The Clock Puzzle and the Incompatibility of Proportional Time Dilation and Remote Non-simultaneity

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Abstract

The slowing down of the rates of clocks by virtue of their motion (time dilation) and the conclusion that events which are simultaneous for one such clock may occur at different times for another (remote non-simultaneity) are two of the most well-known consequences of Einstein's Special Theory of Relativity (STR). A simple algebraic puzzle is presented which shows that these two predictions are fundamentally incompatible with each other. Both effects are derived from the Lorentz transformation (LT), which is the cornerstone of STR, thereby proving that the theory is not physically viable and is therefore in need of comprehensive revision. Another version of the Lorentz transformation (Global Positioning System-LT) is presented which satisfies both of Einstein's postulates of relativity, but which does away with the space-time mixing characteristic of the original LT. The GPS-LT is in agreement with all experimental data as yet observed regarding the variation of clock rates with motion, as expressed in a Universal Time-dilation Law (UTDL). It is also consistent with the Relativistic Velocity Transformation (RVT) and is therefore able to explain many of the effects previously looked upon as unique successes of the LT.

Keywords: Time dilation, remote non-simultaneity, Lorentz transformation (LT), Universal Time-dilation Law (UTDL), alternative Global Positioning System-Lorentz transformation (GPS-LT), relativistic velocity transformation (RVT)

I. INTRODUCTION

The Lorentz transformation (LT) [1-3] revolutionized the way scientists viewed the relationship between space and time. Einstein [3] showed that the LT could be derived on the basis of two postulates of relativity, a version of Galileo's Relativity Principle [RP] and the constancy of the speed of light in free space. It was claimed that the only way to satisfy both postulates is to assume, contrary to the traditional belief of Newtonian classical physics, that space and time are inextricably mixed and are actually a single entity referred to as "spacetime." Poincaré [4] was the first to point out that a consequence of space-time mixing is that events which are simultaneous for an observer in one rest frame may occur at different times for an observer in a different rest frame. The concept of remote non-simultaneity (RNS) was therefore something that needed to be explored experimentally with an open mind [5].

In Einstein's landmark paper of 1905 [3], he derived a different novel consequence of the LT which has come to be known as time dilation (TD). Accordingly, a clock in motion relative to the observer is predicted to run at a slower rate than one which is stationary in the observer's own rest frame. The two rates are expected to differ by a constant proportionality factor which depends only on the relative speed v of the two clocks, namely $\gamma=(1-v^2c^{-2})^{-0.5}$, where c is the speed of light in free space (defined to have a value of 299792458 ms^{-1} by international agreement [6]).

It is helpful to describe the TD and RNS predictions in terms of time differences for the same two events observed in different inertial rest frames S and S' moving with speed v relative to one another [7]. The time difference observed using a stationary clock in S is referred to as Δt, while the corresponding time difference measured with a stationary clock in S' is referred to as $\Delta t'$. In this notation the TD relation between the two time differences is therefore given by the proportionality relation given below:

$$\Delta t = \gamma \Delta t' \tag{1}$$

According to the RP, however, the above equation must be invariant to an interchanging of the S and S' coordinates when the sign of the relative speed is reversed. Hence, it is equally correct to express the relationship between the two time differences as:

$$\Delta t' = \gamma \Delta t \tag{2}$$

since $\gamma(-v)=\gamma(v)$. In both cases, it is seen that Δt and $\Delta t'$ are strictly proportional to one another as long as v remains constant. Examination of the TD derivation from the LT shows that the observer always finds that it is the moving clock which is running slower.

The RNS prediction in the same notation for time differences is as follows. If one assumes that the two events are *simultaneous* based on the stationary clock in S', then $\Delta t'=0$. If the same events do not occur simultaneously based on the stationary clock in S, the corresponding equation is $\Delta t=Y$, where $Y\neq 0$. It will be shown below, however, that there is a simple general relationship between the two time differences which demonstrates that it is actually impossible for both TD and RNS to exist for the same pair of events.

II. THE CLOCK PUZZLE

The fundamental incompatibility of TD and RNS can be demonstrated in a most convincing manner with the help of the following set of two simple equations:

$$A = 0 \tag{3a}$$

$$B = XA\,. \tag{3b}$$

If X is finite, it is clear from eqs. (3a) and (3b) that B=0. The essential point is that the product of any finite number with zero has a unique value of zero.

In order to make the connection between the above equations and relativity theory, it is only necessary to equate the values of A and B to the time differences mentioned in the Introduction. Specifically, in the clock puzzle A is replaced with $\Delta t'$ and B with Δt:

$$\Delta t' = 0 \tag{3a'}$$

$$\Delta t = X\,\Delta t' \tag{3b'}$$

According to the TD prediction of the LT shown in either eq. (1) or eq. (2), the values of the two time differences always occur in strict proportion to one another. With reference to eq. (3b'), this means that X has a value of either $\gamma(v)$ if eq. (1) is used or $1/\gamma(v)$ if eq. (2) is used instead. In both cases X is a finite proportionality constant, so the value of Δt in eq. (3b') is zero since $\Delta t'=0$, in complete analogy to what has been found in eqs. (3a-b). According to the RNS prediction of the LT, however, $\Delta t=Y$, where $Y\neq 0$. Hence, Δt supposedly has a non-zero value in this case, which is inconsistent with the value of $\Delta t=0$ based on TD and eq. (3b'). Changing the meaning of the variables A and B to time differences does not alter the basic conclusion that the only way to make the two equations consistent is to have B=0 in eq. (3b) and/or $\Delta t=0$ in eq. (3b').

