Hamilton's Canonical Equations and Einstein's $E=mc^2$ Relation

Robert J. Buenker[1]

[1]*Fachbereich C-Mathematik und Naturwissenschaften, Bergische Universität Wuppertal,*

Gaussstr. 20, D-42097 Wuppertal, Germany

e-mail: rjbuenker@gmail.com

Abstract

The history of Einstein's landmark introduction of the energy-mass equivalence relation is reviewed. Emphasis is placed on the role that Hamilton's Canonical Equations play in arriving at the famous $E=mc^2$ formula. Although both Einstein and Planck derived the relativistic energy-momentum relationships by considering the effects of electromagnetic interactions, it is shown that a comprehensive theory can be formulated which is based exclusively on Hamilton's $dE=vdp$ relation when used in conjunction with the assumption of light-speed constancy in free space. To this end, it is helpful to return to Voigt's derivation in 1887 of a relativistic space-time transformation that was the precursor of the Lorentz transformation. A parallel derivation of Voigt's space-time transformation which takes account of Hamilton's Canonical Equations is shown to lead in a straightforward manner to both the relativistic energy-momentum transformation and Planck's definition of relativistic momentum as $p=\gamma\mu v$, as well as the $E=mc^2$ relation itself. The associated $E=\gamma E_0$ equation is found to be completely analogous to the relation between atomic clock rates in different inertial systems that is used in the "pre-correction" procedure of the Global Positioning System to insure that atomic clocks on satellites run at the

same rate as their counterparts on the earth's surface. Finally, Einstein's faulty claim of "longitudinal mass" that he used in his original $E=mc^2$ derivation is traced back to his improper application of the Relativity Principle to a non-inertial system.

Keywords: Voigt space-time transformation, Hamilton's Canonical Equations, light-speed constancy assumption, Hamilton-Voigt E-p transformation, Galileo's Relativity Principle, $E=mc^2$

I. INTRODUCTION

The original derivation of Einstein's $E=mc^2$ relation is based on the description of the dynamics of electromagnetic interactions [1]. He took the position that there were two kinds of inertial mass, and his definition of longitudinal mass variant is essential in his derivation of the mass-energy equivalence relation that has become his trademark. Planck subsequently suggested [2] that the same result could be obtained by making a generalized definition of inertial mass, namely as $m= (1-v^2c^{-2})^{-0.5}\mu v=\gamma\mu v$, where μ is the rest mass of the particle and v is its speed relative to the origin of the electromagnetic force responsible for its acceleration. Einstein readily agreed with Planck [2,3] that it was incorrect to argue that there are two different kinds of inertial mass, but the fact remains that there is a definite element of serendipity in his original $E=mc^2$ derivation.

The object of the present work is to show that the mass-energy equivalence relation can be derived without regard to any characteristics of the electromotive force, but rather on the basis of the assumption that the speed of light in free space has a constant value of c relative to its source. The light-speed constancy assumption can be traced to the results of the Fizeau/Fresnel lightdrag experiment [4]. They showed that light is slowed as it moves through a transparent medium but, by extrapolation of the value of the medium's refractive index n to a unit value, that the observed light speed in the laboratory in the limit of free space should be completely independent of the speed v of the medium, i.e. $c(v) = c$. Maxwell's theory of electricity and magnetism published in 1864 also indicated that the speed of light has the same constant value c in every rest frame. The latter result was clearly at odds with the classical (Galilean) space-time transformation which indicates that speeds should be additive, i.e. $c + v \neq c$. Michelson and Morley [5] used their newly developed interferometer to test this theory, but it merely verified the conclusion that the speed of light is independent of the rest frame through which it moves, in particular that it is directionally independent at all times of the year.

