by q_1q_2/m_{01}^2 . The result would clearly be the same as before only if $|q_1| = |q_2|$, however. This is the case for interactions between protons and electrons and their antiparticles, but in a more general formulation capable of dealing with other types of particles as well, it seems preferable to employ the former definitions. In this way the charge of a given particle only appears divided by its own rest mass in the damped Breit-Pauli interactions. This prescription at least formally allows for the treatment of charge-less, mass-less neutrinos with such a Hamiltonian, without introducing the types of singularities which otherwise would arise when coupling constants involving ratios of the charge of one particle and the rest mass of another are employed.

Returning to the immediate focus of attention, the e^+e^- system, it can be seen that the XBPS Hamiltonian of Table I exhibits a special symmetry in this case, namely it commutes with the charge conjugation operation C . This means that the corresponding eigenfunctions must be either symmetric or anti-symmetric with respect to interchanging the electron and positron coordinates.

The same result clearly holds for any particle-antiparticle binary system. This is a different situation than one normally encounters in conjunction with the Pauli principle,⁷⁰ in which permutation symmetry results because the component particles are indistinguishable. As a result it is not necessary to assume that the only physically meaningful solutions are of one symmetry-type, for example, anti-symmetric in the case of fermions. The symmetry in question arises not because the particles are indistinguishable, but rather because their (different) properties can be exchanged without affecting the form of their mutual interaction. This characteristic applies to all states of positronium, including those of hydrogenic type. In view of its special nature, however, there seems little point in incorporating this symmetry into the basis functions employed in explicit e^+e^- calculations with the XBPS Hamiltonian. In general, it will be assumed that the many-particle basis consists of products of Slater determinants⁷¹ for each particle type, i.e. anti-symmetry is assumed only for indistinguishable fermions. Bosons arise naturally as even products of fermions²⁶ and therefore require no additional symmetrization procedure.

V. COMPUTATIONS FOR MASSLESS PARTICLE-ANTIPARTICLE BINARIES

The central hypothesis explored in the previous chapters is that elemental matter can neither be created nor destroyed. Instead, it is argued that there exist strong attractive forces between elementary particles and their antiparticles which lead to mass-less binary systems whose lack of observability is understandable in terms of the Planck frequency relation ($E = m = 0$ implies $\nu = 0$ and $\lambda = \infty$). The XBPS Hamiltonian given in Table I is deduced in accordance with such expectations, and calculations employing it will be considered below, starting with the electron-positron system.

A. DETAILS OF THE COMPUTATIONAL METHOD

The computational methods employed are modeled after those of electronic structure calculations for atoms and molecules. In essence a matrix representation of the XBPS Hamiltonian is formed with the aid of products of one-particle functions. The first step is to compute integrals for each of the original Breit-Pauli operators⁵³ (except for the mass velocity terms) over a set of basis functions. Because of the decision to treat all particles in an equivalent manner, with no assumptions about fixed probability distributions for any of them, it can be noted that the Hamiltonian is symmetric with respect to all operations of the full rotation group

plus inversion. Accordingly, it is reasonable to localize all basis functions at a single center, specifically the origin of the coordinate system. As mentioned in the last chapter, the only one-particle operator under the circumstances is that of the (relativistic) kinetic energy. All the other Breit-Pauli terms in the Hamiltonian, including the Coulomb interaction, are treated exclusively as two-particle operators.⁵³

To construct the most general possible digital computer program it was decided to employ the same set of basis functions to describe each type of constituent particle. This approach seems reasonable because in a system with no net translation, the momentum of the particles should be similar, which implies basis functions of roughly the same extension for each of them. In the last analysis, it is always possible to employ individually optimized basis functions for each particle type by simply combining all such sets and making them available to describe the probability distributions of all the particles involved.

In this approach it is only necessary to compute the various one- and two-particle interaction integrals for electrons in the initial stage of the calculation. These results can then be conveniently adapted for treatment of particles other than electrons at a later stage. It was decided to use real Cartesian Gaussian functions to construct the one-particle basis, although it would also be convenient to use Slater-type exponential functions as more commonly employed in atomic calculations. Since no exact solutions for the Schrödinger equations of primary interest are known, the primary consideration was to choose a basis capable of describing general continuous functions which vanish at infinite distance from the origin.

An option for use of one-center potentials is available in the program which enables conventional atomic calculations to be carried out as well, in which case only two-particle interactions for electrons are assumed, although this restriction could also easily be removed. In such applications the charge of the nuclear center is required as input for the initial integral computations. Otherwise, no input is needed other than the exponents and coefficients of the contracted Cartesian Gaussian basis functions. Formulae for the calculation of Cartesian Gaussian integrals for all the Breit-Pauli operators may be found elsewhere.⁷² The computer program for evaluation of these integrals for the present study has been written by Chandra and implemented by Phillips, Liebermann and the author.⁷³

Because of the presence of spin variables in the one-particle functions it is desirable to carry out the overall treatment by employing a basis of eigenfunctions of the total angular

momentum operators j^2 and j_z . The corresponding spin-orbitals are formed with the help of the ladder operator technique⁷⁴ and conform to the standard Condon-Shortley convention.⁷⁵ The transformation of the Breit-Pauli integrals from a basis of spatial functions to the desired spin-orbitals is effected in two steps: the complex spatial eigenfunctions of l^2 and l_z are employed in the first transformation, followed by a second change of basis to the j^2, j_z eigenfunctions. In the process, the number of basis functions is doubled in the usual way because of the duality of the spin representation for fermions. The final two-particle integrals are classified according to quartets of the four quantum numbers, n, l, j and m_j , whereby n simply numbers the different spatial basis functions from unity upwards. Because of the spherical symmetry of the Breit-Pauli terms it is only necessary to specify three m_j values explicitly, since the only non-zero two-particle integrals are characterized by $\sum_i m_j^i = 0$.

Two other indices are required, referred to as Γ and μ . The index μ simply refers to the different operators employed, while Γ is an ordering index with only two values. A standard order of indices is defined, whereby the integral $\langle \phi_a(1) \phi_b(2) \phi_\mu(1,2) \phi_c(1) \phi_d(2) \rangle$ is characterized by $\Gamma = 1(2)$ if the standard order can be reached by an even (odd) number of permutations of the orbitals of the same particle. As is common practice,⁷⁶ the spin-same-orbit term is treated as two separate operators, one for $\mathbf{1}_1 \cdot \mathbf{s}_1$ and one for $\mathbf{1}_2 \cdot \mathbf{s}_2$. All other Breit-Pauli terms are symmetric with respect to particle exchange prior to multiplication by the various q/m_o coupling constants and can therefore be treated as symmetric sums at this stage of the computations, i.e. before the charges and rest masses of the actual constituent particles are introduced. More details concerning these computer programs will be given elsewhere.⁷⁷

B. TRANSFORMATION TO THE MANY-PARTICLE-TYPE BASIS

In Sect. IV it was argued that the original Breit-Pauli operators need to be adapted so that a Schrödinger equation employing them can lead to a mass-less state of a particle-antiparticle binary system, i.e. one whose binding energy is exactly equal to the sum of the rest masses of its constituents. The suggested changes have always involved functions of the momentum operator, specifically a square-root and several exponentials. These operators lead to integrals needed for their matrix representation which are relatively complicated to evaluate in a direct manner. This is particularly true for the two-particle exponentially-damped Breit-Pauli terms (Table I). Closed

expressions for the relativistic kinetic energy do exist,⁷⁸ but in order to deal with all the required operators in a consistent manner, an approximate integral evaluation technique must be employed.

It was thus decided to apply the matrix procedure referred to above⁶¹ for the treatment of both the square-root and exponential function operators. This involves use of the resolution of the identity formalism⁷⁹ and thus results employing it approach their exact values only as the basis set employed to obtain the matrix representation of the Hamiltonian nears completeness. Numerical tests^{80,81} comparing the exact and matrix representation values for relativistic kinetic energy integrals indicate that this level of approximation is suitable for the purpose at hand. The situation is more complicated for the damped Breit-Pauli terms of a two-particle nature, because it becomes impractical to saturate the basis to a similar extent as in the square-root operator tests. Nonetheless, it is always possible to judge the numerical stability of the final results of the calculations by comparing with analogous findings obtained with basis sets of different size. Since all the operators which are primarily responsible for these difficulties are functions of the momentum operator, it is possible to proceed in a very similar fashion for all of them.

Briefly, one first needs to form the non-relativistic kinetic energy matrix for electrons in the assumed Gaussian basis and then to diagonalize it. If the basis functions were momentum eigenfunctions, the desired relativistic square-root integrals could be obtained exactly by simply replacing the $p^2/2$ eigenvalues by the corresponding results for the operator in question. A related diagonal matrix can be formed even if momentum eigenfunctions are not used, but then it is necessary to subsequently carry out a reverse transformation to the original basis. The resulting non-diagonal matrix is then used for all subsequent computations. In the case of the damped Breit-Pauli terms it is necessary to carry out an additional matrix multiplication⁸⁰ involving four one-electron exponential matrices and the original two-particle Breit-Pauli counterpart discussed in Sect. V.A. The use of this matrix technique has the disadvantage of rendering the overall treatment non-variational, but again use of a suitably flexible basis set minimizes this effect.