The clock puzzle is a simple example of how logical inconsistencies can occur within a pair of equations depending on the physical definitions of their variables. As such, it is little more than a potential exercise for beginning algebra students. When the same analysis is applied for a specific concretization of the variables in eqs. (3a-b) as time differences recorded on two clocks in motion, however, the puzzle becomes far more interesting. This is because the same logical arguments lead to the conclusion that two well-known effects of Einstein's Special Theory of Relativity (STR) [3] are hopelessly incompatible with one another. One claims that B=0 while the other just as surely requires that B≠0. This realization then leads to another conclusion of immense importance to this theory, namely the Lorentz transformation (LT) from which both are derived is self-contradictory. It therefore *cannot be reasonably claimed to provide a true representation of the relationship between space and time.* A related discussion of this general topic may be found in an earlier reference [8].

III. GPS and the Universal Time-dilation Law (UTDL)

Once it is realized that proportional time dilation is incompatible with remote non-simultaneity, the question clearly arises as to which, if either of them, actually occurs in natural processes. For this purpose, it is clear that only well-documented experimental studies can provide a reliable answer. One doesn't have to look further than the dashboard of his/her car to obtain the necessary information. In order for the Global Positioning System (GPS) to ensure sufficiently high accuracy for its distance measurements, it is critical that atomic clocks located on its satellites run at the same rate as those on the earth's surface [9-10]. A study with circumnavigating airplanes carried out by Hafele and Keating [11-12] had shown that the rates of clocks slow by a precisely determined amount as a consequence of their motion relative to the earth's center of mass (ECM). An additional effect due to the gravitational red shift [13] had also been observed.

To account for these influences, a "pre-correction" procedure [9,10] is applied to the atomic clocks prior to their launch into a given orbit around the earth. As a consequence, when the satellite does achieve its prescribed trajectory and the expected effects on the clock rates have occurred, there is an adjusted onboard clock that now runs at the same rate, at least to a suitable degree of approximation, as an identical, un-adjusted, clock left behind at the launch position. As a result, it can safely be assumed that the time read from this adjusted clock when a light pulse is received on the satellite is the same as for the clock at the launch position. Comparing this value with the corresponding earlier time recorded at the origin when the light pulse was emitted there thus enables one to accurately determine the elapsed time Δt for the light to travel between the two positions. Therefore, it is concluded that the corresponding distance travelled by the light pulse is $c\Delta t$ by assuming that its speed during the entire flight is equal to the free-space value of c.

It is generally believed among the physics community that the GPS pre-correction procedure is completely consistent with Einstein's STR [3]. In particular, the clock-rate retardation experienced on the satellite (after correction for gravitational effects) is believed to be in agreement with his time-dilation (TD) prediction [9]. The fact is, however, that the effect actually observed is not consistent with the STR conclusion that it is purely a matter of perspective whether the unadjusted (proper) clock on the satellite runs faster or slower than its counterpart located on the earth's surface [10]. The pre-correction procedure operates on the principle that it is the satellite clock which runs slower. In other words, the *observed* TD effect is expressed in terms of the following proportionality relationship:

$$\Delta t' \;(\text{satellite}) = \frac{\Delta t \;(\text{surface})}{Q} \tag{4}$$

with Q>1. This relationship is perfectly objective in character, not *subjective* as the LT version of the TD prediction in eqs. (1-2) requires. Instead, eq. (4) is consistent with another version of TD discussed in Einstein's 1905 paper [3]. In that case, it was assumed that the clock attached to a particle moving in a closed path would return to its point of origin with less elapsed time than one that stayed behind there. This version is *asymmetric* in character and gave rise to the famous "clock paradox," which among other things led to the speculation that a twin would return from a long high-speed voyage notably younger than his brother who had remained behind at the original position.

It is much easier to see that the "pre-correction" is also inconsistent with the remote non-simultaneity (RNS) LT prediction, however. Accordingly, any event observed from the rest frame of the satellite is assumed to occur at exactly the same time on the adjusted clock as that read from the surface clock. Otherwise, it makes no sense to make the pre-correction if events don't always occur simultaneously for both of the clocks. In summary, neither of the LT predictions actually occurs in nature. The fact that the GPS pre-correction procedure invariably leads to more accurate distance predictions than is possible using unadjusted proper clocks at both locations constitutes very strong evidence to support the result of the clock puzzle in eqs. (3a'-b') that *the LT is not a viable space-time transformation.*

It is interesting to consider one of Einstein's arguments claiming to verify the proposed existence of remote non-simultaneity (RNS). He used the by-now well-known example of two lightning bolts striking opposite ends of a moving train [14], whereby an observer standing on the station's platform finds that they have occurred simultaneously as the midpoint of the train passes his position. Light flashes from the strikes therefore reach the midpoint of the train at the same time from his perspective. If the distance between the positions of each of the strikes and the midpoint is L, he finds that the time required for each of the two light flashes to meet there is $\Delta T = L/c$. Einstein then argued that the situation is qualitatively different for an observer located on the train standing at its midpoint. He has moved relative to his original position by the time the light flashes arrive at the midpoint. If the train is moving with speed v relative to the platform, he finds that he has moved closer to the position of the forward lightning strike by a distance of $v\Delta t^f$ when the corresponding light flash arrives at the midpoint of the train, whereby the time required for this to occur is Δt^f. The distance traveled by this light pulse is therefore $L = c\Delta t^f + v\Delta t^f$. The light flash from the lightning strike at the rear of the train has to move an extra distance of $v\Delta t^r$ to reach the midpoint, so that the total distance traveled by this light flash is equal to $L + v\Delta t^r = c\Delta t^r$, whereby Δt^r is the corresponding elapsed time for this to occur. One therefore finds that the two times of arrival *are not the same* from the vantage point of the observer on the train since $\Delta t^r = L/(c-v)$ and $\Delta t^f = L/(c+v)$. Einstein therefore concluded that the two light flashes do not arrive at the same time for one observer but do so for the other, a clear demonstration of RNS in a practical case.