Rather than search for an "ether" to explain the light-speed constancy observations, Voigt [6] suggested that the matter could be explained by simply altering the classical transformation in a novel way. To this end, he introduced a new term in the latter equations which contains a free parameter that allows for the mixing of the space and time coordinates (the original transformation simply assumes that the value of any elapsed time is the same in all rest frames, i.e. $t=t'$). The new equation is specifically:

$$t' = t + ax\,.$$

On this basis, Voigt was able to derive a new space-time transformation which satisfies the light-speed constancy assumption. It was the precursor of the Lorentz transformation derived in Einstein's work [1] which was originally put forward by Larmor [7] and Lorentz [8]. In the following discussion, it will be shown that a similar approach can be used to define an energy-momentum transformation that allows one in a straightforward manner to obtain all the key relationships obtained by Einstein and Planck in their original investigations without making use of the characteristics of electromagnetic interactions.

II. Einstein and Planck Derivations

Einstein's approach to the description of the dynamics of an electron or other charged particle was first to consider the transformation properties of Maxwell's equations for electromagnetic interactions. He considered the specific case in which the electron has been accelerated along the x axis as a consequence of the application of an electric field $\mathcal{E}$. He *invoked the Relativity Principle* to argue that the equation of force in the new rest frame is:

$$\mu d^2x'/dt'^2 = e\,\mathcal{E}' = e\,\mathcal{E}\,,$$

(primed notation has been used for variables in this rest frame).

He then argued that the corresponding equation of force in the original rest frame can be obtained by a Lorentz transformation between the current rest frame and that in which the force has been applied. He pointed out that this transformation indicated that there are two different kinds of inertial mass, longitudinal and transverse. In the present case, the longitudinal inertial mass, which he concluded was equal to $\gamma^3\mu$ according to the above argument, is required since the motion is along the axis of the electric field. The corresponding equation (using unprimed notation for the original rest frame variables) is thus:

$$\mu\gamma^3 d^2x/dt^2 = e\,\mathcal{E}' = e\,\mathcal{E}\,.$$

He then proceeded to compute the kinetic energy W of the accelerated particle as :

$$W = \mathrm{Int}\,(dE) = \mathrm{Int}\,(Fdx) = \mathrm{Int}\,(vdp) = \mathrm{Int}\,(v\,d(\mathrm{long.mass}\ v) = \mathrm{Int}\,(\mu\gamma^3 v\,dv) = \mu\mathrm{Int}\,(c^2 d\gamma),$$
$$= (\gamma-1)\mu c^2,$$

since $d\gamma = \gamma^3 vc^{-2}\,dv$. The integration is between 0 and infinity, and the longitudinal mass term γ^3 (v) is treated as a constant in the integration. One thus obtains the $E=mc^2 = \gamma\mu c^2$ formula for energy by this route.

Planck was clearly impressed with Einstein's results, but he had some reservations about the way in which he had derived them [2]. Planck was especially skeptical about the need for two different types of inertial mass that Einstein had assumed in order to reach his conclusions. He proceeded instead to make a new generalized definition of momentum as $p=\gamma\mu v$. He then assumed that the electromagnetic force F in Einstein's derivation was equal to dp/dt, in accord with Newton's Second Law. Planck then carried out the differentiation with respect to time in the three spatial directions. For example,

$$dp_x/dt = \gamma\mu dv_x/dt + \mu v_x d\gamma/dt = \gamma\mu a_x + \gamma^3\mu c^{-2} v_x^2 a_x$$

$= \gamma^3\mu a_x(1-v_x^2c^{-2}) + \mu v_x^2\, \gamma^3 a_x c^{-2} = \mu a_x\, \gamma^3[(1-v_x^2c^{-2} + v_x^2c^{-2}] = \gamma^3\, \mu a_x$. The corresponding values in the y and z directions are $\gamma\mu a_y$ and $\gamma\mu a_z$. Einstein had previously referred to the γ^3 and γ factors as longitudinal and transverse masses, respectfully, but he agreed [3] that Planck's derivation was preferable and that there was only one kind of mass after all.