It is at this stage of the computations that the charges and rest masses of the various constituent particle types are first needed. Each of the one-particle matrices for the relativistic kinetic energy and the exponential damping factors requires such input values explicitly. The corresponding matrices are generated for each particle type from the $p^2/2$ non-relativistic counterpart. It should be clearly distinguished between "particle types" and "particles" in this

connection. At this stage it is only necessary to know what kinds of particles are contained in the system at hand, and not how many of each. Once these one-particle matrices are generated it is necessary to carry out a series of four-index transformations for each Breit-Pauli operator μ and pair of particle types ρ . Because of the nature of the total XBPS Hamiltonian, it is necessary to distinguish between $\Gamma = (1, 2)$ and $\Gamma = (2,1)$, i.e. the order of particle-types is significant. This becomes obvious at the next step in the procedure, in which each transformed matrix is multiplied with an appropriate set of coupling constants formed from the charges and rest masses of the constituent particles (see Table I).

These results are then added together to form the final one- and two-particle Hamiltonian matrices, which are stored for further use in the many-particle phase of the theoretical treatment. The four-index (two-particle) integrals are ordered by means of quartets of indices for each basis function, as distinguished by their respective values for the quantum numbers n , l , j and m_j discussed above. In addition there are separate values for each of the two Γ permutations, as well as for each pair of particle-types ρ . The operator index μ has thus effectively been replaced by the particle-pair index ρ at this stage of the treatment as a result of the additions and scalar multiplications of the individual operator matrices to form the Hamiltonian two-particle matrix. The kinetic energy matrix elements are only ordered with respect to n and l because they do not vary with j and m_j . In addition, only diagonal ρ values are needed because of the one-particle nature of this operator.

The method employed to obtain approximation solutions for the Schrödinger equation discussed above is primarily the configuration interaction approach. A self-consistent field calculation is first carried out to generate an orthonormal basis of one-particle functions which allows an optimal description in terms of a single configuration.⁸²⁻⁸⁴ More details of this aspect of the general theoretical treatment are given elsewhere.⁸⁵

C. CALCULATION OF NON-HYDROGENIC STATES OF THE e^+e^- SYSTEM

The basic strategy for obtaining an e^+e^- state with a binding energy of $2m_e c^2$ is to vary the constant A in the XBPS Hamiltonian (Table I) until the lowest energy possible for a given number and type of basis function corresponds to the desired value. Without the exponential damping ($A = 0$), this goal can never be reached because the Hamiltonian is not bounded from below in this case. The first basis chosen contains two s - and two p -type primitive Gaussians

with initial exponents of 1.0×10^7 and $1.0 \times 10^8 \text{ a}_0^{-2}$ for both types. The exponents were multiplied by a scale factor η and a full CI calculation was carried out for symmetries spanned by this basis.

The computations show that the lowest energy results *for a state of 0^- symmetry* ($J=0$ and negative parity). By varying both η and the damping constant A , it is found that a minimum in energy of the desired value ($-2\alpha^{-2} = -37557.7$ hartree) occurs for the lowest 0^- state for $\eta = 0.11$ and $A = 1.054$ a.u. (this result was first obtained in 1987 at the University of Wuppertal⁷⁷). As expected, the value of the minimal energy increases with the magnitude of A for all η and advantage is taken of this relationship in subsequent optimizations. These calculations thus demonstrate that the XBPS Hamiltonian can be subjected to standard optimization techniques and the energies of the corresponding full CI secular equations can be suitably adjusted with a single free parameter.

The next step was to define a larger basis set consisting of five s- and five p-type primitive Gaussians. The exponents were initially assumed to form a geometrical progression: $\beta_N = \beta_1 \Xi^{N-1}$. The value of Ξ was taken to be 2.0 after some initial experimentation. As before, the energy optimizations are carried out with respect to a single scaling parameter η which multiplies all the β_i values. The exponents are taken to be the same for the s and p sets ($\beta_1 = 0.25 \times 10^8 \text{ a}_0^{-2}$). This is already a reasonably large s,p basis, consisting of 80 spin-orbitals or 40 for each particle, considering that a full CI optimization is to be carried out. It is nonetheless considerably smaller than those commonly used to study hydrogenic systems on a definitive basis,^{80,81} but it will serve the present purpose adequately, namely to examine the description of the proposed tight-binding states in the XBPS model.

It is again found that the most stable state of the e^+e^- system has 0^- symmetry. The corresponding wave-function consists of products of $s_{1/2}$ and $p_{1/2}$ (\pm denotes positron or electron function respectively) one-particle functions. Four products of spin orbitals are required for any pair of exponents⁷⁷:

$$\begin{aligned} 0^- = & p_{1/2}^+ (m_j = 1/2) s_{1/2}^- (m_j = -1/2) - p_{1/2}^+ (m_j = -1/2) s_{1/2}^- (m_j = 1/2) \\ & + p_{1/2}^- (m_j = 1/2) s_{1/2}^+ (m_j = -1/2) - p_{1/2}^- (m_j = -1/2) s_{1/2}^+ (m_j = 1/2). \end{aligned} \quad \text{V.1}$$

This function is seen to not only possess singlet spin but also to be symmetric with respect to the charge conjugation operation (see Sect. IV.E for a discussion of the latter symmetry property for

particle-antiparticle pairs). The optimal value for the scale factor η is 0.095 and the correct binding energy of $2\alpha^{-2}$ is obtained for $A = 1.078$ a.u. One should expect that the value of the damping constant increases with improvement in the one-particle basis and this behavior is observed. The change in A relative to its 2s,2p value is 2.3%, which indicates that we are already relatively close to the limit attainable with s and p basis functions. It is thus of interest to look at the results of the 5s,5p calculations in more detail below.

TABLE II. Total full CI energy (in hartree) of the lowest states of various symmetries of the e^+e^- system obtained employing the 5s,5p basis with scale factor $\eta = 0.095$ and exponential damping constant $A = 1.0775$ a.u. for the XBPS Hamiltonian of Table I.

Symmetry	First Root	Second Root
0^+	372025.142	561471.371
0^-	-37656.717	483278.726
1^+	596788.279	646488.246
1^-	592821.494	639727.142
2^+	785793.989	806032.930
2^-	691742.083	763807.974
3^+	853632.944	164188.023

TABLE III. Energy contributions (in hartree) of various operators (see Table I for definitions) for the 0_g^- ground state of the e^+e^- system obtained employing the 5s,5p basis with scale factor $\eta = 0.095$ and exponential damping constant $A = 1.0775$ a.u. for the XBPS Hamiltonian.

Operator	Expectation Value
Kinetic energy	1992262.978
Coulomb	-3233.894
Spin-same-orbit	-378340.098
Spin-other-orbit	-830993.650
Darwin Term	8889.119
Orbit-orbit	-407592.397

Spin-spin	-418648.776
Spin-spin δ	0.000
Total energy	-37656.721

The energies of the most stable e^+e^- states of each symmetry obtained in this treatment are given in Table II, from which it is seen that the lowest-lying 0^- species is favored by a large margin over the other states, being the only one which is bound with respect to the separated particles at this level of treatment. The 0^- total energy is broken down into contributions from each of the terms in the XBPS Hamiltonian in Table III. The total kinetic energy is 1.99×10^6 hartree or 54.2 MeV, so it is clear that the two particles are tightly bound. The main attractive contributions come as expected from the damped Breit-Pauli interactions. The spin-same-orbit, orbit-orbit and spin-spin terms are each in the order of -4×10^5 hartree, while the spin-other-orbit interaction is double this amount. The relative order of these contributions is easily understandable in terms of the constants multiplying each operator in the Breit-Pauli interactions themselves⁵³ (see Table I). The cancellation between these attractive potential terms and the two repulsive quantities is very delicate, as seen by the fact that the binding energy is only 1.89 % of the total kinetic energy. By way of comparison, it should be recalled that the binding energy is exactly equal to the kinetic energy in non-relativistic treatments of atomic systems (by virtue of the virial theorem).

It is important to note that a state composed of only s orbitals must give zero contribution by symmetry for each of the above Breit-Pauli potential terms, so it is not difficult to understand that substantial $p_{1/2}$ character is present in the optimal 0^- wavefunction. It is well known that the $p_{1/2}$ orbital is stabilized by spin-orbit coupling in atomic calculations, so this observation is also not surprising from that point of view. The binding is considerably enhanced for the e^+e^- system relative to the hydrogen atom because of the much larger magnetic moment of the positron compared to that of the proton, as demonstrated in Table III. In this sense the large binding of the 0^- state can be thought of as resulting from a tremendously large increase in the $p_{1/2} - p_{3/2}$ multiplet splitting in the spectrum of hydrogenic systems. It might be thought that a $p_{1/2} p_{1/2}$ configuration would be even more stable on this basis, but the added kinetic energy of $p_{1/2}$ *vis-a-vis* $s_{1/2}$ is decisive in avoiding this result. If the calculations are carried out under the condition of

vanishing translational energy ($\mathbf{P} = 0$; see Sect. V.B), it follows that the corresponding function of the electron-positron separation would be a p orbital (more specifically a $p_{1/2}$ species coupled with the spin of the reference particle to produce a 0^- solution). This point has been verified explicitly in calculations to be discussed subsequently in Sect. V. F.

