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SOME APPLICATIONS OF K-THEORY TO FRAMED BORDISM:

E-INVARIANT AND TRANSFER

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Introduction

The projective spaces play a special rôle in topology. The complex projective space, for example, is closely related to spaces like BU, U, MU or $B\mathbb{Z}/m$. Therefore it is not surprising that the stable homotopy groups of $P_\infty\mathbb{C}$ are important in the study of many interesting geometric and homotopy theoretic problems.

In this paper we study $\pi_*^s(P_\infty\mathbb{C})$ using as the main tools the e-invariant of Adams and various transfer maps. The e-invariant describes the information about $\pi_*^s(X)$ obtainable using K-theory and the transfer maps relate $\pi_*^s(P_\infty\mathbb{C})$ to the stable homotopy of spheres and allow us to construct elements in $\pi_*^s(P_\infty\mathbb{C})$.

Although the e-invariant was used by Adams with notable success in the study of im(J) and $\pi_*^s(S^0)$, not much is known about the e-invariant on other spaces than spheres. The main reason for this is the following. The computation of im(e) in stable homotopy or stable cohomotopy in positive dimensions differs from the case of stable cohomotopy in negative dimensions in a substantial point. In the latter case, one has the J-homomorphism $J: KO^{-i}(X) \longrightarrow \pi_s^{-i+1}(X)$ which gives all of im(e) and the computation of im(e) reduces to a purely K-theoretic problem. In the case of stable homotopy or positive dimensional cohomotopy one still has the restrictions posed on im(e) by the Adams operations but, in contrast to the case of negative dimensional stable cohomotopy, these are not strong enough to describe im(e) completely. In stable homotopy and in positive dimensional cohomotopy there is a sort of mixing between im(J)- and coker(J)-phenomena which is not seen by K-theory.

A better approximation for im(e) is obtained by using BP-homology and its operations. The resulting upper bound for im(e) is then the group $\mathrm{Ext}^{1,2n}_{BP_*BP}(BP_*, BP_*(P_\infty \mathbb{C}))$ of the BP-Adams spectral sequence. But even this is not sufficient to compute im(e), because there are nonzero differentials starting from these groups.

In the case of real projective space a complete determination of im(e) would imply the solution of the Kervaire-invariant problem. So one cannot expect an easy calculation of im(e) on the stable homotopy of a given space, but even from incomplete information on im(e) one can draw interesting consequences.

The S^1-transfer $t: \pi^s_n(P_\infty \mathbb{C}) \longrightarrow \pi^s_{n+1}(S^o)$ relates the stable homotopy of the complex projective space to the stable homotopy of spheres and in particular relates the elements of $\pi^s_*(P_\infty \mathbb{C})$ which can be detected by the e-invariant to elements in $\pi^s_*(S^o)$ of Adams filtration 2 which are at present known only partially.

One of the main advantages of this transfer is that it raises the degree of the Adams filtration, which is a measure of the complicatedness of elements in stable homotopy. This means that t maps less complicated elements to more complicated elements and because t has a good geometric interpretation one retains a geometric description for t(x) if x had one.

There is also a transfer $t: \pi^s_*(P_\infty \mathbb{C} \times P_\infty \mathbb{C}) \to \pi^s_*(P_\infty \mathbb{C})$ which allows one to construct and to describe elements in $\pi^s_*(P_\infty \mathbb{C})$ in geometric terms . Using these transfers several times one can, in a certain sense, resolve a part of $\pi^s_*(S^o)$, that is lift to filtration 0 and then treat the problem by the methods of rational homology. Of course, the depth of the filtration must be paid for with the extent of the computations; one has to calculate in large polynomial rings.

Another important property of the transfer is its relation to the J-homomorphism. The usual J-homomorphism can be stabilized to give a map $J': \pi^s(SO) \longrightarrow \pi^s(S^o)$ of which the complex analogue can be completely described by the transfer.

We begin with the computation of the restrictions imposed on the image of the e-invariant on $\pi_*^s(P_\infty\mathbb{C})$ by the Adams operation ψ^k. It turns out that this is closely related to the algebraic K-theory of finite fields, constructed by Quillen. Let $A_*(-)$ denote the associated homology theory, which has as coefficients the K-groups of a finite field $\mathbb{F}_k$, then the upper bounds given by the ψ^k are essentially the groups $A_*(P_\infty\mathbb{C})$. In chapter 1 we compute the order and the number of cyclic summands of these groups and show how to describe elements in $A_*(P_\infty\mathbb{C})$ using representation theory. As a first application we determine the number of summands in the group $J(P_n\mathbb{C})$.

