Application of Relativity Theory to the Global Positioning System

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Abstract

The Global Positioning System (GPS) measures distances by determining the elapsed time for light to pass between the corresponding endpoints and multiplying this value with the speed of light c. An example is considered in which a metal rod and a clock are carried onboard a satellite. It is shown that Einstein's light-speed postulate is only consistent with the deduction that the rod has increased in length at the same time that the clock has slowed down because of time dilation. Both results are independent of the orientation of the metal rod to an observer on the earth's surface, so the above conclusion indicates that isotropic length expansion accompanies time dilation (clock riddle) on the satellite, contrary to expectations from the Fitzgerald-Lorentz length contraction (FLC) prediction of the Special Theory of Relativity (STR). This logical argument thus shows that the Lorentz transformation (LT) is self-contradictory and therefore invalid. It is also shown that the experimental results of the transverse Doppler effect are only consistent with isotropic length expansion accompanying time dilation. A GPS-compatible space-time transformation (GPSLT) is then derived that removes the contradiction of the LT while still satisfying Einstein's two postulates of relativity.

Résumé

Le Global Positioning System (GPS) mesure les distances en déterminant le temps écoulé pour la lumière de passer entre les extrémités correspondantes et en multipliant cette valeur par la vitesse de la lumière c. Un exemple est examiné dans lequel une tige de métal et une horloge sont réalisés à bord d'un satellite. Il est démontré que Einstein vitesse de la lumière postulat n'est compatible qu'avec la déduction que la tige a augmenté en longueur en même temps que l'horloge s'est ralentie en raison de la dilatation du temps. Les deux résultats sont indépendants de l'orientation de la tige de métal à un observateur à la surface de la terre, de sorte que la conclusion ci-dessus indique que l'expansion de la longueur isotrope accompagne dilatation temporelle (horloge devinette) sur le satellite, contrairement aux attentes de la longueur de contraction Fitzgerald-Lorentz (FLC) prédiction de la théorie de la relativité (STR). Cet argument logique montre ainsi que la transformation de Lorentz (LT) est contradictoire et donc invalide. Il est également démontré que les résultats expérimentaux de l'effet Doppler transversal sont seulement compatibles avec l'expansion de la longueur isotrope accompagnant la dilatation du temps. Une transformation espace-temps GPS compatible (GPSLT) est alors dérivée qui élimine la contradiction de la LT tout en répondant à deux postulats d'Einstein de la relativité.

Keywords: isotropic length expansion, Fitzgerald-Lorentz length contraction (FLC), Lorentz transformation (LT), velocity transformation (VT), GPS-compatible Lorentz transformation (GPSLT).

I. Introduction

There is no question that Einstein's Special Theory of Relativity (STR) played a key role in solving some of the most fundamental engineering problems connected with the Global Positioning System (GPS). However, there are subtle differences between the predictions of relativity theory and the assumptions actually employed in practice by the GPS technology. For one thing, careful discussions of STR always emphasize that it is only applicable to purely translating (inertial) systems. However, GPS satellites are constantly accelerating as they orbit the earth and thus do not meet this qualification, at least not in any strict sense of the term. In addition, STR claims that there is a symmetric relationship between the rates of clocks that are in relative motion. According to Einstein's Symmetry Principle (ESP), observers will always find that clocks in motion run slower than those that are stationary in their own rest frame. That would mean that an observer on a satellite would find that clocks on the earth's surface are running slower than those in his own rest frame, but experience shows that this is definitely not the case. Instead, it has always been found that there is a definite asymmetry for clock rates so that there is never any doubt which clock runs slower as long as sufficient information is available.

The usual reactions to the above discrepancies between theory and experiment fall into two distinct camps. On the one hand, strict "relativists" dismiss them as being of no particular consequence and argue that STR is fundamentally correct in every detail. They resist any and all attempts to criticize the theory, using their considerable influence to keep such discussions out of the mainstream literature. By contrast, scientists and philosophers who do not accept this view, and there are many, have too often concluded that the inconsistencies prove beyond a shadow of doubt that STR is a flawed theory that needs to be discarded entirely. The approach taken in the present work takes a middle road between these two extreme positions by critically examining the theoretical and experimental basis for Einstein's theory.