There is nonetheless an inconsistency with the rest of Einstein's theory, namely with his light-speed constancy postulate. One sees that the speed of the forward light flash relative to its source is not c, but rather $L/\Delta t^f = c+v$. By contrast, the corresponding speed of the light flash coming from the rear of the train is $L/\Delta t^r = c-v$. Thus, Einstein achieves his claimed verification of RNS at the expense of his light-speed postulate, showing that remote non-simultaneity is not even viable within his own Special Theory of Relativity (STR) [3].

The train example can be analyzed in a far simpler way which is consistent with the GPS relationship given in eq. (4). By definition, the platform observer measures the elapsed time for each light flash to arrive at the midpoint of the train to be $\Delta t = L/c$. Because of the clock-rate proportionality expected from GPS, the corresponding elapsed time for *both* events on the proper clocks of the platform observer should be $\Delta t'=L/Qc$, i.e. the light flashes are also expected to arrive *simultaneously* at the train's midpoint from his perspective. This is exactly the result expected from the clock puzzle and eqs. (3a-c), thereby contradicting the RNS predicted by the LT [3].

The failure of the LT to properly describe relationships between elapsed times read from different clocks for the same two events raises the obvious question of what is the true relativistic space-time transformation. The empirical formula in eq. (4) indicates that there is a simple proportionality between such times that contrasts sharply with the space-time mixing expected from the LT. It is obvious that remote non-simultaneity cannot occur as long as this condition holds since a zero value for one of the time differences implies that the other value will be zero as well. Similarly, a non-zero value for one of the clocks also requires a non-zero value for the other.

Moreover, there is another straightforward argument based on Newton's First Law of Kinematics that also is consistent with eq. (4). In the absence of unbalanced external forces, one expects that a clock will move with constant speed and direction indefinitely. It is a seemingly plausible extension of Newton's First Law to expect the values of physical properties of the clock will also not change under this condition, including the rate of the clock. A second such clock may travel along a different straight-line trajectory and its rate may also be different than for the first. Since both rates are expected to be constant, however, it follows that *their ratio must also remain constant indefinitely,* which behavior is therefore also clearly consistent with the proportionality relation in eq. (4).

The experiments carried out by Hafele and Keating with circumnavigating atomic clocks [11,12] were very helpful in answering the important question of how to determine the value of the proportionality factor Q in eq. (4) on a general basis. It was found that the elapsed time Δt on a given clock (after correcting for the effects of the gravitational red-shift [13]) for a given portion of the aircraft's flight path was inversely proportional to $\gamma(v) = (1-v^2c^{-2})^{-0.5}$, *where v is the speed of the clock relative to the earth's center of mass (ECM):*

$$\Delta t = \frac{\Delta t(\mathrm{ECM})}{\gamma(v)}. \tag{5}$$

In this equation, Δt (ECM) is the corresponding elapsed time measured on a *hypothetical* clock which is stationary in the ECM rest frame. This meant, for example, that the eastward flying clock returned to the airport of origin with *less* elapsed time for the entire journey than an identical clock left behind there, because in this case the clock was flying in the *same* direction as that of the earth's rotation so that the aircraft's ground speed needs to be added to the rotational speed to obtain the value of v to be substituted in eq. (5). Conversely, a clock which flew in the westerly direction, and therefore a slower speed relative to the ECM, returned with *more* elapsed time than the stationary clock at the airport.

The constant Δt (ECM) can be eliminated by considering a second clock moving with speed v' for the same distance interval. One can obtain the corresponding elapsed time Δt' from eq. (5), which therefore leads to the following general relation:

$$\Delta t\,\gamma(v) = \Delta t'\,\gamma(v') \tag{6}$$

After correcting for gravitational effects on the various clocks, Hafele and Keating [12] found that eq. (6) gave results which agreed with observed elapsed times for the entire journey to within approximately 10% of the observed values.

Moreover, the same relation was found to accurately describe the results of experiments [15-17] carried out a decade earlier with an x-ray absorber and detector mounted on a rotating disk. The latter can be considered to be clocks in motion with corresponding elapsed times in eq. (6) identified with their respective frequencies ν and ν'. The frequency of the various clocks varied with their speeds v and v' measured relative to the axis of rotation of the disk, as opposed to the ECM used as reference in the Hafele-Keating experiments [11-12]. The standard x-ray frequency ν_0 plays the same role as Δt (ECM) in eq. (5). As in the latter case, the rate of a given clock in the rotor experiments [15-17] *slows* in inverse proportion to γ (v) relative to the standard rest frame. Eq. (6) is also consistent with Einstein's speculation in his original work [3] about the slowing of the rate of a clock attached to an electron moving in a closed path, as well as with his conclusion that a clock at the Equator should run slower than one located at either of the earth's Poles.

At the same time, eq. (6) is clearly inconsistent with eqs. (1-2) derived from the LT. In that case it is concluded that *the relative speed* of two clocks is the sole determining factor for the amount of time dilation. According to eq. (6), however, the relative speed of the clocks is not directly involved in the calculation of the ratio of the two clock rates. As a result, it is not simply a matter of perspective which of two clocks in motion runs slower. There is always a specific rest frame (referred to in earlier work as the objective rest system ORS [18]) from which the speeds of the clocks v and v' are to be measured. One clock runs objectively slower, the other objectively faster. This totally unambiguous relationship was already pointed out by Sherwin [19] in his analysis of the rotor experiments of Hay et al. [14] and was also clearly understood by Hafele and Keating [11] in their analysis of the atomic clock studies on airplanes.