In chapter 2 we introduce transfer maps $t^k: \pi_*^s(P_\infty\mathbb{C}^{kH}) \rightarrow \pi_*^s(S^o)$, where $P_\infty\mathbb{C}^{kH}$ is the Thom space of k times the Hopf bundle, which generalize the transfer $t: \pi_*^s(P_\infty\mathbb{C}) \longrightarrow \pi_*^s(S^o)$ and deduce some of their properties. These transfer maps can be represented by stable maps $\tau^k \in \pi_s^{2k-1}(P_\infty\mathbb{C}^{kH})$ and we show that the representing map τ for the transfer t^o lies in the image of the J-homomorphism. We identify the cofibre spectrum of τ^k with a Thom spectrum and give formulas relating the images of the various t^k.

Next we discuss the relation of the transfer to the bistable

J-homomorphism $J': \pi_*^s(SO) \longrightarrow \pi_*^s(S^o)$. As with the transfer the map J' has an important geometric description. If one identifies stable homotopy with reduced framed bordism then J' is described as follows: An element in $\pi_*^s(SO)$ is given by a triple (M,ϕ,f) where (M,ϕ) is a bounding framed manifold and $f: M \rightarrow SO$ a map. Using this map one can twist the given framing ϕ to obtain a new framing ϕ^f. Then $J'((M,\phi,f))$ is simply (M,ϕ^f). If J' were onto then in each positive dimension n one could find a framed manifold (M^k,ϕ) which bounded and gave all other elements in $\pi_n^s(S^o)$ by twisting the given framing.

That J' should be onto was conjectured by G.W. Whitehead and is known to be true at the prime 2. As one of the main results of this paper, we use the relation between t and J' to show that at odd primes larger than 3 J' is not surjective. The method of proving this is to use the cofibre spectrum of the transfer to transform the computation of t from filtration 1 on $\pi_*^s(P_\infty\mathbb{C})$ to filtration 2 on $\pi_*^s(S^o)$ into a pure filtration 1 problem which can then be treated by means of the e-invariant.

The possibility of computing the transfer t from filtration 1 to filtration 2 is then used in chapter 4 to find representing framed manifolds for some of the elements in $\pi_*^s(S^o)$. Let p be an odd prime. In some range of dimensions all elements in the p-component of $\pi_*^s(S^o)$ can be constructed out of three manifolds: the Hopf bundle $\sigma \in \pi_2^s(P_\infty\mathbb{C})$ and 2 elements x_o, x_1 in $\pi_*^s(P_\infty\mathbb{C} \times P_\infty\mathbb{C})$, which can be described by conditions formulated in rational cohomology. For example, for x_o this reads as follows: x_o is given by a framed manifold of dimension $2(p+1)$ and two complex line bundles ξ_1, ξ_2 on M, such that the Kronecker product $\langle c_1(\xi_1)\cdot c_1(\xi_2)^p, [M]\rangle$

is nonzero mod p. Using the Pontrjagin product structure on $\pi_*^s(P_\infty\mathbb{C}\times P_\infty\mathbb{C})$ - which is essentially given by tensor products of complex line bundles - and the transfer map, the elements σ, x_o, x_1 give all of $\pi_n^s(S^o)_{(p)}$ $(n\leq 2(p-1)\cdot(p^2+1)-4)$ and almost all of $\pi_n^s(P_\infty\mathbb{C})_{(p)}$.
It is even possible to use this method to construct some new elements in $\pi_*^s(S^o)$; we have done this only in one case, namely we have constructed the element β_9 for $p=3$.

In chapter 6 we discuss methods for finding elements in the cokernel of $e: \pi_*^s(P_\infty\mathbb{C})\to A_*(P_\infty\mathbb{C})$. Elements in coker(e) are closely related to values of the transfer maps mentioned above. If the e-invariant mapped surjective by onto $A_*(P_\infty\mathbb{C})$ then all these transfers would have to map $F^1\pi_*^s(P_\infty\mathbb{C}^{kH})$ to higher filtration in π_*^s. Every element of $F^2\pi_*^s(S^o)$ which is in the image of some t^s gives rise to an infinite series of elements in coker(e). By the methods of § 3 we can determine $im(t^s)$ in a certain range of dimensions and use this to compute im(e) and thereby $\pi_*^s(P_\infty\mathbb{C})_{(p)}$ in the same range.