III. Using Voigt's Light-speed Constancy Conjecture to Describe Particle Dynamics

The Voigt space-time transformation [6] can be modified in order to deal directly with the questions considered by Einstein and Planck. The same basic assumption needs to be made as in Voigt's original treatment, namely that the speed of light in free space *relative to its source* is equal to c no matter what the state of motion of the observer might be. Only in this application, speed is not dealt with as a ratio of space and time coordinates. Instead, one defines speed v in terms of Hamilton's Canonical Equations, namely as

$$v=dE/dp.$$

As noted in the previous section, Planck also made use of this relationship in order to obtain the $E=mc^2$ relationship.

In close analogy to Voigt's procedure, one defines the speed of light in free space to have a value of c relative to its source in two different rest frames, i.e.

$$dE/dp = dE'/dp' = c.$$

The starting point is then Hamilton's transformation in terms of E and p coordinates (note that v is the relative speed of the two rest frames):

$$dE = dE' + v dp_x'$$

$$dp_x = dp_x'$$

$$dp_y = dp_y'$$

$$dp_z = dp_z'.$$

In order to satisfy the light-speed constancy condition in this case, an extra term with a free parameter a is added to the second equation;

$$dp_x = dp_x' + a dE'.$$

The value of a is then determined, in complete analogy to Voigt's original procedure, by assuming the above light-speed constancy relation:

$dE/dp_x = c = (dE' + v dp_x')/ (dp_x' + a dE') = [(dE'/dp_x') + v)]/(1 + a dE'/dp_x') = (c+v)/(1+ac).$

One therefore concludes that

$$a = c^{-2} v.$$

Thus, the second E,p equation is changed thereby to

$$dp_x = dp_x' + c^{-2} v dE'.$$

After integration of both sets of differential quantities, one obtains the following relation between the squares of the two sets of E and p_x coordinates:

$$E^2 - p_x^2 c^2 = \gamma^{-2}(E'^2 - p_x'^2 c^2).$$

On this basis, it is clear that $E' = p_x' c$ whenever $E = p_x c$, as required.

In order to obtain the corresponding result for motion of the light in any direction ($p^2 = p_x^2 + p_y^2 + p_z^2$),

$$E^2 - p^2 c^2 = \gamma^{-2}(E'^2 - p'^2 c^2),$$

one must either multiply the right-hand sides of both the p_y and p_z equations (i. e. for a perpendicular direction) by a factor of γ^{-1}, or else multiply both the right-hand sides of the E,p_x equations by a factor of γ. Since we cannot change the p_y and p_z relations because they are fixed by Newton's Second Law (note that Voigt [6] did the opposite for his space-time transformation;

he multiplied the y and z components with γ^{-1} and left the t and x equations unchanged), we are left with only the latter possibility. The result is:

$$E= \gamma (E' +vp_x')$$

$$p_x= \gamma (p_x' + c^{-2}vE').$$

$$p_y= p_y'$$

$$p_z= p_z'.$$

As a consequence, as required by the light-speed constancy condition,

$$E^2 - p^2c^2 = E'^2 - p'^2c^2.$$

The inverse transformation can be obtained by interchanging the primed and unprimed quantities and changing the sign of v (Galilean inversion):

$$E'= \gamma (E -vp_x)$$

$$p_x'= \gamma (p_x - c^{-2}vE).$$

$$p_y'= p_y$$

$$p_z'= p_z.$$

The next step is to consider the inverse transformation from the vantage point of the rest frame in which the accelerating force was applied, in which case, p=0:

$$E' = \gamma E - \gamma v(\gamma c^{-2}vE') = \gamma E - \gamma^2 v^2 c^{-2}E'.$$

Thus, $E'(1+\gamma^2v^2c^{-2}) =$

$$E'\gamma^2(\gamma^{-2}+v^2c^{-2}) = E'\gamma^2 = \gamma E$$

The conclusion is therefore : $E= \gamma E'$. Note that the same result is obtained directly from the forward transformation.