The 0^- state's composition is noteworthy in another way, namely that it possesses an exactly vanishing expectation value for the Darwin δ -function term appearing in the original Breit-Pauli Hamiltonian, as can be verified using the above sample wave-function. This means that the electron and positron never occupy the same spatial position when their spins are identical. The expectation value of the Darwin term in the XBPS Hamiltonian itself (Table I) is non-zero in value (Table III), but this result stems from the form of the exponential damping factors employed in this case (same as for the spin-same-orbit term). The δ -function term in the spin-spin operator also has a vanishing expectation value for the 0^- wave-function, with and without the XBPS adaptations (Table III). This result shows that the avoidance property for electron and positron is not restricted to species of the same spin component since the $\mathbf{s}_1 \cdot \mathbf{s}_2$ scalar product allows α and β spins to have a non-zero interaction in general. In view of the opposite charges of the electron and positron it might be thought that these particles would prefer to be in the same region of space much more than these results would indicate, but one should not forget that the Darwin term itself is repulsive in this case, and thus its influence should tend to be minimized in a variational treatment. This consideration does not completely explain the observed behavior, however, as it must be assumed that such a relationship between the two charge distributions also maximizes the effects of the attractive terms in the XBPS Hamiltonian, such as the spin-spin, spin-orbit and orbit-orbit interactions. Since the vanishing magnitude for the various δ -function terms is so clearly tied up with the 0^- composition for the e^+e^- wavefunction, there is good reason to expect that similar results will be found regardless of the size of the one-particle basis set employed. In the $\mathbf{P} = 0$ limit for this state discussed above, even the expectation value of the exponentially damped Darwin term must vanish exactly since it is forbidden by symmetry from having an admixture of s spatial character, again as demonstrate in the calculations of Sect. V. F. Such a function must also be symmetric with respect to the charge conjugation operation, in agreement with what has been found in the calculations above for the 0^- wave-function with non-vanishing translational energy.

The exceptional nature of the $0^- e^+e^-$ state also gives added support to the hypothesis formulated in Chapter II to the effect that such a mass-less state of positronium is identical with that of the photon at rest. Although this result might be thought to be inconsistent with such an assumption because the photon is normally assigned 1^- symmetry, it should be recalled that the photon state given the latter designation does not correspond to a mass-less system. The experiments which demonstrate that the photon has one unit of angular momentum,^{32,33} for example, are based on a radiative emission process. Since the dominant mechanism in such transitions is electric-dipole in character, it follows that a change in angular momentum of one unit must have occurred in the process, for both the atomic system *and the photon itself* (see Fig. 3). In the creation-annihilation hypothesis, attention is centered on the change in the atom's angular momentum, but the premise in the XBPS model is that a mass-less photon is present prior to the transition and thus that its angular momentum is also altered upon emission. On this basis one can conclude from the rule for the vector addition of angular momentum that the original mass-less photon must have possessed one of three possible J values: 0, 1 or 2. Since it is difficult to imagine how a system with zero momentum, as must be assumed for a photon with $E = 0$, can have other than zero angular momentum, the finding that $J = 0$ is greatly preferred in the present calculations for the proposed e^+e^- tight-binding state is completely in line with the results of the photon angular momentum measurements.

Since the electric dipole moment has negative parity, one is also led to conclude that *the photon state after emission has a different parity than the initial mass-less state*. This deduction would appear to contradict the presently calculated finding that the symmetry of the latter is 0^- , but closer consideration shows that *the assignment of negative parity to the state of the emitted photon is perfectly arbitrary*. Parity designations of particles are always based on assigning one of two possible values to some standard system, and as such it is not possible to speak of absolute parity determinations⁸⁶ on the basis of experimental evidence alone. If one believes that the photon is created from nothing, it is perhaps natural to assign even parity to the initial photon state, and consequently odd parity to a photon generated as a result of an electric-dipole transition. The present calculations suggest a definite structure for the mass-less photon state, with 0^- symmetry, so that in this view one is led to conclude that the state of a photon observed after an electric-dipole transition is actually 1^+ , i.e. the opposite parity as conventionally assumed.

At the same time, one can point to more general discussions⁸⁷ of the dynamics of relatively light particles such as electrons or neutrinos when confined to small (nuclear-like) volumes which clearly suggest that it is highly unlikely that they exist in other than the lowest possible angular momentum state under these circumstances, which again in the case of a photon would be $J = 0$. Higher values of J are easily conceivable in conjunction with the translation of a tightly-bound system, however, which corresponds to the natural condition of a photon with non-zero energy. With regard to the positronium decay process, it is interesting to note that the present assignment for the photon's mass-less state implies that the most common process involving a singlet initial state corresponds to a $0^+ \rightarrow 0^-$ transition (see Fig. 4). Such a process is well known to be forbidden by any radiative mechanism involving only a single photon, consistent with what is observed (Sect. II.A). A possible two-photon transition would proceed with the aid of a 1^\pm virtual photon state, in which case the two decay photons would possess respectively 1^+ and 1^- symmetry, i.e. of opposite parity to one another but of the same total J value. Since there is no net change of total angular momentum in the overall process, it follows that the two photons must have complementary polarizations, again as observed. On the other hand, there appears to be no definitive means of establishing their relative parity experimentally. All in all, it can be concluded that the calculations appear to be perfectly consistent with both experimental observations and fundamental theoretical considerations with regard to the symmetry properties expected for particles of light.

The variation of the energy of the 0^- state with the scaling parameter η and the exponential damping constant A is shown in Fig. 7. The minimal energy always becomes higher as A is increased, as already noted. The E vs. η curve is closely related to the schematic total energy diagram given in Fig. 5, whereby η plays much the same role as the reciprocal of the square root of the inter-particle distance r by virtue of the scaling properties of Gaussian functions. For example, the expectation value $\langle r \rangle$ approaches zero to the right in Fig. 7 as η increases. The damping of the Breit-Pauli terms (Table I) produces a sharp minimum consistent with this interpretation, and the depth of the minimum is seen to be very sensitive to the value of the constant A . Qualitatively, it is easy to imagine from this diagram that a system trapped in such a deep potential well would be extremely stable. At distances to the left of the minimum's location the energy increases sharply, passing well beyond the zero value for the separated particles.

Eventually as the Gaussian exponents are decreased to their hydrogenic values ($\eta \cong 10^{-8} - 10^{-9}$), the total energy peaks and then the positronium 1s minimum is reached on the basis of the same Hamiltonian. A change of state occurs along the way, however, so that the $s^{1/2} s^{1/2}$ configuration becomes most stable. At this point the mean values of the exponential damping factors are very nearly unity, but they have the advantage of allowing a variational treatment of the hydrogenic states while still retaining a form of the Breit-Pauli interactions in the Hamiltonian. At distances smaller than that of the location of the deep potential minimum in Fig. 5, the energy is seen to increase very sharply as a result of the steep decrease in the magnitudes of the exponential damping factors in this region combined with the corresponding increase in the magnitude of the kinetic energy (see Fig. 6). Altogether, a consistent picture emerges of a tightly bound e^+e^- state with a binding energy of exactly $2m_{oe}c^2$, resulting primarily from an exponentially damped attractive potential of relatively short range.