The computation of the e-invariant on $\pi_*^s(P_\infty\mathbb{C})$ is connected with the problem of determining the image of the Hurewicz map $h_m: \Omega_{2n-1}^{fr}(B\mathbb{Z}/m) \longrightarrow \Omega_{2n-1}^{U}(B\mathbb{Z}/m)$ from framed bordism to complex bordism or K-theory for the classifying space of a finite cyclic group. The image of h_m essentially describes the values which the α-invariant, an invariant derived from the equivariant signature, can take on free equivariantly framed $\mathbb{Z}/m$-manifolds. For m a prime, this is known. We compute as an application the image of h_m for $m=p^2$ and $n\not\equiv 0(p)$.

In the last chapter we deal with the problem of computing some James numbers. The James number $U(n,k)$ describes how far away the projection $\pi: W_{n,k} \longrightarrow S^{2n-1}$ of the complex Stiefel manifold to the sphere is from having a section. There is a conjecture that these numbers are determined by K-theory. Using the properties of the transfer maps and the e-invariant we verify this conjecture in some cases, precisely we compute $U(n+M_k,k)$ where M_k is the order of the Hopf bundle in $J(P_{k-1}\mathbb{C})$ and $n=1,2,k-1,k-2$ and $k+2$ for all k.

The numbers $U(n,k)$ are closely related to the values of the transfer maps t^k on $F^o\pi_*^s(P_\infty\mathbb{C}^{k\tilde{H}})$. As an application it is shown that the elements μ_r in the 2-component of $\pi_*^s(S^o)$ are in the image of the transfers t^2 and $t^{(-2)}$.
Another application of the computation of $U(n,k)$ is the construction of infinite families of elements in the matastable homotopy groups $\pi_{2m+2t}(U(2m))$ of the unitary group.

Content

§ 1 The algebraic K-groups of $P_\infty\mathbb{C}$

In this chapter we define the e-invariant on $\pi_*^S(P_\infty\mathbb{C})$ and show how it is related to the Hurewicz map $h: \pi_*^S(P_\infty\mathbb{C}) \longrightarrow A_*(P_\infty\mathbb{C})$, where $A_*(X)$ is the homology theory defined by the K-theory of a finite field. We then compute the order and the number of cyclic summands of $A_*(P_\infty\mathbb{C})$ and show how to describe the elements of $A_*(P_\infty\mathbb{C})$. As an application we determine the number of cyclic summands in the group $J(P_n\mathbb{C})$.

The e-invariant of a stable map $f\colon S^{m+k} \longrightarrow S^k_{\wedge} X$ is defined if $f^* = 0$ in K-theory and classifies the extension

$$0 \longrightarrow \tilde{K}^*(S^{m+k+1}) \longrightarrow \tilde{K}^*(C_f) \longrightarrow \tilde{K}^*(S^k X) \longrightarrow 0$$

induced by the cofibre sequence of f (C_f = cofibre of f).

In the case of a space X with torsion-free homology groups, this definition can be reformulated using the functional Chern character (see [2]). The Thom-Pontrjagin construction identifies stable homotopy $\pi_n^s(X^+)$ with framed bordism $\Omega_n^{fr}(X)$, and so an element in $\pi_n^s(X)$ may be given by a framed manifold rather than by a stable map. For this situation a third definition is appropriate. The Hurewicz map $h\colon \Omega_n^{fr}(X) \rightarrow K_n(X)$ can be interpreted directly in terms of framed manifolds. The following diagram

$$\begin{array}{ccccccc}
 & & \Omega^{fr}_{2n}(X;\mathbb{Q}) & \longrightarrow & \Omega^{fr}_{2n}(X;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\beta} & \Omega^{fr}_{2n-1}(X) & \xrightarrow{r} & \Omega^{fr}_{2n-1}(X;\mathbb{Q}) \\
 & & \downarrow & & \downarrow{\scriptstyle h_{\mathbb{Q}/\mathbb{Z}}} & & \downarrow{\scriptstyle h} & & \\
 & \longrightarrow & K_0(X;\mathbb{Q}) & \xrightarrow{j} & K_0(X;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & K_1(X) & &
\end{array}$$

defines the functional Hurewicz map h_r as a map from the subgroup $\ker(r) \cap \ker(h)$ of $\Omega^{fr}_{2n-1}(X)$ into $K_{2n}(X;\mathbb{Q})/(K_{2n}(X;\mathbb{Z})+H_{2n}(X;\mathbb{Q}))$. For X with $H_*(X;\mathbb{Z})$ torsion-free, the last group can be identified via the K-theory Kronecker product with the range of the functional Chern character. A proof that h_r is the same as the functional Chern character and so gives the e-invariant can be found in [21].