The second of the inverse transformation equations under this $p_x'=0$ condition is:

$$0= \gamma(p_x - c^{-2}vE)= \gamma(p-c^{-2}vE).$$

Recalling the definition of momentum in terms of inertial mass m and speed v, this equation leads directly to the mass-energy equivalence relation, since it shows that

$$p=mv= c^{-2}vE.$$

Upon solving for m, we obtain the desired relationship:

$$m=E/c^2.$$

The key innovation that Planck made to arrive at the above results in Sect. II was to introduce the definition of relativistic momentum as $p= \gamma\mu v$. This result can also be easily derived from the above E-p transformation as follows (μ is the rest mass of the particle):

$E/E' = \gamma = m/\mu$.

Again, from the original definition of momentum as p=mv, one therefore finds that $p= \gamma\mu v$, as desired.

It is important to focus on the $E=\gamma E'$ relation, especially to recall how it was obtained. It results by considering the special case of p=0. In other words, E' is the rest energy of the particle, i.e. $E'=\mu c^2$. However, there are three different speeds that are to be taken into account in the general case. They involve two rest frames in which the particle is stationary at any one time, as well as the other from which the accelerating force originates. The latter has been referred to in previous work as the Objective Rest System or ORS [9]. The former two rest frames moves relative to another with speed v, as indicated explicitly in the E=p transformation. It is this speed which is used to define γ (v) in the transformation equations.
Their corresponding speeds relative to the common ORS are given by Hamilton's Canonical Equations as $dE/dp = v_0$ and $dE'/dp' = v_0'$, respectively.

IV. Addendum to Galileo's Relativity Principle

According to the above analysis in terms of the Hamilton-Voigt energy-momentum (E-p) transformation, the energy of a particle in a stationary position within a given rest frame is obtained as $E=\gamma (v_0) E_0 = \gamma (v_0) \mu c^2$. In other words, one assumes that the particle energy increases relative to its rest value by a factor of γ (dE/dp) or γ (dE'/dp') to $\gamma (v_0) \mu c^2$ or $\gamma (v_0') \mu c^2$, respectively. This relationship is essential for understanding the overall effect of the application of force to the particle at two different stages of acceleration (note that $v_0=v_0'=c$ in the case of a light pulse, consistent with the derivation of the E-p transformation).

This relationship can be expressed in the above notation as a direct proportionality:

$$E/\gamma (v_0) = E'/\gamma (v_0').$$

The above equation is closely related to the inverse proportionality relation for elapsed times Δt and $\Delta t'$ measured in the same two rest frames (referred to elsewhere as the Universal Time-Dilation Law [10]), namely:

$$\Delta t\, \gamma (v_0) = \Delta t'\, \gamma (v_0').$$

Of course, the latter can also be converted into a direct proportionality by using the periods of clocks τ and τ' instead of the corresponding elapsed times. What one concludes then is that energy and time scale in exactly the same manner with the application of force to the corresponding particle.

One can simplify these relationships further by looking upon the various quantities as *units of a physical property*. In other words, what we see is that both the unit of energy and time vary in the same proportion as the particle is accelerated. It is helpful to refer to the proportionality factor as $Q = \gamma (v_0')/\gamma (v_0)$. Thus, $\Delta t' = \Delta t/Q$ and $\tau' = Q\tau$. Similarly, $E' = QE$, using the same value of Q, which will be referred to in the following as the *kinetic scale factor*.

It is obviously important to use exact definitions in order to make correct use of these relationships. The key definition is that Q is the scale factor in going from inertial rest frame S to S', i.e. where S has a relative speed to the rest frame where the force is applied of v_0 and S' has the corresponding speed of v_0'. It is helpful to refer to the latter rest frame as the Objective Rest System (ORS), as noted above. The value of Q is greater than unity if $v_0' > v_0$, but it is less than unity if $v_0' < v_0$. This fact distinguishes the present theory in a crucial manner from the relativity theory espoused by Einstein in his 1905 paper [1]. The time dilation in his theory, as well as other physical properties, is *symmetric* (whereby two clocks can each be running slower than the other, for example). This characteristic makes his theory *subjective*, i.e. measured values depend on the *perspective of the observer*, whereas the present theory is perfectly *objective*. The GPS navigation system makes direct use of the scaling property for elapsed times [11-13], and thus serves as an important experimental verification of these concepts.