Consequently, it is appropriate to refer to eq. (6) as the Universal Time-dilation Law (UTDL) [20-22]. To apply it, it is first necessary to identify the aforementioned ORS rest frame relative to which the speeds v and v' of the clocks are to be measured. It is the ECM in the atomic clock study on circumnavigating aircraft [11-12], the rotor axis in the x-ray absorption experiments [15-17] and the point at which the force is applied to the electron in Einstein's original example [3].

It is a simple matter on the basis of the UTDL to determine the value of the proportionality constant Q in the completely general version of eq. (4) given below for any two clocks in motion:

$$\Delta t' = \frac{\Delta t}{Q}. \tag{7}$$

Rearrangement of eq. (6) leads directly to the desired result:

$$Q = \frac{\gamma(v')}{\gamma(v)}. \tag{8}$$

It is assumed thereby that the ORS is the same for both clocks. If the ORS is not the same for both, as for example if one clock is orbiting the moon while the other is orbiting the earth, it is also necessary to take account of the ratio of the rates of two hypothetical clocks that are stationary in the different ORS frames in order to obtain the correct value for Q.

The experimental data [12, 15-17] which are in support of the UTDL therefore indicate that eq. (7) is the true relativistic space-time transformation. It clearly is not consistent with RNS and it also eliminates any ambiguity about which of the two clocks runs more slowly, contrary to what is expected on the basis of eqs. (1,2) and the space-time mixing characteristic of the LT.

There is also a clear distinction between the UTDL and the LT version of time dilation with regard to the consequences of the Relativity Principle (RP). As discussed in the Introduction, it is essential because of the RP that the inverse of eq. (7) be obtained by interchanging the primed and unprimed variables and reversing the sign of the relative speed v of the two clocks. When this procedure is applied to eq. (7), the result is:

$$\Delta t = \frac{\Delta t'}{Q'}. \tag{9}$$

Contrary to the situation for eqs. (1,2) of the LT, it can be assumed that eqs. (7,9) are related by standard algebra. In the case of the LT, this is impossible because γ(v)=γ(-v), but combining eqs. (7,9) leads directly to the unique result for Q':

$$Q' = \frac{1}{Q}. \tag{10}$$

Applying the same RP procedure to eq. (8) then leads to a perfectly consistent result, namely:

$$Q' = \frac{\gamma(v)}{\gamma(v')} = \frac{1}{Q}. \tag{11}$$

The above reciprocal relationship can be understood quite succinctly by looking upon Q and Q' as *conversion factors* between the different units of time in the two rest frames S and S'. A stationary observer in S employs the standard unit of 1 s for time. He finds, however, that the corresponding unit in S' is Q s. This is why his measured value for a given elapsed time Δt is Q times greater [if Q>1; note that Q<1 is also possible based on eq. (8) if v>v'] than the corresponding value Δt' obtained for the same event by his counterpart in S', as indicated in eq. (7). Because of the RP, however, the S' observer also thinks that the standard unit of time in his rest frame is 1 s, but experiment shows that it is actually not the same as the unit employed by the S observer. Thus, when the tables are reversed, and the S' observer wants to reconcile his measured elapsed times with those obtained with proper clocks that are stationary in S, he must apply the reciprocal conversion factor Q' in accord with eq. (9). The situation is completely analogous to what one meets in conventional conversions between different units for the same quantity. For example, a conversion factor of 100 is needed to change a value of distance from m to cm, whereas in the reverse direction the corresponding factor is 1/100 to go from cm to m.

IV. The true relativistic space-time transformation

Newtonian clock-rate proportionality and the UTDL of eq. (6) is assumed in the following set of equations known as the Alternative Lorentz Transformation (ALT) or the Global Positioning System-Lorentz transformation (GPS-LT) [23-26]:

$$\Delta t' = \eta \frac{\Delta t - vc^{-2}\Delta x}{Q} = \frac{\Delta t}{Q} \tag{12a}$$

$$\Delta x' = \eta \frac{\Delta x - v\Delta t}{Q} \tag{12b}$$

$$\Delta y' = \left(\frac{\eta}{\gamma Q}\right)\Delta y \tag{12c}$$

$$\Delta z' = \left(\frac{\eta}{\gamma Q}\right)\Delta z \tag{12d}$$

,

where $\eta = (1\text{-}vc^{-2}\Delta x/\Delta t)^{-1}$ and $\gamma=(1\text{-}v^2c^{-2})^{-0.5}$. Note that eq. (12a) is identical to eq. (7) discussed in the previous section. Division of the space variables $\Delta x'$, $\Delta y'$ and $\Delta z'$ in eqs. (12b-d) by $\Delta t'$ leads to the same relativistic velocity transformation (RVT) that is derived from the LT:

$$u'_x = \left(1 - vu_x c^{-2}\right)^{-1}\left(u_x - v\right) = \eta\left(u_x - v\right) \tag{13a}$$

$$u'_y = \gamma^{-1}\left(1 - vu_x c^{-2}\right)^{-1} u_y = \eta\gamma^{-1}u_y \tag{13b}$$

$$u'_z = \gamma^{-1}\left(1 - vu_x c^{-2}\right)^{-1} u_z = \eta\gamma^{-1}u_z, \tag{13c}$$

where the velocity components are defined as $u_x' = \Delta x'/\Delta t'$ etc. This characteristic of the GPS-LT proves that all relationships between velocity components obtained with the LT are also valid for this transformation, including consistency with Einstein's second postulate of relativity, the constancy of light in free space. The LT can be obtained from the GPS-LT by multiplying each of its four equations on the right-hand side by $\gamma Q/\eta$, so the fact that the velocity relationships are the same for both transformations is obvious. Lorentz [27] has pointed out that there is a degree of freedom in the most general definition of the LT that needs to be eliminated in order to obtain a unique solution for the relativistic space-time transformation [24], so the GPS-LT is simply the result of multiplying with a different factor than either Lorentz or Einstein [3] did in their derivation of the LT.