Stable operations act trivially on classes in the image of h, so they give upper bounds for the image of e. The Adams operations ψ^k are not compatible with Bott periodicity and so not stable. To get a stable operation from the Adams operation ψ^k, one has to introduce coefficients R, in which k is invertible. The stable operation

$$\psi_n^k\colon K^n(X;R) \longrightarrow K^n(X;R)$$

is then defined for even values of n by $\psi_{2m}^k = k^{-m}\cdot\psi^k$ and for n odd via suspension. Because of the stablility of the operations ψ_n^k, they induce operations in the K-homology groups, also denoted by ψ_n^k.

Immediately from the definition of the Hurewicz map one sees that (ψ_n^k-1) vanishes on im $(h\colon \pi_n^s(X) \to K_n(X))$. So the subgroups $\ker(\psi_n^k-1) \subset K_n(X;\mathbb{Q}/\mathbb{Z})_{(p)}$ for $k \not\equiv 0(p)$ constitute upper bounds for the p-component of im(e) in dimension n. Our principal tool for calculating these upper bounds is the algebraic K-theory introduced by Quillen: Seymour [36] has constructed a multiplicative cohomology theory $Ad_a^n(X)_R$ (a a prime , $1/a \in R$, $R \subset \mathbb{Q}$) which fits into the exact sequence

$$\longrightarrow Ad_a^n(X)_R \xrightarrow{k} K^n(X;R) \xrightarrow{\psi_n^a-1} K^n(X;R) \xrightarrow{j} Ad_a^{n+1}(X)_R \longrightarrow \tag{1.1}$$

$Ad_a^o(X)$ is based on vector bundles invariant under the Adams operation ψ^a.

In the following we choose a fixed odd prime p and a prime number k generating the group of units $(\mathbb{Z}/p^2\mathbb{Z})^*$. We set $R = \mathbb{Z}_{(p)}$, the integers localised at p, and denote $Ad_a^n(X)_R$ simply by $Ad^n(X)$. The coefficient groups $Ad^n(*)$ are computed directly from (1.1) using

$$\nu_p(k^t-1) = \begin{cases} 0 \text{ if } t \not\equiv 0(p-1) \\ 1+\nu_p(t) \text{ if } t \equiv 0(p-1) \end{cases} \tag{1.2}$$

where $\nu_p(n)$ is the power of p in n (see [1]).

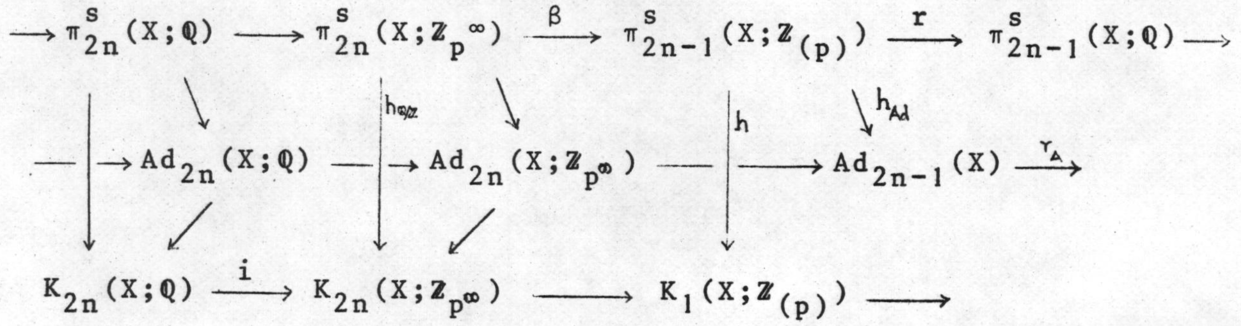
To the cohomology theory Ad^* there correspond a homology theory Ad_*

and connected versions, denoted by A^* and A_*. The groups A^* are isomorphic to the higher algebraic K-groups for the finite field $\mathbb{F}_k$ constructed by Quillen [32]. For p = 2 it is possible to choose a better A_* see [36].

The Hurewicz map $h: \pi_n^s(X^+) \to K_n(X)$ factorizes through $A_n(X)$, but also the e-invariant is determined by $h_A: \pi_*^s(X) \to A_*(X)$.

<u>Lemma 1.3</u>: Let $H(X;\mathbb{Z}_{(p)})$ be torsion-free, then the p-component of the e-invariant on the torsion subgroup of $\pi_{2n-1}^s(X)$ is determined by the Hurewicz map $h_A: \pi_{2n-1}^s(X) \to A_{2n-1}(X)$.