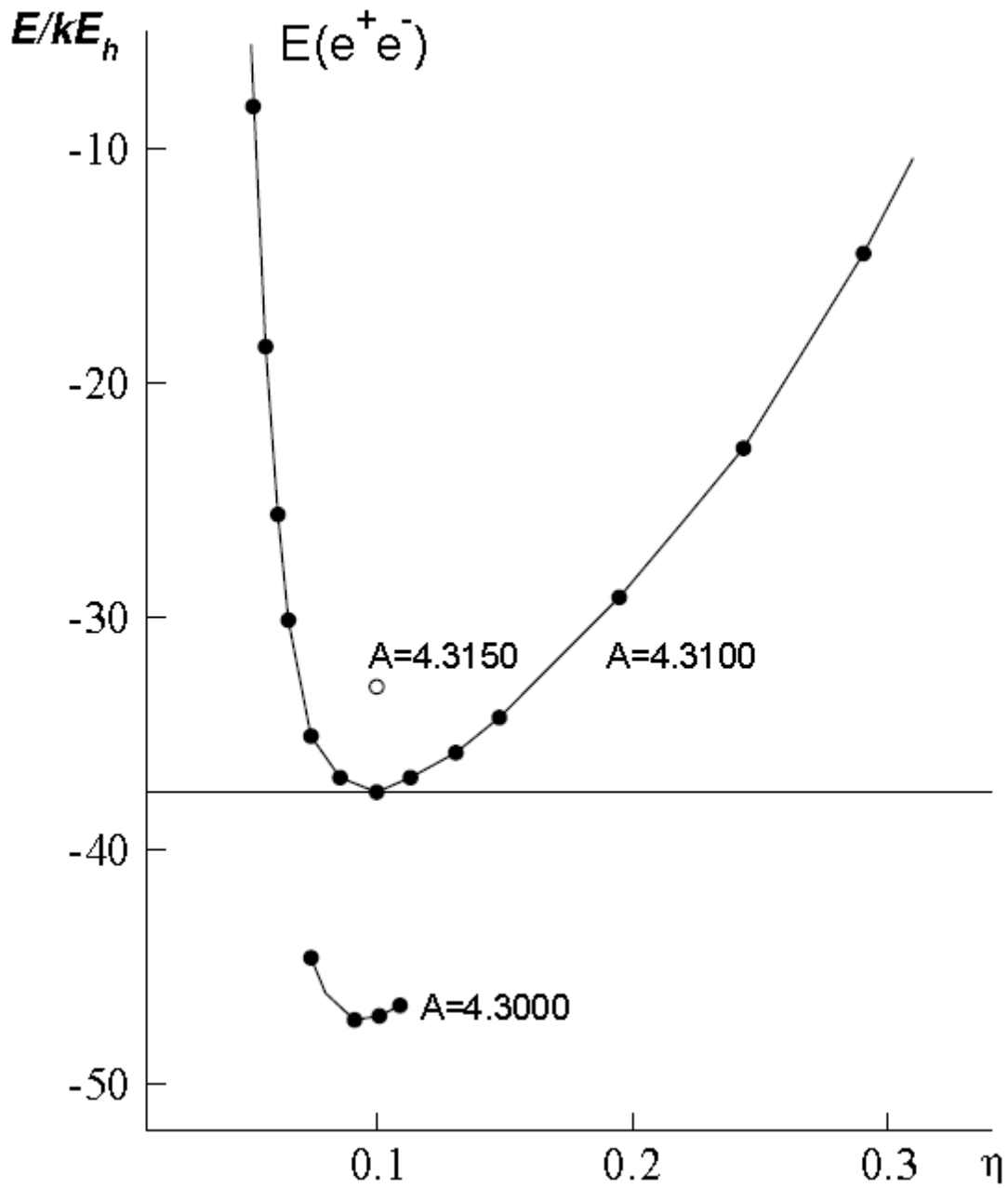


FIG. 7. Variation of the computed total energy (in hartree) of the e^+e^- system as a function of the 5s,5p basis set scaling factor η in the XBPS treatment for various values of the damping constant A . The horizontal line at the center of the diagram corresponds to the negative of the rest energy ($2m_{0e}c^2$) of the system. A value of A is sought which leads to this energy result for the optimum choice of η . Results for several other A values are also shown for comparison. Note that the values of A given in the diagram are 4.0 times larger than those given in the text due to a difference in definition.

D. CONSIDERATION OF TRANSLATIONAL EFFECTS IN THE XPBS CALCULATIONS

The calculations discussed thus far have yet to consider the magnitude of the translational component of the total energy. This can be done in several ways, the simplest of which in the present context involves the computation of the expectation value of $(\Sigma \mathbf{p}_i)^2$ for the wave-function already obtained. On this basis it is found that the translational energy is 1.3483×10^6 hartree, a very considerable amount. This result needs to be kept in perspective, however, when comparing with conventional calculations in which the center-of-mass motion is factored out. We have therefore carried out the analogous treatment for the hydrogen atom (employing a slightly larger $10s,5p$ basis), i.e. by also treating the translational and internal motion together explicitly. A total energy of -0.4660 hartree results, which is 0.0340 hartree higher than the non-relativistic Schrödinger equation value. The expectation value of the translational energy $\langle T \rangle$ obtained with this wavefunction is 0.015 hartree, which corresponds to a mean center-of-mass momentum of 7.42 a.u. Assuming that momentum increases proportionally with r^{-1} and comparing the value of the $0^- e^+e^-$ expectation value Table III (Coulomb term, -3233.89 hartree) to the unit value known for the H atom ground state leads to an estimate of the translational momentum of 2.400×10^4 a.u., which upon multiplication with $c = \alpha^{-1}$ corresponds to a translational energy for a mass-less system of 3.29×10^6 hartree, roughly two-and-one-half times larger than the above computed value.

The $p_{1/2}s_{1/2}$ form for this state helps to minimize the expectation value of T because it leads to a $\langle \mathbf{p}_1 \cdot \mathbf{p}_2 \rangle$ cross term which is necessarily negative (it is proportional to $-|\langle p_{1/2} \mathbf{p}_{s_{1/2}} \rangle|^2$). In other words, it is also possible to look upon the relatively high stability of the 0^- state as resulting from its ability to minimize the translational energy with this type of wave-function. More details regarding further calculations to study the effects of translation on the present XPBS results are given elsewhere.⁸⁸

E. CALCULATIONS WITH ELIMINATION OF TRANSLATIONAL ENERGY EFFECTS

It is possible to carry out the calculations in a different way using the Hamiltonian of Table I in order to eliminate the effects of translation on the wavefunctions. As discussed in Sect. IV.B, this can be done by simply applying the condition that the momentum difference of the two

particles always be equal to zero, that is by requiring that $\mathbf{p}_1 = -\mathbf{p}_2$. There remains only one spatial co-ordinate as a result, the \mathbf{r}_{12} inter-particle vector. In effect the calculations correspond to those carried out for atoms with a single center. It is merely necessary to replace the two kinetic energy terms in the original calculations with a single square-root operator in which the reduced mass ($\mu = 0.5$) of the electron-positron pair is used.

As a result, it is possible to employ a much larger Gaussian basis set in this series of calculations than in those already discussed in Sect. V. B. It contains 70 functions for each l value (s,p,d,...) whose exponents range in a geometric series from 0.00158 to 1.00000×10^{10} . In contrast to the first set of calculations, this basis set is large enough to describe the lowest-lying Rydberg states of positronium in the same treatment as for those with much smaller average radial distances. The energy results obtained for the lowest states of 0^+ and 0^- symmetry are given in Table IV. The corresponding results for $J=1$ and 2 have also been obtained and they produce no low-energy states other than the Rydberg species shown for the 0^+ and 0^- symmetries. It would be possible to obtain accurate energies for many more Rydberg states by simply adding more small-exponent Gaussian basis functions to the theoretical treatment, e.g. by expanding the above progression by several orders of magnitude downward to 1.0×10^{-7} .

The point which needs to be emphasized, however, is that the present calculations demonstrate explicitly that it is possible to construct a Hamiltonian operator whose eigenvalues not only include those of all the positronium Rydberg states, but in addition a single value with energy equal to $2m_0c^2$, i.e. the 0^- value that corresponds to complete elimination of rest mass from the e^+e^- system. This result comes out most clearly in the above calculations in which the energy of the center-of-mass of the system has been completely removed, i.e. no translation. To do this quantitatively, it is necessary to increase the value of A in the XBPS Hamiltonian from 1.0775 a.u. in Table II to 1.54666 a.u. to obtain the results of Table IV given below. This change demonstrates that the amount of exponential damping must be significantly increased in order to make up for removing the effects of translational motion in the lowest 0^- state.

Root No.	0^+	0^-
1	-0.250041 (n=1)	-37561.376228 ($2m_0c^2$)
2	-0.062507 (n=2)	-0.062502 (n=2)
3	-0.027779 (n=3)	-0.027778 (n=3)

4	-0.015626 (n=4)	-0.015625 (n=4)
5	-0.010000 (n=5)	-0.010000 (n=5)
6	-0.006943 (n=6)	-0.006944 (n=6)
7	-0.004686 (n=7)	-0.004903 (n=7)

TABLE IV. Total full CI energy (in hartree) of the lowest states of 0^+ and 0^- symmetry of the e^+e^- system obtained in the calculations with the condition of $\mathbf{p}_1 = -\mathbf{p}_2$ using the XBPS Hamiltonian of Table I.. In the 0^+ case 70 s-type functions are employed, while the same number of p-type functions are employed in the corresponding 0^- treatment. The value of the exponential damping constant A in the XBPS Hamiltonian has been varied so as to obtain a minimal energy for the 0^- states of $2m_e c^2 = 1.02$ Mev. The value of $A = 1.54666$ a.u. is found to fulfill this condition to a satisfactory degree. The principal quantum number n of each Rydberg state is indicated in parentheses next to the corresponding energy value.

An analysis of the energy contributions to the lowest-energy state is given in Table V. These results can be compared with the corresponding values given in Table III for the calculations in which translational effects are not excluded. For example, the Coulomb energy is

Operator	Expectation Value
Kinetic energy	1262707.2926853
Coulomb	-2662.0961756246
Spin-same-orbit	-230447.7644808
Spin-other-orbit	-460895.52896168
Darwin Term	0.000000
Orbit-orbit	-375815.5148147
Spin-spin	-230447.7644808
Spin-spin δ	0.000000
Total energy	-37561.396228

TABLE V. Energy contributions (in hartree) of various operators (see Table I for definitions) for the 0_g^- ground state of the e^+e^- system obtained with exponential damping constant $A = 1.54666$ a.u. for the XBPS Hamiltonian under the $\mathbf{p}_1 = -\mathbf{p}_2$ (zero translational energy) condition.

increased by 571.798 hartree, showing that the distance between the electron and positron is greater in the present treatment excluding translational effects. Because of the definition of the Coulomb operator in Table I, it is possible to obtain an average value for this separation simply by taking the reciprocal of the corresponding energy value. The inter-particle distance is therefore estimated to be 3.756×10^{-4} bohr in the present treatment, as opposed to the corresponding value of 3.09228×10^{-4} bohr when translational effects are included. The corresponding value for the positronium $0+$ ground state is 2.00 bohr, which is 5324.8 times larger. This value in turn is equal to $7.05 \alpha^2$ bohr. It is still more than an order of magnitude larger than a typical separation between nuclear particles ($\alpha^2/2$ bohr). The smaller distances in the latter case are reasonable in view of the much smaller kinetic energies expected for proton and neutron constituents of nuclei as opposed to that of the electron and positron in the present XBPS treatment.