II. Lack of Internal Consistency in STR

The key to understanding the "disconnect" between GPS and STR is found in Einstein's original paper in 1905 [1]. He had two theories of time dilation. One is symmetric in nature, as exemplified in the ESP, but the other is completely asymmetric. He used the example of a clock attached to an electron moving in a closed path and returning to its original location with less elapsed time showing than for its identical counterpart which remained stationary there ("clock paradox"). He went on to conclude that this meant that clocks at the Equator will run systematically slower than those at either of the Poles, speculation which has since received experimental verification and is essential to understanding the mechanics of the GPS methodology. Engineers simply ignored the ESP and used Einstein's asymmetric theory to make the necessary adjustment in the rates of clocks carried onboard satellites to allow the *quantitative* determination of elapsed times for light signals to arrive from the earth's surface. Something quite important has been ignored in this discussion, however. As shown in the author's previous work [2], the decision to use this version of time dilation leads directly to the conclusion that the lengths of objects carried on the satellites have increased by the same fraction as clock rates have slowed. That is also in opposition to what one expects from STR and the ESP, which claims that there should be length contraction on the satellite.

The argument of the clock riddle is quite simple. It relies entirely on the light-speed postulate. Accordingly, the length L of a metal rod is measured by determining the amount of elapsed time T for a light pulse to travel between its two end points. It follows from STR that $L = cT$, where c is the speed of light in free space. This method of distance determination results from the modern definition of the meter [3] as the distance travelled by light in free space in c^{-1} s (c is defined to have a value of 2.99792458 x 10^8 ms^{-1}). If the rod is located on a satellite moving with speed v relative to the observer O on the earth's surface, it follows from STR (time dilation) that his elapsed time T is greater than that (T') measured locally by an observer O′ on the satellite by a factor of $\gamma = \left(1 - v^2 c^{-2}\right)^{-0.5}$, i.e. $T = \gamma T'$. The light-speed postulate demands that the speed of the light pulse has the same value of c for both O and O′, from which one concludes that $L = cT = \gamma c T' = \gamma L'$, where $L' = cT'$ is the length of the rod measured by O′.

This result is directly opposed to the prediction of the Fitzgerald-Lorentz length contraction (FLC) effect of STR [4,5]. It states that if there is time dilation in the rest frame of the metal rod ($T = \gamma T'$), observer O will measure its length to be $L = \gamma^{-1} L'$ if it is oriented parallel to velocity **v** and $L = L'$ if it is oriented in the perpendicular direction. The argument given above based on the light-speed postulate makes no distinction regarding the orientation of the rod. This is because the speed of light is expected to be the same in all directions for each observer and the rates of clocks are also expected to be independent of direction. The prediction for this theoretical approach is thus *isotropic length expansion* of the rod as opposed to the type of anisotropic length contraction claimed by the FLC. Since both predictions result from application of STR, this state of affairs reveals a clear problem with this theory. In particular, it shows that the Lorentz transformation (LT), which is the centerpiece of STR, is self-contradictory. The derivation of the FLC is based directly on the LT [4,5], and the same holds for time dilation. Moreover, the LT is perfectly consistent with Einstein's light-speed postulate. In the past, textbooks [4,5] have discussed relativistic length variations exclusively in terms of the FLC, but it is clear that the light-speed postulate can be used just as well for this purpose.

Some basic principles of logic come into play in this situation. While the self-contradiction of the LT renders it invalid, this does not mean that all of its many predictions are necessarily incorrect. That is reassuring since physicists have consistently sung the praises of the LT over more than a century and presented numerous examples from experiment that justify this enthusiastic support. What the contradiction does show is that relativity theory needs to be amended in such a way as to eliminate the LT while still remaining true to the many successes of the unrevised theory (STR). Secondly, the contrasting results for relativistic length variations do not in themselves tell us which of the two possibilities, anisotropic length contraction or isotropic length expansion, is actually correct. Both of them are consistent with Einstein's two postulates of relativity. It is simply clear that they both cannot be correct for the same set of circumstances. The obvious solution to this problem is to carry out definitive experiments that can clearly distinguish between length expansion and length contraction.

Length measurements are fundamentally different than time measurements, however. This is because clocks have "memory" whereas metal rods do not. When a clock returns to its original position as in Einstein's famous example [1], it is possible to compare its elapsed time with that of its formally identical counterpart that has remained stationary throughout the process. An analogous criterion does not exist for the lengths of objects, however. Just because the dimensions of the returning object are exactly the same as those of identical objects left behind clearly does not prove anything about possible changes that might have occurred during its flight. One way or the other, one must rely on logical deductions based on sound physical principles to make such judgments. A convincing means of accomplishing this objective is discussed in the next section.