<u>Proof</u>: We have a commuting diagram of Bockstein sequences



which shows, that $h_r = i^{-1} \circ h_{\mathbb{Q}/\mathbb{Z}} \circ \beta^{-1}$ factorizes through $Ad_{2n}(X;\mathbb{Z}_{p^\infty})/\mathrm{im}\, Ad_{2n}(X;\mathbb{Q}) \cong \ker(r_A)$ and is thus determined by h_{Ad}; clearly h_{Ad} factories through h_A.

Because of (1.3) we will also use the name e-invariant for the map $h_A: \pi_{2n-1}^s(P_\infty\mathbb{C}^+) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$.

A first upper bound for im(e) on $\pi_{2n-1}^s(P_\infty\mathbb{C})_{(p)}$ is thus given by the group $A_{2n-1}(P_\infty\mathbb{C})$. Let $\alpha_p(n)$ denote the sum of the coefficients in the p-adic representation of the number n.

<u>Proposition 1.4</u>: Let $n = t(p-1)+s$ with $0<s\leq p-1$, then the order of $\tilde{A}_{2n-1}(P_\infty\mathbb{C})$ is p^a, where

$$a = \sum_{i=1}^{t} \nu_p(i) + (\alpha_p(n)-s)/(p-1)$$

<u>Proof</u>: We first calculate $Ad_{2n-1}(P_{n-1}\mathbb{C})$. As a basis of $K_o(P_{n-1}\mathbb{C})$ we choose the elements b_i $(i\geq 0)$ dual to $(L-1)^i\in K^o(P_{n-1}\mathbb{C})$, where L is the universal line bundle. Via the Kronecker product the operation ψ^k_n induces a dual operation $c(\psi^k_n)$ in K-homology:

$$< y, c\psi^k_n(x) >: = < \psi^k_n(y), x >$$

for all $y\in K^o(P_\infty\mathbb{C})$. It is easy to see that $c\psi^k_n = (\psi^k_n)^{-1}$. To calculate kernel or cokernel of $\psi^k_{2n}-1$ we can therefore equally well use $c(\psi^k_{2n})-1$. Using $\psi^k_o(x^i) = (L^k-1)^i = ((x+1)^k-1)^i$, one sees that $c\psi^k_{2n}-1$ is given by an upper triangular matrix B with respect to the basis $b_1,\ldots,b_{n-1}$. The diagonal entries are $k^{i-n}-1$. The determinant of B gives the order of $\operatorname{coker}(B) = Ad_{2n-1}(P_{n-1}\mathbb{C})$. By (1.2) we have

$$\nu_p(\det B) = \sum_{i=1}^{\left[\frac{n-1}{p-1}\right]} (1+\nu_p(i)) .$$

The group $A_{2n-1}(P_\infty\mathbb{C})$ is defined as the image of $Ad_{2n-1}(P_{n-1}\mathbb{C})$ in $Ad_{2n-1}(P_n\mathbb{C})$ and is therefore isomorphic to

$$Ad_{2n-1}(P_{n-1}\mathbb{C})/\operatorname{im}(\partial : A_{2n}(P_n\mathbb{C};P_{n-1}\mathbb{C}) \to Ad_{2n-1}(P_{n-1}\mathbb{C}))$$

Because of $Ad_{2n}(P_n\mathbb{C};P_{n-1}\mathbb{C}) = Ad_{2n}(S^{2n}) \cong \mathbb{Z}_{(p)}$ we only need to calculate the image of $Ad_{2n}(P_n\mathbb{C}) \to Ad_{2n}(P_n\mathbb{C};P_{n-1}\mathbb{C})$. For this we need $Ad_{2n}(P_n\mathbb{C})$. The multiplication map of $P_\infty\mathbb{C}$ makes the groups $Ad_*(P_\infty\mathbb{C})$, $A_*(P_\infty\mathbb{C})$ and $K_*(P_\infty\mathbb{C})$ into Pontrjagin rings.