The average value of the momentum p in the lowest root can be estimated by dividing the kinetic energy by in Table V by α^{-1} , the speed of light in a.u. The result is 9214.42 a.u., which is 18428.8 times the corresponding value in the 0^+ positronium ground state. The product of this value and the above distance is 3.46, which may be compared to the corresponding product in the positronium ground state of 1.0, which is the same ratio as found in the hydrogen atom calculations. Thus, it is found that the momentum of the lowest-energy state in Table V increases notably faster than the corresponding inter-particle distance decreases. This relationship is expected on the basis of Fig. 6, which shows that the kinetic energy, which is proportional to the momentum in this region, is increasing linearly toward shorter distances while the mean distance is increasing at a slower rate because of the effects of exponential damping.

The Breit-Pauli terms have an interesting relationship to one another, namely exactly the same value is obtained for the spin-same-orbit and spin-spin operators, which in turn is exactly one-half of that found for the spin-other-orbit term. The value of the orbit-orbit term lies between these two values. The values shown in Table III for the calculations which do exclude translational effects on the energy eigenvalues are much more irregular. Finally, the value of the Darwin term's energy is exactly zero in Table V, which is the expected result given the $l=1$ value of the (p) basis functions in this case. The non-zero value in Table III shows that translational effects tend to allow the wavefunctions to avoid having a node at the origin, although they do not

change the fact that the spin-spin delta term must be of vanishing magnitude for $L=1$ symmetry, also as found in Table V, i.e. with or without consideration of the effects of translation.

F. THE PROTON-ANTIPROTON BINARY

As pointed out in Sect. IV.D, there is a scaling property for the XBPS Hamiltonian (Table I) which requires that for every e^+e^- wavefunction, there exists a corresponding p^+p^- solution with an energy eigenvalue which is exactly $m_{op}=1836$ times greater in magnitude. The desired p^+p^- eigenvector can always be obtained from the relation: $\Psi_{e^+e^-}(r) = \Psi_{p^+p^-}(m_{oe}r/m_{op})$. In other words for the p^+p^- system everything is played out in a coordinate system which is contracted by a factor of 1836 relative to that of e^+e^- . The Gaussian exponents in the e^+e^- basis must each be multiplied with 1836^2 in order to obtain the desired scaling relationships for each energy quantity. The same scaling property exists for the translational energy operator T , so all energy results obtained above for the electron-positron system can be converted over to those of p^+p^- simply by multiplying the corresponding e^+e^- value with 1836. In particular, this scaling relationship allows one to employ the same value for the exponential damping constant as before, thereby giving this quantity more of a general character than might otherwise be assumed.

It might come as a surprise to see that the required binding energy of $2m_{op}c^2 = 1.876$ GeV comes from short-range effects which are generally associated with magnetic interactions, i.e. Breit-Pauli terms, because the magnetic moment of the proton is so small compared to that of the electron. Closer examination of the effects involved, however, underscores the fact that the relatively large mass of the proton is actually quite beneficial in forming a tight-binding state which takes extensive advantage of such short-range interactions. To begin with, there is the obvious fact that the kinetic energy associated with a given momentum value is substantially smaller for a proton than for an electron. The occurrence of the mass in the denominator of the exponential arguments of the XBPS Hamiltonian (Table I) is an even more significant factor, however, since it leads to a drastic reduction in the exponent of the damping function for a given momentum *vis-a-vis* that for a lighter particle (Sect. IV.D). This means the short-range potential terms are still going down in energy for inter-particle distances much shorter than the $r \cong 7\alpha^2$ value favored by the e^+e^- system (Fig. 5). As usual, the exponential damping ultimately prevents the situation from getting out of hand, but the corresponding p^+p^- energy minimum occurs for $r =$

$7.05 \alpha^2/m_{\text{op}}$ (see the discussion after Table V) or $2.044 \times 10^{-7} a_0$ ($1.08 \times 10^{-17} \text{ m}$), making this probably one of the shortest-range interactions which occurs in nature.

The small magnetic moment of the proton only means that magnetic-type interactions occurring at typical atomic separations (Bohr radius a_0) are nearly negligible compared to those of electrons. By contrast, at the much smaller inter-particle distances preferred by the p^+p^- system, even the Coulomb attraction is far from negligible. For $r = 7.05\alpha^2/m_{\text{op}}$ the expectation value for this relatively long-range interaction would be -133 MeV, for example, which would be 0.071 times the total binding energy of the p^+p^- system (1876 MeV). From this point of view, the corresponding Breit-Pauli energy contributions, though considerably larger at such distances, do little more than counter the system's enormous kinetic energy, which at $r = 7.05\alpha^2/m_{\text{op}}$ can be estimated to be 63.1 GeV based on the result of Table V for the e^+e^- kinetic energy value of 34.3 MeV.

By either of the above measures the cancellation of the kinetic energy due to the Breit-Pauli terms is seen to be almost total, with a net binding energy which is only 2.97 % of the total kinetic energy in either system. As is evident from Fig. 6, it is difficult to imagine that anything other than an exponentially damped potential could achieve such a delicate balance on a general basis.

On the other hand, the main argument against assuming that electrons are actually present in nuclei has been that no such potential can supposedly be found. The present experience strongly suggests that this position should be reevaluated. The Hamiltonian employed in the above investigation allows for a quantitatively equivalent variational treatment of systems with binding energies varying between 1.0 MeV and nearly 2.0 GeV, respectively, as well as a reliable description of conventional atomic systems by virtue of its close association with the Dirac and Breit-Pauli formulations of quantum electrodynamics interactions.

G. THE INTERACTION OF A PROTON AND AN ELECTRON

It remains to be considered whether a strongly bound state also results from an analogous treatment of the electron-proton system. This is done by simply inserting a very large mass for the proton so that the reduced mass of the electron is 1.0. The answer is clearly that *no such second minimum is found which corresponds to an inter-particle distance smaller than 1.0*

bohr (Table VI). The corresponding 0^+ results for the treatment in which translational effects are eliminated are shown in Table VII.

Moreover, the reason for this distinction between the e^+e^- and p^+e^- systems is easily understandable from the previous calculations with the XBPS Hamiltonian. The results of Table III for the e^+e^- system are changed dramatically by the substitution of a proton for the positron because of the disparity in the masses of these two particles. The magnitudes of the various Breit-Pauli terms are substantially reduced upon making this substitution because of the importance of the q/m_0 factors in the corresponding operators (Table I). Only the spin-same-orbit and Darwin terms with the electron's $(e/m_{oe})^2$ pre-factor survive for all practical purposes. By contrast the kinetic energy is doubled to a value of 0.5 a.u. because of the change in particle mass, thereby destroying the delicate balance mentioned in the last section between the attractive and repulsive components of the total energy for both the e^+e^- and p^+p^- binary systems.

η	E
0.30	710408.14
0.20	560400.24
0.10	374890.12
0.08	329689.40
0.07	305334.70
0.06	279471.535
0.02	148434.705
0.005	64715.8077
5.0×10^{-4}	12839.8600
5.0×10^{-5}	1623.358959
5.0×10^{-6}	159.118214
5.0×10^{-8}	0.173309
3.0×10^{-8}	-0.195469
2.0×10^{-8}	-0.339412
1.0×10^{-8}	-0.425769
9.0×10^{-9}	-0.428911

8.0×10^{-9} (*)	-0.430521
7.0×10^{-9}	-0.430349
6.0×10^{-9}	-0.428057
5.0×10^{-9}	-0.423152
4.0×10^{-9}	-0.414828
3.0×10^{-9}	-0.401501
1.0×10^{-9}	-0.333257
5.0×10^{-10}	-0.281758
1.0×10^{-10}	-0.168834

TABLE VI. Total full CI energy E (in hartree) for the lowest state of the p^+e^- hydrogen atom system obtained employing the 3s,2p,2d basis for various scale factors η and a fixed value of the exponential damping constant $A = 1.2648$ a.u. for the XBPS Hamiltonian of Table I (* indicates minimum).

Root No.	0^+
1	-0.4999727 (n=1)
2	-0.1249970 (n=2)
3	-0.0555547 (n=3)
4	-0.0312496 (n=4)
5	-0.0199998 (n=5)
6	-0.0138887 (n=6)
7	-0.0102039 (n=7)

TABLE VII. Total full CI energy (in hartree) of the lowest states of 0^+ symmetry of the p^+e^- (H atom) system obtained in the calculations with the condition of $\mathbf{P}_1 = -\mathbf{P}_2$ using the XBPS Hamiltonian of Table I.. As in the e^+e^- calculations in Table III, 70 s-type functions are employed. The value of the exponential damping constant A in the XBPS Hamiltonian is the same as for the e^+e^- treatment, although this choice is not critical for the H- atom. The principal quantum number n of each Rydberg state is indicated in parentheses next to the corresponding energy value.