III. The Transverse Doppler Effect and the Relativity Principle

The transverse Doppler effect was predicted by Einstein in his original work [1]. It was first verified experimentally by Ives and Stilwell in 1938 [6]. It was proclaimed as an example of time dilation, but the actual experiment only involves direct measurements of the wavelength of light emitted from an accelerated light source. It was assumed that the wavelength λ measured in the laboratory is related to the standard value $\lambda_0 \equiv \lambda'$ obtained for a stationary source as follows:

$$\lambda = \left(1 - \mathrm{v}^2 \mathrm{c}^{-2}\right)^{-0.5} \lambda' \left(1 - \mathrm{v}\mathrm{c}^{-1} \cos \Theta\right), \qquad (1)$$

where Θ is the angle between the light beam's direction and the relative velocity $\mathbf{v}$ of the light source. The transverse Doppler effect is second-order in vc^{-1} and is independent of angle Θ. The actual experiment was done by carrying out measurements for diametrically opposed directions of the light source (Θ and $\pi + \Theta$) and recording the respective wave patterns on the same photographic plate. Consideration of eq. (1) shows that the averaging of these two results gives the desired second-order effect: $\lambda = \gamma\lambda'$. Subsequent investigations [7] improved the error limits of the experiment to 2-3 %. The average value of λ is interpreted [8] as the wavelength of light that would be measured if the diffraction grating in the laboratory could be transported without change to the rest frame of the light source. The results of the experiment thus speak clearly in favor of length expansion ($\lambda > \lambda'$) in the rest frame of the light source, in quantitative agreement with the above approach which makes use of the light-speed constancy to measure the lengths of objects in accelerated rest frames. Moreover, the effect is completely independent of the orientation of the light source, also as indicated by the theoretical result of the latter method.

The frequency of light is expected to be inversely proportional to its wavelength, again as a consequence of Einstein's light-speed postulate. For this reason the Ives-Stilwell experiment [6,7] has been interpreted [9] as a confirmation of time dilation in the rest frame of the light source. It is only because of strong belief in the FLC that there has not been a similar consensus regarding length variations. At the same time, there have been other confirmatory experiments for time dilation, such as the investigations of muon decay rates [10,11]. The latter work has also emphasized that the degree of acceleration of the particles does not contribute to the amount of time dilation [11,12]. This is especially significant in the interpretation of the transverse Doppler experiments since it indicates that the second-order wavelength shift would be exactly the same if the measurements had been carried out after the accelerated light source had reached a state of pure translation.

One conceivable argument against the conclusion of relativistic length expansion is that wavelengths somehow do not qualify as material objects. It is interesting that the same argument is not used for time dilation, but there is a more fundamental reason for rejecting this view. The relativity principle (RP), Einstein's first postulate of relativity, demands that the wavelength measured locally in the rest frame of the light source in the transverse Doppler experiment must have the standard value (λ_0) determined when the same source is stationary in the laboratory rest frame. The aforementioned negative result for the effect of acceleration on muon decay rates again indicates that the observed wavelength shift would be the same if the rest frame of the light source were inertial, thereby satisfying the conditions for applying the RP in this case. On this basis it can be concluded that the diffraction grating co-moving with the light source must have increased in size in exactly the same proportion as the wavelength. Otherwise, it is impossible to find a rational explanation for the fact that the result of the wavelength measurement in that rest frame is no different than when the analogous measurement is made in the laboratory for the identical light source at rest there. Indeed, the RP stipulates that everything in the formerly accelerated rest frame must have increased in dimension by the same amount in all directions, including the observers themselves. Otherwise, the latter would be able, at least in principle, to distinguish their current inertial system from that of the laboratory where the light source was accelerated initially, which is in clear contradiction to the RP. This line of thought also shows that the same situation exists for any other experiment in which observers in different rest frames measure the dimensions of the same object.

Examination of previous claims of FLC observations [13] shows that they involve distributions of a large ensemble of particles such as electrons. As such, they ignore the effects of de Broglie wave-particle duality [14] which is known to produce a decrease in the wavelength of the distribution in inverse proportion to the momentum of the particles ($p = h\lambda^{-1}$) [15]. It should be noted that the FLC has a substantially different dependence on the speed of particles than does the de Broglie duality. For example, doubling v in the latter case leads to a reduction in the de Broglie wavelength of the particle by 50%, where if the FLC is invoked a much smaller decrease is expected, namely by a maximum factor of $\frac{\gamma(2v)}{\gamma(v)} \approx 1 + 1.5v^2c^{-2}$. The overriding conclusion is thus that the lengths of objects expand isotropically in the same proportion as the rates of co-moving clocks slow down when an external force is applied to cause their mutual acceleration.