It is less obvious that the GPS-LT also satisfies the RP, but this characteristic is nonetheless easily verified by noting that the new transformation can be obtained directly from the RVT, which of course also satisfies the RP, by multiplying each of its three equations with eq. (7), i.e. changing eqs. (13a-c) to the corresponding GPS-LT eqs. (12b-d) since $\Delta x'=u_x'\Delta t'$ etc. The proof can also be obtained using the identity [28]:

$$\eta\eta' = \gamma^2 . \tag{14}$$

Since QQ'=1 in the GPS-LT, changing the sign of v and interchanging the primed and unprimed variables changes eq. (12c) into its inverse, for example.

In summary, the GPS-LT is the only space-time transformation which is consistent with both of Einstein's postulates of relativity and at the same time also agrees with the experimentally observed strict proportionality of clock rates as described by the UTDL of eq. (6).

V. Comparison of the experimental predictions of the GPS-LT and the LT

One of the most pressing issues that arises when it is realized that the LT is not a valid space-time transformation is how its application could nonetheless have led to so many accurate experimental predictions. It is a simple matter to answer this question, however. A survey of the literature [29] shows that all experimental tests of the LT space-time relationships *only involve ratios of these quantities*. As such, they do not require the LT at all and thus cannot be properly cited as *unique* verifications of this version of the general Lorentz space-time transformation [27]. For example, von Laue’s derivation of the Fresnel light-drag effect [30] is based *exclusively on the RVT* of eqs. (13a-c). Similarly, the aberration of starlight at the zenith can be explained entirely by comparing the parallel and perpendicular velocity components of light in the rest frames of the star and the earth [31]. The Thomas precession effect [32] for atomic spins only involves an angle and therefore also does not require the LT for its derivation [33]. Thus, the GPS-LT, because of its direct relationship to the RVT, is every bit as compatible with the above observations as the LT. Moreover, the Sagnac effect can be explained entirely on the basis of Einstein's light-speed postulate and the RVT and therefore also does not require the LT for its explanation [29]. Finally, Maxwell's equations are also invariant under the application of the GPS-LT, again because of the degree of freedom in the general version of the Lorentz transformation [3,27].

Much has been made of the fact that that the LT leads to the following highly symmetric invariance relation:

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \,. \tag{15}$$

known simply as the condition of Lorentz invariance. It is obviously consistent with both of Einstein's postulates of relativity and has sometimes been used to derive the LT directly [34]. The problem is that the only space-time relation which is compatible with eq. (15) leads to the unavoidable contradiction associated with the "clock puzzle" discussed in Sect. II:

$$\Delta t' = \gamma \left(\Delta t - v \Delta x c^{-2} \right) = \gamma \eta^{-1} \Delta t \,. \tag{16}$$

It is impossible to satisfy the Lorentz invariance condition without requiring that both remote non-simultaneity and the Einsteinean symmetric version of proportional time dilation must occur in natural processes. The elegant symmetry of eq. (15) therefore must be discarded as a physical principle for the simple reason that it leads to a logical contradiction.

It is important to see how time dilation (TD) is derived from STR [3, 7]. An example is considered in which a stationary clock in one of the rest frames (S') described via the LT is used to measure a time difference Δt' for a given pair of events. There is no restriction on the types of events to be considered. The time difference could be a lifetime, a period of a clock or an elapsed time of any kind. It is stipulated that the clock remain in the same position during the entire course of the measurement, which means that $\Delta r'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = 0$ in eq. (15).

The corresponding measurement is also made in the other rest frame (S), and this time difference is designated as Δt. The stationary clock in S' is moving relative to S with speed v in the x direction, however. Consequently, $\Delta r = \Delta x = v\Delta t$. The right-hand side of eq. (15) is equal to $\Delta r^2 - c^2\Delta t^2$, so upon substitution one therefore has the following result for this application:

$$-c^2\Delta t'^2 = v^2\Delta t^2 - c^2\Delta t^2 . \tag{17}$$

Rearrangement thus leads to the TD formula in eq. (1):

$$\Delta t' = \left(1 - v^2c^{-2}\right)^{0.5} \Delta t = \gamma^{-1}\Delta t , \tag{1'}$$

whereby the key proportionality constant is seen to be γ^{-1}.

There is an interesting twist to this derivation, however. If the tables are turned and the measured times of the stationary clock in S are studied by the observer in S', one must change the derivation so that now $\Delta r = 0$ and $\Delta r' = -v\Delta t'$ are to be substituted in eq. (15). The result is therefore the same as the TD formula in eq. (2):

$$\Delta t = \left(1 - v^2c^{-2}\right)^{0.5} \Delta t' = \gamma^{-1}\Delta t' . \tag{2'}$$

As has been discussed in the Introduction, eq. (1) cannot be obtained from eq. (2) by a simple inversion. Instead, there is a symmetry between the two equations that can be conveniently summarized by noting that *it is always the moving clock* relative to the observer that runs slower.

This is a completely *subjective* relationship, since it indicates that it is a matter of perspective which of two moving clocks has the slower rate.

The point which has been emphasized in Sect. II is that the relationships of eqs. (1,2) are not consistent with remote non-simultaneity otherwise derived from the LT, specifically from eq. (16). The latter indicates that remote non-simultaneity must occur when v and Δx both have non-zero values, whereas the proportionality in the TD relation of eqs. (1,2) indicates unequivocally that events which are simultaneous for one observer will be simultaneous for all others as well.