<u>Lemma 1.5</u>: The n^{th}-power of b_1 in the Pontrjagin product generates

$$Ad_{2n}(P_\infty\mathbb{C}) = A_{2n}(P_\infty\mathbb{C}) \cong \mathbb{Z}_{(p)}$$

Proof: $c\psi_o^k(b_1^n) = (c\psi_o^k b_1)^n = k^n b_1$, so $b_1^n \in \ker(c\psi_{2n}^k - 1) = Ad_{2n}(P_\infty\mathbb{C})$. The Chern character maps the eigenspace of $c\psi_o^k$ for the eigenvalue k^n in $K_*(P_\infty\mathbb{C};\mathbb{Q})$ bijectively onto $H_{2n}(P_\infty\mathbb{C};\mathbb{Q})$. So every element in $\ker(c\psi_n^k-1)$ is a rational multiple of b_1^n. An elementary calculation gives b_1^n as a linear combination of the elements b_i:

$$b_1^n = \sum_{k=1}^{n} k!S(n,k)b_k \tag{1.6}$$

where

$$S(n,k) = (1/k!)\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}j^n$$

is a Stirling number of the second kind. Since $S(n,1) = 1$, there is no element z in $K_o(P_\infty\mathbb{C})$ with $b\cdot z = b_1^n$ and $b\in\mathbb{Z}$ and $b\neq\pm1$ and b_1^n must generate $Ad_{2n}(P_\infty\mathbb{C})$.

We now can finish the proof of (1.4):
Since $S(n,n) = 1$, we have $b_1^n = n!\cdot b_n + \ldots$ and because $Ad_{2n}(P_n\mathbb{C},P_{n-1}\mathbb{C})$ is generated by the image of b_n, the map $Ad_{2n}(P_n\mathbb{C})\to Ad_{2n}(P_n\mathbb{C},P_{n-1}\mathbb{C})$ is multiplication by n! . Im ∂ is therefore isomorphic to $\mathbb{Z}/n!\mathbb{Z}$ and the number of elements in $\tilde{A}_{2n-1}(P_\infty\mathbb{C})$ is p^a where

$$a = \sum_{i=1}^{t} 1+\nu_p(i) - \nu_p(n!)$$

Applying the well known formula for $\nu_p(n!)$ gives then (1.4).

Remarks:

1. For low values of n, one can use the theory of elementary divisors to calculate the structure of coker $c(\psi_{2n}^k-1)$, which is $Ad_{2n-1}(P_n\mathbb{C})$. The exact sequence of $(P_n\mathbb{C},P_{n-1}\mathbb{C})$ then shows, that $A_{2n-1}(P_\infty\mathbb{C})$ is the tosion subgroup of $Ad_{2n-1}(P_n\mathbb{C})$.
2. To calculate in $A_{2n-1}(P_\infty\mathbb{C})$ one thinks of $A_{2n-1}(P_\infty\mathbb{C})$ as

$Ad_{2n}(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})/\mathrm{im}\ Ad_{2n}(P_n\mathbb{C};\mathbb{Q})$ (The group $Ad_{2n}(P_nC;\mathbb{Q}/\mathbb{Z})$ is the kernel of $c\psi_o^k-k^n$ on $K_o(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})$ and $\mathrm{im}\ Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}) \cong b_1^n\cdot\mathbb{Q}/\mathbb{Z}$) that is, one calculates in $\ker(c\psi_o^k-k^n) \subset K_o(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})$ modulo $b_1^n\cdot\mathbb{Q}/\mathbb{Z}$.

To describe elements in $A_{2n-1}(P_\infty\mathbb{C})$ explicitly and to avoid the theory of elementary divisors, we study the natural map $\pi_*: A_{2n-1}(B\mathbb{Z}/p^r) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$. For fixed n and sufficiently high value of r π_* becomes surjective. This is also true for π_*^S as shown in [21]. The same proof applies for A_*. There is a description of $K_1(B\mathbb{Z}/p^r)$ which leads to an easy computation of $\ker(c\psi_{2n-1}^k-1) = Ad_{2n-1}(B\mathbb{Z}/p^r)$. This is done in [22], we summarize here results from [22], which we shall need.

Let $R(\mathbb{Z}/p^r) = \mathbb{Z}[\rho]/(\rho^{p^r}-1)$ denote the representation ring of $\mathbb{Z}/p^r$, where ρ is a primitive one-dimensional complex representation. In [45] and [22] it is shown that $K_1(B\mathbb{Z}/p^r)$ is isomorphic to $\overline{R(\mathbb{Z}/p^r)}\otimes\mathbb{Z}/p^\infty$, where $\overline{R(\mathbb{Z}/p^r)}$ is the quotient of $R(\mathbb{Z}/p^r)$ by the ideal given by the regular representation of $\mathbb{Z}/p^r$

We define the element $x(i,k)$ to be $p^{-k}\cdot\sum s^n\cdot\rho^{s\cdot p^i}$ where the sum is taken over all s , $1 \leq s \leq p^{r-i}$ $(i<r)$ with $s \not\equiv 0(p)$.