IV. The Universal Time-dilation Law

The GPS methodology relies on an asymmetric theory of time dilation. The first experiments with atomic clocks carried onboard aircraft were carried out by Hafele and Keating in 1971 [16]. In the introduction of their paper describing the results of these experiments, the authors noted that there was considerable uncertainty among physicists of the day as to whether the so-called “clock paradox” was fact or fiction. The burning question was whether the onboard clocks of circumnavigating airplanes would lose or gain time relative to their counterparts left behind at the common origin of the flights. Hafele and Keating succeeded in answering this question quantitatively within error limits of about 10% and were able to express their timing results in terms of the formula given below:

$$\tau_1 \gamma\left(\mathrm{v}_{10}\right) = \tau_2 \gamma\left(\mathrm{v}_{20}\right) \tag{2}$$

In this equation τ_i is the elapsed time on a given clock and v_{i0} is the speed of the clock relative to a specific origin, namely the rest frame of the non-rotating polar axes of the earth. Since the earth's center of mass (ECM) is stationary in this rest frame, one could just as well use this position as the reference point. As such, their result is identical to what Einstein had claimed in his purely speculative treatment of the respective rates of clocks located at the Equator and Poles [1]. The key point that is easily overlooked is that their result indicates that the rates of the clocks conform to a purely *objective* theory of measurement. It is the antithesis of the ESP, which comes from Einstein's LT, that claims it is purely a matter of one's perspective as to which of two clocks in relative motion runs slower than the other. Their results show once again, as the Ives-Stilwell investigation did 23 years earlier [6,7], that the predictions of the LT are violated by experiment.

Moreover, it is found that eq. (2) also corresponds quantitatively to the empirical formula discovered by Hay et al. [17] in their Mössbauer study of the transverse Doppler effect for x-rays. These authors mounted an x-ray source and an absorber on a high-speed rotor and measured the frequency shift as a function of the speeds of the two devices relative to the rotor axis. Analogous experimental results were later obtained by Kündig [18] and Champeney et al. [19]. They found that their results could be understood in terms of the time dilation experienced in the respective rest frames of the absorber and light source: the greater the distance of a given clock from the axis of the rotor, the faster is its speed and the slower is its rate. A shift to higher frequency (blue shift) was observed in the experiments because the absorber was located at the rim of the rotor, causing its clock to run slower than its counterpart at the x-ray source nearer the axis. As a consequence more wave maxima were registered per second by the absorber than were emitted at the x-ray source based on a stationary clock located there. The τ_i values in eq. (2) are the respective periods of the measured radiation and the v_{i0} values are the corresponding speeds relative to the rotor axis.

There are in fact no experimental studies that are in conflict with the above analysis. In view of this state of affairs, it is reasonable to refer to eq. (2) as the Universal Time-Dilation Law (UTDL). In order to apply it, it is necessary to specify the reference for computing the speeds in its $\gamma\left(v_{i0}\right)$ factors. It has been referred to in past work [20] as the objective rest system (ORS). It is the point of acceleration for Einstein's electron [1], the rotor axis in the Hay et al. experiments [17-19], and it is the ECM in the Hafele-Keating atomic clock study [16] as well as in Einstein's original example [1] of clocks located at various latitudes on the earth's surface. The remaining question is how this asymmetric theory of time dilation can be seamlessly incorporated into relativity theory in general.

V. The GPS-compatible Version of the Lorentz Transformation

There is a simple reason why the LT has escaped scrutiny over the past century. It is because it is only one of an infinite number of space-time transformations that satisfy Einstein's light-speed postulate [21-23]. As such, it is consistent with the relativistic velocity transformation (VT) [1], but so are all the other possibilities. The VT in turn is directly responsible for all the confirmed experimental results that have previously been attributed to the LT. These include the Michelson-Morley experiment [24], aberration of starlight at the zenith

[1,25], the Fresnel light-drag experiment [25,26], and the many other investigations such as of the transverse Doppler effect [6-7,17-19] that find that the speed of light is indeed independent of both the observer and the light source.