The corresponding invariance relation obtained from the GPS-LT,

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2 = \eta^2 \gamma^{-2} Q^{-2} \left(\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \right) \tag{18}$$

is not as simple in appearance as eq. (15), but it also satisfies both of Einstein's postulates without standing in contradiction to Newton's First Law and the Law of Causality (see Sect. III). This raises the question as to whether the same lack of consistency with regard to simultaneity occurs for the GPS-LT. To check this possibility, we can again consider the above example of two clocks in motion.

If one makes the same substitutions as previously ($\Delta r' = 0$ and $\Delta r = v\Delta t$), the result is:

$$-c^2 \Delta t'^2 = \eta^2 \gamma^{-2} Q^{-2} \left(v^2 \Delta t^2 - c^2 \Delta t^2 \right). \tag{19}$$

The constant η has a value of $\left(1 - v^2 c^{-2}\right)^{-1} = \gamma^2$ in this case, so evaluation of eq. (19) gives:

$$\Delta t' = \left[\gamma^4 \gamma^{-2} Q^{-2} \left(1 - v^2 c^{-2} \right) \right]^{0.5} \Delta t = \frac{\Delta t}{Q}, \tag{20}$$

which is seen to be identical to eqs. (7,12a) of the GPS-LT, proving that there is consistency in this case.

Reversing the roles of the two stationary clocks, i.e. substituting $\Delta r = 0$ and $\Delta r' = -v\Delta t'$ in eq. (18), one obtains (note that $\eta = 1$ in this case since $\Delta x = 0$):

$$\left(v^2\Delta t'^2 - c^2\Delta t'^2\right) = \gamma^{-2}Q^{-2}\left(-c^2\Delta t^2\right). \tag{21}$$

Rearrangement then gives the same result as in eqs. (7, 12a), which shows that the GPS-LT is self-consistent in this application, unlike the LT.

Since Lorentz invariance is the cornerstone of Minkowski's four-vector formalism [35,36], it also is clear that this version of Einstein's STR is flawed. It depends wholly on the LT. This was the point that Einstein was making when he dismissed it [36] as "superfluous learnedness." All that is done is to put STR in the framework of linear/affine spaces. One defines the spatial variables in the LT as follows: $x_1=\Delta x$, $x_2=\Delta y$, $x_3=\Delta z$. Then, instead of using elapsed time directly, a fourth vector is defined as $ic\Delta t$. The Lorentz invariance condition of eq. (15) is obtained by summing the squares of the four LT relations. In terms of the Minkowski four-vector, $\mathbf{x}=(x_1,x_2,x_3,x_4)$, this equation becomes a relation between scalar products:

$$x \bullet x = x' \bullet x'. \tag{22}$$

The beautiful simplicity of eq. (22) doesn't change the fact that the LT on which it is based is invalid.

The energy E and momentum **p** of the particle also combine to form a four-vector which satisfies the following relationship in STR [3] between different rest frames:

$$E^2 - p^2c^2 = E'^2 - p'^2c^2 = \mu^2c^4, \tag{23}$$

where μ is the rest mass of the particle. This equation can be derived from the experimental results obtained by Bucherer [37] for the variation of the inertial mass m of accelerated electrons with speed v relative to the laboratory:

$$m = \gamma(v)\,\mu\,. \tag{24}$$

When combined with Einstein's mass-energy relation [3], this equation can be converted to

$$E = mc^2 = \gamma\mu c^2 = \gamma E_0\,, \tag{25}$$

where E_0 is referred to as the rest energy of the particle. Squaring eq. (25) leads back to eq. (23) since

$$E^2\gamma^{-2} = E^2\left(1 - v^2c^{-2}\right) = E^2 - E^2v^2c^{-2} = E^2 - \left(E^2c^{-4}\right)v^2c^2 = E^2 - m^2v^2c^2 = \tag{26}$$

$$E^2 - p^2c^2 = E_0{}^2 = \mu^2c^4 = E'^2 - p'^2c^2\,.$$

Note that E' and **p**' in the last term correspond to a different rest frame (ORS) than the original and therefore to a different value of the particle's speed (v') relative to the laboratory.

Eq. (25) for energies is completely analogous to eq. (5) for elapsed times in which γ (v) plays the role of a conversion factor between energy values obtained in the rest frame of the ORS and those in a rest frame moving with speed v relative to the ORS. It leads to a counterpart of the UTDL of eq. (6) for two speeds v and v' relative to the ORS:

$$\frac{E}{\gamma(v)} = \frac{E'}{\gamma(v')}\,. \tag{27}$$

A completely analogous result is obtained for inertial masses because of the mass-energy relation:

$$\frac{m}{\gamma(v)} = \frac{m'}{\gamma(v')}\,. \tag{28}$$

Both energies and masses in different inertial rest frames are therefore governed by the same *conversion factor Q* as that derived for elapsed times in eq. (8):

$$\frac{E'}{E} = \frac{m'}{m} = Q\,. \tag{29}$$

As a final topic in this section, consider the four-vector relationship for frequencies ν and wavelengths γ. For this purpose, it is convenient to use the definitions of circular frequency ω=2πν and wave vector **k**=2π/**λ.** There is again an invariance condition for the associated scalar product, in this case:

$$\omega^2 - k^2c^2 = 0\,. \tag{30}$$

This relationship only holds for light in free space, however, in which case ω/k=λν=c. It has special significance [38] because of the quantum mechanical relationships for photons: E=hν and p=h/λ. There is thus a close connection between the E,**p** and ω,**k** four-vectors for this case.

The point to emphasize from the above discussion is that the Minkowski four-vector relations for space-time and energy-momentum are replaced in the GPS-LT version of relativity theory by the simple proportionality relations of eqs. (6, 28, 29) for elapsed times, energy and mass in two inertial rest frames. An analogous relation can be found for all other physical properties, as discussed in Ref. [39], including those occurring in electromagnetic interactions [40].