<u>Lemma 1.6 ([22]):</u> $\widetilde{Ad}_{2n-1}(B\mathbb{Z}/p^r)$ is isomorphic to

$$\bigoplus_{i=o}^{r-1} \mathbb{Z}/p^{r-i+\nu_p(n)}$$ and the i^{th}-summand is generated by $x(i,r-i+\nu_p(n))$

For the rest of this section we identify the groups $K_1(B\mathbb{Z}/p^r)$, $\tilde{K}_o(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z})$ and $\overline{R(\mathbb{Z}/p^r)}\otimes\mathbb{Z}/p^\infty$.

The map $\pi_*: K_o(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}) \longrightarrow K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ can be calculated as follows: we choose ρ, so that $\pi^*(L) = \rho$. If we write

$\pi_*(x(i,k)) = p^{-k} \sum_j a_j^{(i)} \cdot b_j$ we have

$a_j^{(i)}/p^k = < \pi^*(L-1)^j, x(i,k) >$, where $<,>$ is the Kronecker product between $\tilde{K}^o(X)$ and $\tilde{K}_o(X;\mathbb{Q}/\mathbb{Z})$ with values in $\mathbb{Q}/\mathbb{Z}$. For $X = BG$ this is calculated in [22]. One has

$$<(L^i-1), \rho^k/p^s> = \begin{cases} 1/p^s & \text{if } i=k \\ 0 & \text{if } i\neq k \end{cases}$$

By the binomial theorem

$$a_j^{(i)}/p^k = \sum_{\substack{1\leq m\leq p^{r-i} \\ m\not\equiv o(p)}} \binom{j}{mp^i} \cdot (-1)^{j+m} \cdot m^n/p^k \tag{1.7}$$

This formula shows that the element $y_o = \sum_i x(i,k-in)$ is mapped onto $p^{-k} \cdot b_1^n$ in $K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$. To compute $A_{2n-1}(B\mathbb{Z}/p^r)$ it is only necessary to find out which linear combinations of $x(i,j)$ $(i>o,\ j\leq r+\nu_p(n)-i$ are in

$\mathrm{im}(Ad_{2n-1}((B\mathbb{Z}/p^r)^{(2n-1)}) \to Ad_{2n-1}((B\mathbb{Z}/p^r)^{(2n)})) = A_{2n-1}(B\mathbb{Z}/p^r)$.

The next proposition allows one to calculate $A_{2n-1}(B\mathbb{Z}/p^r)$.

<u>Proposition 1.8</u>: Let $\omega = \sum_{i>o} d_i \cdot x(i,j_i)$ be an element of

$Ad_{2n-1}(B\mathbb{Z}/p^r) \cong Ad_{2n}(BZ/p^r;\mathbb{Q}/\mathbb{Z})$ and

$\pi_*(\omega) = \sum_j c_j \cdot b_j$ it's image in $Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}/Z)$.

Then $\omega \in A_{2n-1}(B\mathbb{Z}/p^r)$ iff $c_j = o$ for $j>n$

<u>Proof</u>: If $\omega \in \tilde{Ad}_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z})$ has $\pi_*(\omega) = \sum_{j=o}^{n} c_i \cdot b_i$, then $\pi_*(\omega) \in Ad_{2n}(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})$. We will show that $\pi_*(\omega)$ lies in the image

of $Ad_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to Ad_{2n}(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})$. The sphere bundle of $L^{p^r} \longrightarrow P_n\mathbb{C}$ is the $(2n+1)$-skeleton of $B\mathbb{Z}/p^r$. We look at the Gysin sequence of L^{p^r} in K-homology with $\mathbb{Q}/\mathbb{Z}$-coefficients:

$$0 \longrightarrow K_o(S(L^{p^r});\mathbb{Q}/\mathbb{Z}) \xrightarrow{\overline{\pi}_*} K_o(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z}) \xrightarrow{\frown e(L^{p^r})} K_o(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \qquad (1.9)$$

$\overline{\pi}^*(L^{p^r}) = 1$ implies $\overline{\pi}^*(e(L^{p^r})) = \overline{\pi}^*(1-L^{p^r}) = 0$ and so $\pi_*(k(\omega)) \frown e(L^{p^r}) = \pi_*(k(\omega) \frown \overline{\pi}^*(e(L^{p^r})) = 0$, where $k: Ad_{2n}(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z}) \longrightarrow K_o(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})$ is the canonical map. Thus by exactness of (1.9) there is an element $z \in K_o(S(L^{p^r});\mathbb{Q}/\mathbb{Z})$ with $\overline{\pi}_*(z) = k\pi_*(\omega)$. The exact sequence of the pair $((B\mathbb{Z}/p^r)^{(2n)},(B\mathbb{Z}/p^r)^{(2n+1)})$ shows that $j : K_o((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to K_o((B\mathbb{Z}/p^r)^{(2n+1)};\mathbb{Q}/\mathbb{Z})$ is isomorphic. By the injectivity of j_* and $\overline{\pi}_*$ we have $c(\psi_o^k - k^n)j_*^{-1}(z) = 0$, because this is true for $k(\omega)$.
Hence $j_*^{-1}(z) \in \text{im}(Ad_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to K_o((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}))$.
Because k and π_*: $Ad_{2n}(B\mathbb{Z}/p^r,\mathbb{Q}/\mathbb{Z}) \longrightarrow Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ are injective it follows by naturality, that