The results of Table III correspond to an e^+e^- inter-particle distance on the order of $r = 7 \alpha^2$, but as we have seen, the maximum binding for the p^+p^- system occurs when the corresponding separation is m_{op}/m_{oe} times smaller. At this distance all energy values in Table III can be multiplied by a factor of 1836 to obtain the corresponding p^+p^- results because of the

scaling property discussed in Sect. IV.D. If the mass of the electron is substituted for that of the antiproton at such a small distance, the kinetic energy is halved according to the classical Einstein free-particle expression. On the other hand, the corresponding damping factors for the electron are very close to zero for such high momentum, and so only the two Breit-Pauli terms with the $(e/m_{\text{op}})^2$ pre-factor are left (relatively) unaffected.

As a result the required cancellation of the attractive and repulsive terms no longer takes place and an extremely large positive total energy results for the p^+e^- system in this range of inter-particle separation. No choice of basis set succeeds in binding the electron and proton more strongly together than is the case for the hydrogenic 1s state. For example, the use of electronic functions which are optimal for the strongly bound e^+e^- system along with proton functions which are 1836 times more compact (corresponding to their optimal p^+p^- counterparts) also produces only negative binding energies.

These results can again be understood on a qualitative basis by considering the p^+e^- system in its own center-of-mass coordinate system. The condition of equal and opposite momentum (see Sect. V.E) now requires that the electron move at a speed which is 1836 times greater than that of the proton. This requirement forces the particles to avoid each other by wide margins if high speeds are to be maintained, making the type of short-range momentum-dependent interaction represented by the Breit-Pauli interactions very ineffective in producing binding under these circumstances. Such distinctions in the relative particle masses for binary systems are of only minor importance when the Coulomb interaction dominates, however, *which explains why we normally regard positronium as just another hydrogenic system*. By simply decreasing the values of the exponents employed for the Gaussian basis functions, one eventually finds that the variational treatment of the XBPS Hamiltonian leads to binding energies in the 0.5 hartree range (Table VI) expected for the ground state of the hydrogen atom. No amount of exponent optimization produces a second minimum at shorter inter-particle separations for this system, in marked contrast to what is observed when employing the analogous Hamiltonian for the e^+e^- and p^+p^- systems.

XI. CONCLUSION

In the present investigation attention has been centered primarily on the pervasive assumption in modern-day physical theories that matter can be created or destroyed by means of

a suitable addition or loss of energy. It has been emphasized that it is impossible to distinguish experimentally between a particle which is unobservable in its present state and one which has gone out of existence entirely. The concept of all material particles being composed of atoms or elements which are impervious to the application of any force has played a crucial role in the development of the physical sciences over a period of several millennia. It has been argued in the present work that since the antithesis of this view, the creation-annihilation hypothesis, can never be proven by direct experimental observation, it is quite important to see if an alternative theory of physical transformations can be formulated which gives a plausible interpretation of all measured phenomena without giving up the principle of the indestructability of material elements.

Consideration of the decay of positronium and the subsequent production of photons, which has hitherto been assumed to involve the annihilation of an electron and positron, suggests a different explanation in terms of the formation of particle-antiparticle binary systems with exactly zero rest mass. In order to give quantitative substance to this alternative hypothesis, attention is turned to the goal of finding a suitably concrete form for the system of interactions which would be capable of binding an electron and positron so strongly together that the energy lost in the process is exactly equal to the sum of their rest masses, $2 m_{oe} c^2$ or 37557.73 hartree (1.02 MeV). Instead of simply deducing this result with the help of the Einstein mass-energy equivalence relation, it is proposed to consider the positronium decay as a conventional radiative emission process in which the binding energy of the final state is to be computed with standard quantum mechanical methods once the nature of the associated interaction mechanism is identified.

In the course of studying other modern physics experiments on a qualitative basis, it has been concluded that such a massless e^+e^- structure can plausibly be attributed to the photon itself, since it is known to have zero rest mass and to interact electromagnetically in a way that is at least consistent with a dipolar composition of this kind. Because of the well-known fact that photon emission processes occur at all locations in the universe, i.e. wherever a given excited state of a particle is found to undergo radiative decay, it follows that the proposed massless e^+e^- binary systems must exist everywhere in space with sufficiently high density in order to explain these phenomena without the creation-annihilation hypothesis. Support for this assumption of ubiquitous photons with zero mass can be found in the black-body radiation experiment (Sect.

II.D). The original quantum hypothesis of Planck holds that for every frequency of radiation ν absorbed by a blackbody at thermal equilibrium, there must be a higher population of photons with $E = 0$ than for $E = h\nu$ or any other allowed energy value. Since there are an unlimited number of such frequencies possible, it follows that the number of massless photons in the thermodynamical system is essentially boundless. It is then only a matter of theoretical interpretation whether occupation of such an $E = 0$ state is taken to correspond to a photon which has suddenly ceased to exist or one which simply defies experimental detection.

In this view the radiative emission process is not seen as involving the creation of a photon with $\Delta E = h\nu$, but rather as an exchange of energy between the original excited system and a photon in its neighborhood which initially possesses zero energy. The photon simply takes on an amount of energy which is lost by the other system in a transition to one of its lower-lying states. Furthermore, the requirements of conservation of energy and linear momentum are shown to be directly responsible for the quantization of such processes, since an arbitrary exchange of energy would require a smaller increase in the photon's momentum than that lost by the heavier system with which it interacts. Only by changing to a lower-lying internal state with nearly the same momentum as prior to the transition can the heavier system satisfy the $\Delta E = pc$ relation required by the photon's zero rest mass. The explanation of the Mössbauer effect is based on the same considerations, with the distinguishing feature that for the very large energies of nuclear emission processes, the two internal states of the heavier system involved in the transition are associated with momenta which differ far more greatly from another than those of different electronic states in an atomic transition.

One of the most critical aspects of the massless photon hypothesis is its questioning of the off-stated belief that zero energy and/or mass for any system necessarily implies its lack of existence (Sect. II.C). Instead it is pointed out that the Planck frequency and de Broglie relations have zero frequency and infinite wavelength as limiting values when the energy and momentum of a photon approach vanishing magnitudes. Before this limit can be achieved by systematically reducing the energy of a single photon, frequency and wavelength values must be reached which by virtue of the above two relations already lie outside the range of experimental observation. Since photons whose (finite) de Broglie wavelengths are too large to measure are nevertheless assumed to exist, it is not unreasonable to expect that their zero-momentum counterparts may also be present, *despite the impossibility of observing them directly.*

In this connection a flaw is pointed out in the argument which holds that a system with zero rest mass must move with the speed of light based on the dependence of the relativistic mass m on velocity. The latter relation is satisfied for any velocity v smaller than c as long as $m \neq 0$, including $v = 0$. The ratio m_0/m is not uniquely defined under these circumstances and thus may exceed zero, *implying that $\gamma^{-1} = (1 - v^2/c^2)^{1/2}$ also can be non-vanishing in this limit.*

Consideration of other types of observed phenomena arising in the field of modern physics is found in no way to contradict the above hypothesis of ubiquitous massless photons. These processes include the photoelectric effect, Compton and Raman effects, Bremsstrahlung and electron production from photon collisions. At the same time, this line of argumentation calls into question standard interpretations of related elementary processes which involve the creation or annihilation of individual particles. The primary example considered in this connection is that of the beta decay of a neutron.

In order to give quantitative substance to the above theoretical model, it is imperative that one clearly identify the nature of the interactions responsible for high-energy processes, especially the prototype example in which an electron and a positron combine to form a tightly bound binary complex with exactly zero rest mass. Emphasis is placed thereby on the fact that no corresponding state of a proton and an electron is known, i.e. the $1s_{1/2}$ state of the hydrogen atom is perfectly stable when left in an isolated condition, whereas that of positronium has only a short lifetime. Similarly, a massless p^+p^- binary must also be assumed to exist based on the analogous experimental results for the interaction of a proton and an antiproton.

The high energies associated with the formation of these two systems suggest that similar short-range interactions are involved as in the binding of nuclei, in which case an exponential form for the corresponding potential has been deduced from scattering experiments. Because of the participation of electrons, positrons and photons in the electromagnetic interaction, it is suggested that a good starting point in the search for such a potential is the relativistic Dirac equation or some approximation to it. It is noted that the Breit-Pauli reduction of this equation contains short-range terms varying inversely as the cube of the inter-particle distance which are of the order of α^2 (10^{-4} hartree) for typical electron-proton separations in an atomic system. The unbounded character of these terms in the limit of vanishingly small separations, particularly in relation to the corresponding relativistic kinetic energy, makes it clear, however, that the desired short-range bonding of elementary particles can only be satisfactorily described by interactions

of this type *if they are somehow modified to become considerably less attractive at extremely small interparticle distances.*

An exponential damping of the Breit-Pauli interactions is thus suggested (Sect. IV.C). This is in analogy to that proposed by Yukawa fifty years earlier for the description of nuclear binding, except that in that case it was applied to an r^{-1} potential. Similar quantities of r^{-3} type present in the $\mathbf{p} \cdot \mathbf{A}$ cross terms of the Maxwell-Lorentz Hamiltonian are kept in check by the $\mathbf{A} \cdot \mathbf{A}$ component of a perfect square, which in turn varies as r^{-4} . This suggests that the exponent of the damping function for the Breit-Pauli interactions should vary as r^{-1} or p , and that it should contain a factor of the charge-to-mass ratio of a given particle multiplied by a constant of order α^2 .