VI. Invariance Conditions and the Lorentz Force Law

The standard relativistic treatment of electromagnetic interactions is based on the premise that the components of the electric **E** and magnetic **B** field vectors transform according to the following equations [41] (c is the speed of light in free space, 299792458 ms^{-1}):

$$\begin{array}{ll} E_x{}' = E & B_x{}' = B_x \\ E_y{}' = \gamma\left(E_y - vc^{-1}B_z\right) & B_y{}' = \gamma\left(B_y + vc^{-1}E_z\right). \\ E_z{}' = \gamma\left(E_z + vc^{-1}B_y\right) & B_z{}' = \gamma\left(B_z - vc^{-1}E_y\right) \end{array} \tag{31}$$

Einstein derived this set of relations [3] by assuming that Maxwell's equations must be invariant to a Lorentz transformation (LT) of spatial and time coordinates between different rest frames. However, it has been shown in Sect. II that the LT is self-contradictory, thereby at least raising doubts about the validity of the above transformation. It was further assumed that the components of the electromagnetic force **F** on charged particles e are given in terms of the above field components by the Lorentz Force equation:

$$\mathbf{F} = \mathrm{e}\left(\mathbf{E} + \mathbf{v}\mathrm{c}^{-1} \times \mathbf{B}\right) \tag{32}$$

In this equation **v** is the velocity of charged particles relative to the observer, a point which will prove worthy of further discussion subsequently.

There is ample evidence [42] that the Lorentz Force satisfies the equation of motion expected from Newton's Second Law, namely:

$$\mathrm{e}\left(\mathbf{E} + \mathbf{v}\mathrm{c}^{-1} \times \mathbf{B}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\gamma\mu\mathbf{v}\,, \tag{33}$$

i.e., the force **F** equals the time rate of the relativistic momentum $\mathbf{p} = \gamma\mu\mathbf{v}$, with $\gamma=(1\text{-}v^2c^{-2})^{-0.5}$ and μ, the rest mass of the particle/electron. Nonetheless, as will be seen from the following concrete example which makes use of this equation, there is still an uncertainty in the definition of **v** therein when the observer is located in a different rest frame than that of the field origin/laboratory.

Consider the effects of an electromagnetic field with only the two components, E_x and $B_{y,}$ acting on an electron. From the point of view of an observer located at the origin of the field, the electron will initially move along the x axis. This is because the force **F** in eq. (32) only depends on E_x at the instant the field is applied since the value of v=0 negates any effect from the corresponding magnetic field component B_y. This situation changes as time goes by as the electron is accelerated to non-zero speeds. The $\mathbf{v} \times \mathbf{B}$ term in eq. (32) gradually produces a force component in the z direction, causing the electron to veer away from its initial path. Depending on the relative strengths of the constant values of E_x and B_y, the amount of deflection can be quite significant over time. This situation is easily reproduced in the laboratory and there is no doubt that it is consistent with the Lorentz Force Law.

Next, consider the same example from the perspective of an observer co-moving with the electron. Since the speed v of the electron relative to the observer is zero at all times, it follows according to the transformation law of eq. (31) as well as eq. (32) that the *magnetic field has no effect.* As a result one expects that, from the perspective of this observer, the electron continues *indefinitely along a straight line* parallel to the x axis. This predicted trajectory is therefore clearly distinguishable from that discussed first from the vantage point of the laboratory observer. This behavior raises the question of whether it is reasonable to expect that the two observers would disagree about the electron's path through space. No one has ever ridden along with an accelerated electron or other charged particle to verify that the predicted straight-line trajectory would actually be found by such an observer. Since the curved path expected from the laboratory perspective is routinely observed, however, it would therefore seem, on the contrary, *that the straight-line result is pure fiction*, an artifact of a physically unrealistic theory.

Does this example prove that Galileo's RP does not apply to electromagnetic interactions? Clearly not. The reason is because there is another quite straightforward way to satisfy both Maxwell's equations and the RP at the same time, namely to insist that all observers, regardless of their state of motion, *see exactly the same results of any given interaction*. In particular, the hypothetical observer co-moving with the accelerated electron must record the same curved trajectory as is viewed from the laboratory perspective.

The measured values for the parameters of the electron's path may still differ for the two observers, however. This is *because the units in which they express their respective measured values may not be the same.* We know, for example, from the time-dilation experiments [12,15] mentioned in Sect. III that the clocks they employ to measure elapsed times can run at different rates. This fact does not change the above conclusion about the trajectory of the electron in the above example, however. There is no reason to doubt that all observers should agree that a curved path is followed as a consequence of the interaction of crossed electric and magnetic fields.

The relativity principle (RP) was originally intended by Galileo to apply exclusively to inertial systems, i.e. under the influence of no unbalanced forces. Einstein and his contemporaries sought to extend the RP in the above example to apply to electrons undergoing acceleration due to an electromagnetic field. His first postulate of relativity, which states in broad terms that the laws of physics are the same in all inertial systems, falls short of this objective and has been a source of confusion for physicists ever since its inception. Examination of his arguments with regard to electromagnetic interactions shows that what he actually did was to assume that Maxwell's equations must hold in the *non-inertial* rest frame of an accelerated electron. He was led under this assumption to conclude that the electric and magnetic fields must undergo continuous mixing as the electron increases its speed.

The resulting transformation, as well as the Lorentz Force Law itself, contain a parameter **v.** It has generally been assumed that this implies that the forces acting on the electron vary with the perspective of the observer because ***v*** *is assumed to be the velocity of the electron relative to the observer.* However, in the preceding discussion, it has been shown that this interpretation leads to a physically untenable result with regard to the trajectory of the electron. It implies that an observer co-moving with the electron will find that it moves continuously along a straight line since v=0 from his perspective, whereas his counterpart who remains stationary in the rest frame of the laboratory where the electromagnetic force originates finds instead that the electron follows a curved path.