Continuing along this line of thought, it is easy to determine the corresponding scale factor for velocity/speed in a given rest frame. It must be independent of the state of motion of the particle because that condition must be satisfied in order to be consistent with the constancy of light speed assumed in the derivation of the E-p transformation. We can bring this result into the basic system outlined above for energy and time by simply noting that this constancy requirement translates into having the velocity scale factor vary as Q^0, i.e. also as a power of the fundamental scaling parameter Q. This choice in turn forces one to assign the scale factor for distance to also have a value of Q, since speed is defined as a ratio of distance traveled to corresponding elapsed time, i.e. that distance in time scale in exactly the same manner upon acceleration of the particle. This conclusion stands in stark contrast to Einstein's claim of length contraction accompanying time dilation because it means *that lengths expand when clocks slow* *down*, not get smaller by different proportions depending on the orientation of the object as Einstein claimed (and also what FitzGerald and Lorentz had said before he did).

Experimental measurements of the inertial mass of an electron [14] are in complete accord with the above analysis. They showed that mass increases with speed relative to the laboratory by the same factor as expected for lifetimes, from which one concludes that the scale factor for this property also varies in direct proportion to Q. Taken together, all these results are found to be consistent with the $E=mc^2$ formula; both E and m scale as Q, while c remains constant. In general, by simply noting its composition in terms of the three fundamental properties of inertial mass, distance and time, one can predict in a quite easy manner the way in which the corresponding scale factor for a given property varies. This information by itself is sufficient to allow one to determine what the corresponding integer exponent of Q will be for it. For example, the exponent for angular momentum is +2 because it is a product of linear momentum p (+1) and distance (+1). This explains why the energy-frequency law holds in all inertial frames since Planck's constant h has units of angular momentum: $E(+1)=h(+2)\nu(-1)$. More discussion of these property scaling characteristics may be found elsewhere [15].

It is important to note that the above procedures are perfectly in line with Galileo's Relativity Principle. Even though the units for the various physical properties vary upon application of a force to the particle, the fact remains that there is no way based on *in situ* measurements alone that a stationary observer co-moving with the particle can be aware of such changes. This is because a change in the value of a given property is always perfectly matched by a proportional change in the unit employed to express it. Observers in different inertial systems have every reason to believe that the units they are independently using are standard, even though it can be shown experimentally that they differ from one rest frame to another. They each think that their meter stick has a length of exactly 1.0 m, and that their standard of energy is exactly equal to 1.0 J.

This being the case, it is still true that the passengers locked below the hull of a ship cannot know whether they are underway on a perfectly calm sea or have actually never left the port. Galileo used this example to help contemporaries to accept the truth that the earth is orbiting the sun at the "unbelievable" speed of 30 km per second. The above scaling arguments indicate, however, that there should be an addendum to the Relativity Principle, namely [16]: The laws of physics are the same in every inertial rest frame, *but the units in which they are expressed vary in a completely systematic manner from one frame to another.*

V. Einstein's Mistaken Use of the Relativity Principle

In his derivation of the mass-energy equivalence relation, Einstein [1] invoked the Relativity Principle to deduce the electromagnetic force equation in the rest frame of the accelerating charged particle. As discussed in Sect. II, he then used the Lorentz transformation to obtain the corresponding force equation in the rest frame in which the force was applied. It is generally overlooked thereby that the latter rest frame is not freely translating, and therefore that Einstein's claim that the Relativity Principle is relevant to this situation is not correct. In his 1905 paper, he explicitly states on p.895 (see point #1) that it applies to "freely translating systems," without listing any exceptions beyond this. As a consequence, there is no reason to accept his conclusion about the above force equation as an unavoidable consequence of the Relativity Principle. In particular, his derivation of the expression for integral mass is certainly faulty.