Since the motion of charged particles is involved in such interactions, it seems reasonable to choose a momentum-dependent exponential damping whose argument is always negative. Ultimately these arguments lead to the following explicit form for the damping function: $\exp(-A \alpha^2 |\mathbf{p} q/m_0|)$. At the same time, the unbounded correction to the Breit-Pauli kinetic energy, which varies as p^4 , is introduced via the Einstein relativistic operator $(p^2 + m_0^2 c^4)^{1/2} - m_0 c^2$ (with $c = \alpha^{-1}$ in atomic units). The resulting set of interactions is subsequently referred to as the exponentially damped Breit-Pauli Hamiltonian and is employed in a Schrödinger equation (XBPS) of the standard $H \Psi = E \Psi$ form. The potentially crucial advantage of this Hamiltonian is that it is bounded from below and thus can be treated using standard variational techniques, unlike the un-damped Breit-Pauli terms themselves. The explicit form of the XBPS Hamiltonian is given in Table I.

The above Schrödinger equation has an interesting scaling property of relevance to the positronium decay process. The total energy of any state of a particle-antiparticle system with a given charge and rest mass is exactly M times larger than that obtained by a coordinate scaling for an analogous binary system with the same charge but a rest mass which is M times smaller. This means that the same Hamiltonian which produces the $2m_{oe}c^2$ binding energy assumed for the e^+e^- system leads to the corresponding value of exactly $2m_{op}c^2$ for the p^+p^- counterpart. The latter system's average inter-particle separation is smaller by a factor of the proton-electron rest-mass ratio m_{op}/m_{oe} than that for the e^+e^- system. The form of the XBPS Hamiltonian is thus seen to guarantee the condition required by the Einstein mass-energy equivalence relation, namely that the maximum energy lost in a particle-antiparticle interaction is directly proportional to the

rest masses of the isolated systems (Sect. IV.D). This condition places a restriction on the form of the exponential damping factor in the XBPS Hamiltonian, specifically that the momentum p be divided by the particle's rest mass in the corresponding argument.

The use of the relativistic kinetic energy operator in the XBPS model raises a fundamental question of a different nature, however, namely how to deal with the translational motion of the combined system being treated (Sect. IV.B). Analysis indicates that, contrary to the non-relativistic case, a transformation to center-of-mass coordinates for such a Hamiltonian does not lead to a total separation of the internal and translation motion. This fact is related to the requirement of the theory of special relativity that distances measured in an inertial system moving with constant velocity relative to the observer are found by him to be different. Such findings clearly cannot result if it is simply assumed that the center-of-mass motion is completely separable, since properties involving only internal variables are thereby forced to be independent of the state of translation of the combined system. It was therefore decided to carry out calculations with the XBPS Hamiltonian in terms of the Cartesian coordinates of the constituent particles without introducing a center-of-mass transformation, so as not to prejudice the results of the treatment in this respect, despite the fact that this procedure brings with it certain computational difficulties otherwise avoided by such a change in coordinates.

Allowing the internal and center-of-mass motion to remain coupled in the theoretical treatment gives insight as to how the e^+e^- system can have a tightly bound ground state below the familiar $1s$ state of positronium, unlike the case for the hydrogen atom of proton-electron composition. The essence of this argument is contained in the observation that the Σp_i condition for a binary system whose center of mass is at rest in the origin of the coordinate system implies that $\mathbf{p}_1 = -\mathbf{p}_2$ or $\mathbf{v}_1 = -(m_2/m_1) \mathbf{v}_2$. Since the mass of the proton is so much greater than that of the electron, this condition requires that their respective velocities be quite different from one another in the hydrogen atom. This circumstance is incompatible with the need to keep the particles close to one another while each is moving at high speed in order to take maximum advantage of the short-range momentum-dependent attractive terms in the XBPS Hamiltonian such as the damped spin-orbit coupling (Sect. V.G).

By contrast, the analogous condition for the electron-positron and proton-antiproton systems leads to a perfectly correlated relative motion of the two particles. This relationship lends considerable support to the proposition that such systems possess non-hydrogenic states of

exceptionally large binding energies. As the translational energy increases beyond zero the correlated motion is gradually disturbed, however. At first glance the smaller q/m_o value of the proton relative to the electron appears to be inconsistent with the requirement that p^+p^- have a much larger binding energy than the corresponding e^+e^- system, especially when one only considers the magnitude of their respective coupling constants in the short-range XBPS interactions. The same quantity also causes the argument of the exponential damping factors to be correspondingly smaller for the proton-antiproton system, however, which in turn allows its two particles to approach each other far more closely than is preferable for the lighter electron-positron pair.

A full CI treatment of the Schrödinger equation discussed above for the e^+e^- system employing various primitive Cartesian Gaussian one-particle basis sets has indicated that the lowest-energy state for this system possesses 0^- symmetry and is symmetric with respect to the charge conjugation operation, which commutes with the corresponding Hamiltonian (Tables II-III). The constant A in the exponential damping factors has been chosen so as to obtain the desired e^+e^- binding energy of $2 m_{oe}c^2$ for an optimal choice of orbital exponents in the full CI treatment (Sect. V.C).

Additional calculations have also been carried out using the $\mathbf{p}_1 = -\mathbf{p}_2$ condition that eliminates translational energy from the theoretical treatment (Tables IV-V). In this case, the 0_g^- symmetry of the lowest-energy e^+e^- (ground) state is found to possess a vanishing expectation value for the conventional (undamped) Darwin term, as well as for the corresponding δ -function term in the spin-spin interaction. This result shows in a particularly striking manner that the two constituent particles must effectively avoid one another completely in order to achieve maximum stability.

The value of the damping constant A is found to be 1.54666 a.u., using the above criterion. As a result, the range of the damping interaction conforms to the expectations discussed previously, becoming significant for inter-particle distances smaller than or equal to $r = 10 \alpha^2$. In accordance with the XBPS scaling property, the optimum proton-antiproton wave function is characterized by Gaussian exponents which are larger by a factor of $(m_{op}/m_{oe})^2$ relative to those for e^+e^- , giving the desired $2m_{op}c^2$ binding energy relative to the separated proton and antiproton for the same choice of A as for the electron-positron system, as discussed in Sect. IV. D.

The well-known Rydberg states of positronium are obtained with high accuracy using the same XBPS treatment. This experience shows that it is quite feasible that the suggested e^+e^- state does indeed exist as the true ground state of the e^+e^- system. The value of the Coulomb energy in the latter state indicates that the average separation between the electron and positron is 3.756×10^{-4} bohr or $7.05 \alpha^2$, whereas the corresponding value for the 1s Rydberg ground state is 2.0 bohr.

Finally, analogous calculations have been carried out using the XBPS Hamiltonian for the electron-proton system. The results are shown in Tables VI and VII. As expected, no low-energy state comparable to that for the electron-positron system is found in this case. Instead, only an accurate calculation of the H-atom Rydberg spectrum is obtained, especially when the effects of translation are eliminated (Table VII). There is no question that the lowest-energy state obtained is the completely stable ground state of the hydrogen atom.

Sept. 27, 2020

Acknowledgement

The author wishes to express his sincere gratitude to Prof. P. Chandra of the Banaras Hindu University, and Prof. Dr. B. A. Hess, Dipl.-Phys. H.-P. Liebermann, Dipl.-Phys. P. Funke and Dr. R. A. Phillips of the Bergische Universität Wuppertal for their efforts in carrying out the calculations of the present study as well as for many useful discussions during the course of this work.

REFERENCES

1. P.A.M. Dirac, *Proc. R. Soc. London A* **133**, 80 (1931).
2. C.D. Anderson, *Phys. Rev.* **43**, 491 (1933).
3. O. Chamberlain, E. Segre, C. Wiegand and T. Ypsilantis, *Phys. Rev.* **100**, 947 (1955)

4. A. Einstein, *Ann. Physik* **17**, 891 (1905); H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity* (Dover, New York, 1952), p. 35.
5. M. Planck, *Ann. Physik* **4**, 553 (1901).
6. I. Newton, *Philosophiae Naturalis Principia Mathematica* (London, 1686); *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World*, Vols. 1 and 2, A. Motte translation, revised and edited by F. Cajori (University of California Press, Berkeley, 1962).
7. Galileo Galilei, *Il Saggiatore* (1623); *The Controversy on the Comets of 1618*, S. Drake and C.D. O'Malley translation (1960).
8. Titus Lucretius Carus, *De Rerum Natura*, Book I, W.H.D. Rouse translation (Loeb Classical Library), 1. 155.
9. W. Shakespeare, *King Lear*, Act I, Sc. 1. 1. 82; *William Shakespeare, The Complete works*, edited by S. Wells and G. Taylor, (Clarendon Press, Oxford, 1986), p. 1027.
10. Democritus, Fragment 125.
11. Titus Lucretius Carus, *De Rerum Natura*, Book I. W.H.D. Rouse translation (Loeb Classical Library), 1. 518.
12. Robert Boyle, *The sceptical chymist* (London. 1661).
13. G. Shortley and D. Williams, *Elements of Physics*, Third edition (Prentice Hall, Englewood Cliffs, N.J., 1960. p. 302.
14. J. Chadwick, *Nature* 129, 312 (1932); *Proc. R. Soc. London A* 136, 692 (1932).
15. I. Newton. *Philosophiae Naturalis Principia Mathematica* (London, 1686); G. Shortley and D. Williams, *Elements of Physics*, Third edition (Prentice Hall, Englewood Cliffs, N.J., 1960. p. 52.
16. H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin. 1957), p. 114.
17. N. Bohr, "On the Constitution of Atoms and Molecules, Part I," *Phil. Mag.* 26 (151), 1-24 (1913).
18. R.T. Weidner and R.L. Sells, *Elementary Modern Physics* (Allyn and Bacon, Boston, 1960), pp. 113-116.
19. A.C. Doyle, *The Complete Sherlock Holmes* (Hamlyn Publishing, Twickenham, 1984), p. 101.
20. L. de Broglie, *Phil. Mag.* **47**, 446 (1926); *Ann. Physik* **3**, 22 (1925).
21. R.L. Mössbauer, *Z. Physik* **151**, 124 (1958); *Naturwissenschaften* **45**, 538 (1958).
22. W. Heitler, *The Quantum Theory of Radiation*, Third edition (Clarendon Press, Oxford, 1954).
23. R.T. Weidner and R.L. Sells, *Elementary Modern Physics* (Allyn and Bacon. Boston, 1960), pp. 12-14.
24. G. Shortley and D. Williams, *Elements of Physics*, Third edition (Prentice Hall, Englewood Cliffs,