The equations of the VT are given below:

$$u'_x = \left(1 - vu_x c^{-2}\right)^{-1}\left(u_x - v\right) = \eta\left(u_x - v\right) \tag{3a}$$

$$u'_y = \gamma^{-1}\left(1 - vu_x c^{-2}\right)^{-1} u_y = \eta\gamma^{-1} u_y \tag{3b}$$

$$u'_z = \gamma^{-1}\left(1 - vu_x c^{-2}\right)^{-1} u_z = \eta\gamma^{-1} u_z \,. \tag{3c}$$

The VT expresses the relationship between the respective values of the speed components $u_x = \frac{\Delta x}{\Delta t}$, $u'_x = \frac{\Delta x'}{\Delta t'}$ etc. of an object as measured by two observers who are moving with velocity **v** relative to each other along the common x,x′ axis. In order to obtain the desired transformation between the space-time variables of the two observers, it is necessary to add another piece of information that is consistent with experimental data. The UTDL of eq. (2) is an obvious candidate for this purpose. As mentioned in previous work [21-23,27,28], it is first necessary to convert the UTDL into a form which is consistent with the primed-unprimed notation of the VT. This can be done quite easily by equating τ_1 with $\Delta t'$ and τ_2 with Δt and setting their ratio equal to the proportionality constant $Q = \frac{\gamma\left(v_{10}\right)}{\gamma\left(v_{20}\right)}$, i.e. $\Delta t' = Q^{-1}\Delta t$. Melding this equation with each of the VT relationships in eqs. (3a-c) then gives the desired alternative transformation (ALT):

$$\Delta t' = Q^{-1}\Delta t \tag{4a}$$

$$\Delta x' = \eta Q^{-1}(\Delta x - v\Delta t) \tag{4b}$$

$$\Delta y' = \eta (\gamma Q)^{-1} \Delta y \tag{4c}$$

$$\Delta z' = \eta (\gamma Q)^{-1} \Delta z\ , \tag{4d}$$

with η having the same definition as in the VT. In view of the rationale which has led to the above equations, it is appropriate to refer to them collectively as the Global Positioning System Lorentz transformation (GPSLT). It satisfies both of Einstein's postulates of relativity while ensuring that the proportionality of clock rates assumed in the GPS methodology is guaranteed as well. The GPSLT avoids the contradiction inherent in Einstein's LT (clock riddle [2]) by eliminating the FLC from consideration. It also restores the concept of remote absolute simultaneity since it is clear that Δt must vanish whenever $\Delta t'$ does because of eq. (4a). Most importantly, it conforms to the asymmetric theory of time dilation which is observed in all experimental investigations reported to date.

VI. Conclusion

There are two ways to use the Lorentz transformation (LT) to predict changes in the length of an object as it moves away from a stationary observer. One is based on Fitzgerald-Lorentz length contraction (FLC), which states that the length will *decrease in various proportions* depending on its orientation at the same time that clock rates slow down because of time dilation. The other way is to measure the elapsed time ΔT for a light pulse to move between the two termini of the object. In this approach, the length L is determined by multiplying the elapsed time with the speed c of the pulse ($L = c\Delta T$), consistent with the modern-day definition of the meter. An observer O′ co-moving with the rod will not notice any change in its length (D m) nor in the elapsed time required for the light pulse to traverse it $\left(\frac{D}{c} s\right)$, no matter what its speed v is relative to the stationary observer O. Because of time dilation, however, O will measure a larger value for the elapsed time, namely by a factor of $\gamma = \left(1 - v^2 c^{-2}\right)^{-0.5}$, i.e. $\gamma \frac{D}{c}$ s. According to the light-speed postulate, O will therefore measure the length of the rod to be γ D m, which means that it has increased in length. By contrast, according to the FLC, O will measure the length of the rod to be γ^{-1}D if it is oriented in a parallel direction and D if it is oriented in a perpendicular direction.

Since both results are obtained from the LT, the disagreement between these values shows that this space-time transformation is self-contradictory and is therefore invalid. The question as to which of the two methods is actually proper is answered by examining the results of the transverse Doppler effect. After eliminating the v-dependent factor in eq. (1) it is found that wavelengths on a moving source increase by the same factor γ defined above. This result is clearly in violation of the FLC and in quantitative agreement with the method that uses elapsed times for light-pulse travel to make such determinations. The Hafele-Keating experiments [16] as well as the investigations of the transverse Doppler effect using high-speed rotors [17-19] indicate that eq. (2) can be used quite generally to determine the ratios of clock rates and hence is referred to as the Universal Time-dilation Law (UTDL). In using eq. (2), it is necessary to assign a specific rest frame (Objective Rest System, ORS) for determining the relative speeds that appear in it. Finally, an alternative Lorentz transformation (GPSLT) has been derived that satisfies both of Einstein's postulates as well as the UTDL, and is therefore directly applicable to the GPS methodology.

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