The fact is that the question of integral and transverse masses was settled once and for all by Planck's intercession, so there is no need to further discuss that conclusion. There still remains a various consequence of Einstein's version of the Relativity Principle, however. This has to do with the value of the speed in the momentary rest frame of the accelerated electron. According to the Hamilton-Voigt transformation, the energy measured in this rest frame is equal to $\gamma\mu c^2$ and the corresponding inertial mass is γ, whereby the argument v for γ in both cases is the *speed of the particle relative to the rest frame (ORS) in which force was applied.* The corresponding momentum is $p=\gamma\mu v$.

The energy and mass in the rest frame in which the particle is momentarily stationary is μc^2 and μ, respectively. The reason this is so according to the E-p transformation is not because the appropriate value of the speed v is 0, however. Rather, these results are the consequence of the fact that the increase in both quantities is perfectly matched by a change in the corresponding units. In other words, the numerical value of both stays the same as prior to acceleration, whereas the *absolute* values increased by a factor of γ.

Ultimately, this example unveils a fundamental deficiency in Einstein's interpretation of the Relativity Principle. It makes a clear assumption that the speed v is to be taken relative to the state of motion of the co-moving observer, which therefore means that it has a null value, whereas the corresponding reference in the E-p transformation is the original rest frame in which the force was applied (ORS). If one accepts the former definition, one is left with the conclusion that the particle *has no momentum* in this state. This is a strange result, indeed, when it is realized that the particle in question *is reacting to the application of a possibly extremely large force.*

The resolution of the above question might seem to be deserving of nothing more than a minor footnote in the history of Einstein's relativity theory, but it is considerably more than that. This can be seen by considering a different example in which the charged particle is subjected to both an electric field along the x direction and a perpendicularly oriented magnetic field. According to the definition of the Lorentz force, the particle must start out in the x direction, and then gradually follow an increasingly curved path because of the factor of v multiplying the magnetic field vector. According to Einstein's interpretation of the Relativity Principle, the magnetic field can have no effect on the particle in its rest frame, because there the value of v is always equal to zero. As a consequence, it must be expected that an observer co-moving with the charged particle finds that it continues to move indefinitely in a straight line along the x axis. As discussed elsewhere [17], this is an impossible situation because it simply cannot be that two observers in their respective freely translating states could possibly disagree as to whether the path an object follows is curved or not. The only conclusion remaining is therefore that Einstein's first postulate of relativity is not consistent with experimental reality.

VI. CONCLUSION

It is possible to derive the relativistic energy-momentum transformation on the basis of Hamilton's Canonical Equations (dE=vdp) alone, without any consideration of electromagnetic interactions. One force is as good as another for this purpose. The fundamental assumption is the same as for the Lorentz transformation, namely that the speed of light relative its source is always equal to c. Voigt was the first one to derive a space-time transformation using this assumption, and his method is simply adapted to the treatment of energy-momentum interactions. The resulting set of equations is referred to as the Hamilton-Voigt energy-momentum transformation.

In this way, one is led directly to the key results of Einstein's and Planck's dynamics theory, $p=\gamma\mu v$ and $E=mc^2$, as well as to the true explanation for Einstein's faulty distinction between "longitudinal" and "transverse" mass. The latter conclusion is seen to be directly related to Einstein's improper use of Galileo's Relativity Principle, namely to apply it to a rest frame that is not inertial.

These considerations lead to a purely objective theory of particle dynamics, unlike the version Einstein developed in his 1905 paper (according to which two clocks can each be running slower than one another, for example). The key result in the present theory is that the energy of a particle increases by a factor of $\gamma = (1-v^2/c^{-2})^{-0.5}$ after being accelerated to speed v relative to the rest frame (ORS) in which the force is applied, i.e. $E=\gamma E'$. There thus exists a direct proportionality between the energy E of a particle and γ (v). An observer co-moving with the particle does not notice this change, however, because the standard he uses to express the numerical value of the particle's energy has also increased by exactly the same factor. An analogous proportionality relationship holds for both the inertial mass m of the particle and its momentum $p=mv$, whereby it is important to specify that the speed v is measured relative to the ORS in all cases.