- N.J., 1960, pp. 871-872.
25. R. J. Buenker, "Evidence from the special relativity and blackbody radiation theories for the existence of photons possessing zero kinetic energy, *Mol. Phys.* **76**, 277-291 (1992); see also <http://arxiv.org>, physics/0501054.
 26. S. N. Bose, *Zeit. Physik* **26**, 178-181 (1924).
 27. E. Fermi and C.N. Yang, *Phys. Rev.* **76**, 1739 (1949).
 28. C. Davisson and L.H. Germer, *Phys. Rev.* **30**, 705 (1927).
 29. H. Paul, *Photonen: Experimente und ihre Deutung* (Vieweg, Braunschweig, 1985), p. 27.
 30. J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
 31. E.P. Wigner, *Rev. Mod. Phys.* **29**, 255 (1957).
 32. R.A. Beth, *Phys. Rev.* **50**, 115 (1936).
 33. P.J. Allen, *Am. J. Phys.* **34**, 1185 (1964).
 34. I. Newton, *Opticks* (London, 1704).
 35. R.T. Weidner and R.L. Sells, *Elementary Modern Physics* (Allyn and Boston, 1960), pp. 35-37.
 36. A. Einstein, Letter to M. Besso, 1951, cited in: H. Paul, *Photonen: Experimente und ihre Deutung* (Vieweg, Braunschweig, 1985), p. 7.
 37. The German word Einstein actually used was "Lump", a humorously derogatory term more accurately translated as "rascal" or "scoundrel".
 38. H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950), p. 191.
 39. W. Pauli, in: *Proceedings of the Meeting of the American Physical Society* (Pasadena, 1931); *Proceedings of Solvay Congress* (Brussels, 1933), p. 324.
 40. A. Einstein, *Ann. Physik* **17**, 132 (1905).
 41. Under these conditions a relativistic form for the kinetic energy needs to be employed to obtain quantitative results.
 42. R.M. Eisberg, *Fundamentals of Modern Physics* (Wiley, New York, 1961), Sect. 5.5.
 43. P.A.M. Dirac, *The Principles of Quantum Mechanics*, Fourth edition (Clarendon Press, Oxford, 1958), pp. 116-118.
 44. H. Eyring, J. Walter and G.E. Kimball, *Quantum Chemistry* (Wiley, New York, 1944). p. 114.
 45. H. Paul, *Photonen: Experimente und ihre Deutung* (Vieweg, Braunschweig, 1985), p. 17.
 46. A.H. Compton, *Phys. Rev.* **21**, 715 (1923); **22**, 409 (1923).
 47. C.V. Raman and K.S. Krishnan, *Nature* **121**, 501 (1928); **121**, 619 (1928).
 48. A. Smekal, *Naturwissenschaften* **11**, 873 (1923).
 49. E. Schrödinger, *Ann. Physik* **79**, 361 (1926); **79**, 489 (1926); **80**, 437 (1926); **81**, 109 (1926).
 50. P.A.M. Dirac, *Proc. R. Soc. London A* **117**, 610 (1928).

51. G. Breit, *Phys. Rev.* **34**, 553 (1929).
52. W. Pauli, *Z. Physik* **43**, 601 (1927).
53. H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957), p. 181.
54. R. J. Buenker, "Exponentially damped Breit-Pauli Hamiltonian for the description of positronium decay and other high-energy processes, *Mol. Phys.* **77**, 1095-1122 (1992); see also <http://arxiv.org, physics/0504070>.
55. J.C. Slater, *Quantum Theory of Atomic Structure*, Vol. 11 (McGraw-Hill, New York, 1960), p. 191.
56. P.A.M. Dirac, *The Principles of Quantum Mechanics*, Fourth edition (Clarendon Press, Oxford, 1958), p. 312.
57. M. Goeppert Mayer, *Phys. Rev.* **74**, 235 (1948); **75**, 1969 (1949); **78**, 16 (1950).
58. O. Haxel, J.H.D. Jensen and H. Suess, *Phys. Rev.* **75**, 1766 (1949); *Z. Physik* **128**, 295 (1950),
59. A.M. Feingold, *Thesis*, Princeton University (1952).
60. N. Bohr, *Nature* **121**, 580 (1928).
61. R.J. Buenker, P. Chandra and B.A. Hess, *Chemical Physics* **84**, 1 (1984).
62. H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957), p. 25.
63. L. Pauling and E.B. Wilson, Jr., *Introduction to Quantum Mechanics* (McGraw-Hill, New York, 1935), p. 113.
64. R. J. Buenker, *New Synthesis of Elementary Particle Physics: A Theory of Elemental Balance in Physical Transformations, 1st Edition* (University of Wuppertal Library, Item 23, Wuppertal, 2015), pp. 231-247.
65. A. Ore and J. Powell, *Phys. Rev.* **75**, 1696 (1949); **75**, 1963 (1949).
66. R.H. Dicke and J.P. Wittke. *Introduction to Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1960), p. 97.
67. H.A. Bethe and E.E. Salpeter. *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957), pp. 195-196.
68. R.G. Sachs, *Nuclear Theory* (Addison-Wesley, Cambridge, Mass., 1953), p. 28.
69. H. Yukawa, *Proc. Phys. Math. Soc. Japan* **17**, 48 (1935).
70. W. Pauli, *Z. Physik* **31**, 765 (1926); W. Heisenberg, *Z. Physik* **38**, 411 (1926); **41**, 239 (1927); P.A.M. Dirac, *Proc. R. Soc. London A* **112**, 661 (1926).
71. J.C. Slater, *Phys. Rev.* **34**, 1293 (1929).
72. P. Chandra and R.J. Buenker, *J. Chem. Phys.* **79**, 358 (1983); **79**, 366 (1983)
73. P. Chandra, R.A. Phillips, P. Liebermann and R.J. Buenker, results of this laboratory.

74. M. Tinkham, *Group Theory and Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 178.
75. E.U. Condon and G.H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935).
76. S.R. Langhoff and C.W. Kern, "Molecular Fine Structure", in: *Modern Theoretical Chemistry*, Vol. 4: *Applications of Electronic Structure Theory*, edited by H.F. Schaefer III (Plenum, New York, 1977), pp. 381-437.
77. R.J. Buenker, H.- P. Liebermann, P. Funke and R.A. Phillips, results of this laboratory.
78. F. Mark, 21. Symposium für Theoretische Chemie (Altenberg, W. Germany, 1985).
79. P.A.M. Dirac, *The Principles of Quantum Mechanics*, Fourth edition (Clarendon Press, Oxford, 1958), p. 64.
80. B.A. Heß, *Habilitation Thesis*, University of Wuppertal, 1986.
81. B.A. Heß, *Phys. Rev. A* **32**, 756 (1985); B.A. Heß, R.J. Buenker and P. Chandra, *Int. J. Quantum Chem.* XXIX, 737 (1986).
82. J.C. Slater. *Phys. Rev.* **32**, 339 (1928).
83. C.C.J. Roothaan, *Rev. Mod. Phys.* **23**, 69 (1951); G.G. Hall, *Proc. R. Soc. London A* **205**, 541 (1950).
84. M. Tinkham, *Group Theory and Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 162-167.
85. R. J. Buenker, *New Synthesis of Elementary Particle Physics: A Theory of Elemental Balance in Physical Transformations, 1st Edition* (University of Wuppertal Library, Item 23, Wuppertal, 2015), pp. 77-80.
86. H. Frauenfelder and E.M. Henley, *Subatomic Physics* (Prentice-Hall, Englewood Cliffs, N.J., 1974), pp. 201-202.
87. E.J. Konopinski and M.E. Rose, "The Theory of Nuclear β -Decay" in: *Alpha-, Beta - and Gamma - Ray Spectroscopy*, edited by K. Siegbahn (North - Holland, Amsterdam, 1965), p. 1329.
88. R. J. Buenker, *New Synthesis of Elementary Particle Physics: A Theory of Elemental Balance in Physical Transformations, 1st Edition* (University of Wuppertal Library, Item 23, Wuppertal, 2015), pp. 89-95.