There is a straightforward way to eliminate this dilemma, namely to remove the observer from *active* participation in the interaction. The measurement process becomes completely objective as a consequence. This is accomplished in the present case by assuming that the velocity of the accelerated particle is *uniquely defined relative to the origin of the electromagnetic field.* All observers, regardless of their own state of motion, agree on the value of this velocity *on an absolute basis,* and also on all other results of the interaction. The only source of disagreement must be due to the fact that different systems of physical units are employed by the various observers. This requirement suggests an amended version of the RP [43]: The laws of physics are the same in all inertial systems *but the physical units in which their results are expressed can and do vary from one system to another*.

The same conclusion applies to the definition of momentum $\mathbf{p}=\gamma\mu\mathbf{v}$ for an accelerated particle; the velocity $\mathbf{v}$ of the particle should always be determined relative to the rest frame in which the force causing acceleration has been applied. Again, there is no disagreement among different observers as to the absolute value of the momentum, both in its magnitude and direction, even though differences in the numerical value occur because of their respective choices of the fundamental units of inertial mass, distance and time.

The above discussion also raises questions about the transformation properties of physical laws in general. The essential point is that the laws of physics must be accurate for the stationary observer at the origin of an applied force. His rest frame plays a unique role in the physical description of forces. Observers in other rest frames must simply agree on the results of the interaction (after appropriate changes in units are made) by virtue of another basic physical principle: the objectivity of the measurement process. The latter is distinct from Galileo's RP but perfectly consistent with it. As noted above, Einstein's derivation of the electromagnetic field transformation assumes that Maxwell's equations and the Lorentz Force Law must be invariant to a Lorentz space-time transformation. However, the consequences of this assumption with regard to the prediction of electron trajectories from different vantage points suggests that this interpretation of the RP is overly restrictive.

VII. Conclusion

The clock puzzle in eqs. (3a-c) is a simple exercise that can be solved by anyone with an elementary knowledge of algebra. The essence of the puzzle is to demonstrate that if two quantities are always strictly proportional to one another, it is impossible for one of them to vanish without the other doing so as well. The puzzle takes on far more importance, however, when one takes this lesson from the abstract to concrete applications in which the numbers represent *time differences* for events measured by two observers in different rest frames. *It is impossible for two clocks to disagree whether events occur at the same time or not and still have them running at rates which are strictly proportional to one another.* The GPS pre-correction technique used to adjust the rates of atomic clocks carried onboard satellites verifies this conclusion in a thoroughly convincing manner. Making a proportional change in the rate of the clock to cause it to agree with the time registered on an identical clock on the earth's surface ensures that any conceivable event occurs at the same time for both clocks.

The fact that this GPS procedure is universally effective in everyday applications constitutes indisputably strong evidence that clock rate proportionality and remote non-simultaneity are incompatible with one another. Since both effects are predicted by the Lorentz transformation (LT), it therefore becomes clear that the latter cannot be a viable component of relativity theory. It is not sufficient to satisfy both of Einstein's postulates in order to guarantee that the resulting theory will always be consistent with experimental reality. There is a third condition that has to be satisfied as well, one that is evident from consideration of Newton's First Law of Kinematics: the rates of inertial clocks must be strictly proportional to one another for an indefinite period of time as long as no external forces are applied. Consideration of available experimental evidence indicates that there is a Universal Time-dilation Law (UTDL) which is in complete agreement with the above conclusion. This condition is violated by the LT since it demands that space and time be inextricably mixed, i.e. the ratio of the rates of two such clocks can vary with their respective positions in space. *No experiment was ever needed to prove that the LT is not a viable component of relativity theory*. It is already evident from its lack of internal consistency.

There is another space-time transformation, the GPS-LT, which satisfies all three conditions, however. It assumes that the ratio of elapsed times for a given event has a constant value for any pair of inertial systems. This is the antithesis of "space-time mixing." It is in complete agreement with the classical Newtonian view that space and time are totally separate entities, one measured with a clock, the other with a meter stick. Unlike the LT, it is in full agreement with the empirical evidence embodied in the UTDL. The GPS-LT is also consistent with Poincaré's relativistic velocity transformation (RVT), which is responsible for many of the reported confirmations of the original LT. This includes the aberration of starlight at the zenith and the Fresnel light-drag effect, in addition to the myriad of experimental verifications of the light-speed constancy postulate itself. The GPS-LT and the UTDL also suggest new experiments that can be carried out to further cement the belief in proportional time dilation, for example, by carrying out Hafele-Keating experiments with atomic clocks orbiting the moon.

Finally, the belief that the laws of physics should be invariant to the LT has been exposed as myth. In this case, all one has to do is compare the different trajectories of a charged particle in an electromagnetic field as viewed from different rest frames. Since the speed of the particle in its own rest frame is zero, it follows that only a straight-line trajectory can be observed from this vantage point, in marked contrast to the curved trajectory always observed in the rest frame of the laboratory in which the fields originate. The contradiction is removed by simply assuming that the speed v of the particle in the Lorentz Force Law is always measured relative to the field origin rather than to the rest frame of the observer. *This association removes the observer from the measurement process.* The results of any experiment are objectively the same for all observers independent of their respective states of motion or locations in a gravitational field. However, their numerical values can disagree because they employ different standard units to express their results.

A key objective of relativity theory thus becomes the determination of the *conversion factors* between the various units employed by observers in different rest frames. This approach is perfectly consistent with Galileo's RP. It simply recognizes that, although the physical laws in each inertial rest frame are exactly the same, the units in which they are naturally expressed can and do differ by virtue of the different states of motion of the observers.

(August 27, 2017)

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