Observers in two different rest frames therefore do not use the same standard units for these quantities. There is a conversion factor Q for each pair of rest frames which allows one to deduce the value of a property in one rest frame based on the actual measured result in the other. The value of Q is equal to γ $(v_0')/\gamma$ (v_0), whereby v_0'are v_0 the respective speeds of the two rest frames relative to the ORS. The values of scale factors for other physical properties are always integral powers of Q. The Relativity Principle needs to be modified to incorporate the above relationships by noting that the standard units employed to express the numerical values of a given quantity vary from one rest frame to the other, in accord with the above considerations.

Finally, it has been demonstrated that the *absolute* value of a given physical quantity is the same in all rest frames. The only reason it can differ is because *the units in which the numerical results are expressed in different rest frames are not the same.* This principle is violated in the long-accepted interpretation of electromagnetic interactions because the latter leads, for example, to the totally unphysical conclusion that the path of an electron can be curved from the standpoint of an observer in one freely translating rest frame but not from the standpoint of a counterpart co-moving with the electron. This discrepancy is traced to the fact that latter rest frame is not inertial, contrary to the prescriptions for use of Galileo's Relativity Principle.

References

[1] A. Einstein, "Zur Elektrodynamik bewegter Körper," Ann. Physik **322 (10)**, 891 (1905).

[2] R. D. Sard, *Relativistic Mechanics*, W. A. Benjamin, New York, 1970, pp. 138-140.

[3] A. Einstein, *Jahrb. Rad.Elektr*. **4**, 411 (1907).

[4] A. Pais, *'Subtle is the Lord…' The Science and Life of Albert Einstein*, Oxford University Press, Oxford, 1982, pp. 118-119.

[5] A. A. Michelson and E. W. Morley, "On the Relative Motion of the Earth and the Luminiferous Ether," Am . J. Sci. **34**, 333 (1887); L. Essen, "A New Æther-Drift Experiment," *Nature* **175**, 793 (1955).

[6] W. Voigt, "Ueber das Doppler'sche Princip," Goett. Nachr., 1887, p. 41.

[7] J. Larmor, *Aether and Matter*, Cambridge University Press, London, 1900, pp. 173-177.

[8] H. A. Lorentz, "De draaiing van het polarisatievlak in lichamen die zich bewegen," Versl. K. Ak. Amsterdam **10**, 793 (1902); Collected Papers, Vol. 5, p. 139.

[9] R. J. Buenker, "Time dilation and the concept of an objective rest system," *Apeiron* **17**, 99-125 (2010).

[10] R. J. Buenker, *Relativity Contradictions Unveiled: Kinematics, Gravity and Light Refraction,* Apeiron, Montreal, 2014, p. 50.

[11] C. M. Will, *Was Einstein Right?: Putting General Relativity to the Test.* 2nd ed. Basic Books Inc; New York, 1993, p. 272.

[12] T. Van Flandern, in: *Open Questions in Relativistic Physics*, ed. F. Selleri, Apeiron, Montreal, 1998, p. 81.

[13] R. J. Buenker, "The role of simultaneity in the methodology of the Global Positioning Navigation System," *J. App. Fundamental Sci.* **1 (2)**, 150-156 (2015).

[14] A. H. Buecherer, *Phys. Zeit.* **9**, 755 (1908).

[15] R. J. Buenker, *Relativity Contradictions Unveiled: Kinematics, Gravity and Light Refraction,* Apeiron, Montreal, 2014, pp. 71-77.

[16] R. J. Buenker, *Relativity Contradictions Unveiled: Kinematics, Gravity and Light Refraction,* Apeiron, Montreal, 2014, pp. 58-60.

[17] R. J. Buenker, *Relativity Contradictions Unveiled: Kinematics, Gravity and Light Refraction,* Apeiron, Montreal, 2014, pp. 93-96.

Aug. 11, 2019