$$\omega \in \text{im}(Ad_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \longrightarrow Ad_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}))$$

which means $\omega \in \text{im}(A_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}) \to Ad_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}))$.

Remarks:

1. By (1.7) the conditions $c_r = o$ for $r > n$ can be expressed explicitly as

$$\textstyle\sum_i (a_r^{(i)} \cdot d_i)/p^{j_i} \in \mathbb{Z}_{(p)}$$

for $r > n$.

Example: $p=3$, $x(1,1) = (\rho^3 + 2\cdot 5\cdot\rho^6)/3$ has $\pi_*(x(1,1)) = (b_3 - b_4 + b_5)/3$ in $A_9(P_\infty\mathbb{C}) = \mathbb{Z}/3$ and gives the first torsion element in $A_*(P_\infty\mathbb{C})$

2. Whereas the elements b_n and so $\sum_i x(i,k-in)$ have skeleton filtration 2n, the individual elements $x(i,k)$ are not closely related to filtration. Modulo the summand generated by $y_o = \sum_i x(i,r+\nu_p(n)-in)$ the group $A_{2n-1}(B\mathbb{Z}/P^r)$ becomes stationary as a function of r if $r \geq r_o$, namely $A_{2n-1}(B\mathbb{Z}/p^r)/\langle y_o\rangle \cong A_{2n-1}(P_\infty\mathbb{C})$

The Adams operations ψ^m of $K_*(P_\infty\mathbb{C};\mathbb{Z}/p^\infty)$ with $m \not\equiv o(p)$ induce operations in $Ad_*(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$, because they commute with ψ^k. To see that the use of these ψ^m cannot improve the upper bound for im(e) given by $A_*(P_\infty\mathbb{C})$, we show

<u>Lemma 1.1o</u>: Let $m \not\equiv o(p)$, then $c\psi^m_o$ acts on $A_{2n-1}(P_\infty\mathbb{C})$ as multiplication by m^n.

<u>Proof</u>: By surjectivity of $\pi_*: A_{2n-1}(B\mathbb{Z}/p^r) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$, r large, the above statement follows from the corresponding statement for $A_*(B\mathbb{Z}/p^r)$ and because $A_{2n-1}(B\mathbb{Z}/p^r)$ is a subgroup of $Ad_{2n-1}(B\mathbb{Z}/p^r)$ from that for $Ad_{2n-1}(B\mathbb{Z}/p^r)$. The effect of ψ^m on elements of $Ad_{2n-1}(B\mathbb{Z}/p^r) \subset K_1(B\mathbb{Z}/p^r)$ is calculated in [22]

To determine the number of cyclic summands in $A_{2n-1}(P_\infty\mathbb{C})$ we require some preparation. We set $a(n)$ = number of cyclic summands in $A_{2n-1}(P_\infty\mathbb{C})$. Multiplication by p on the H-space $P_\infty\mathbb{C}$ induces a map $m_p : A_{2n-1}(P_\infty\mathbb{C}) \to A_{2n-1}(P_\infty\mathbb{C})$.

<u>Lemma 1.11</u>: For $x_i = \pi_*(x(i,1)) \in Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ we have

$$m_{p*}(x_i) = x_{i-1} \; (i \geq 1) \quad \text{and} \quad m_{p*}(x_o) = o$$

<u>Proof</u>: The natural projection pr: $\mathbb{Z}/p^k \longrightarrow \mathbb{Z}/p^{k-1}$ induces a map $Bpr : B\mathbb{Z}/p^k \longrightarrow B\mathbb{Z}/p^{k-1}$ which fits into a commuting diagram

$$\begin{array}{ccc} B\mathbb{Z}/p^k & \xrightarrow{\ \pi\ } & P_\infty\mathbb{C} \\ \downarrow{\scriptstyle Bpr} & & \downarrow{\scriptstyle m_p} \\ B\mathbb{Z}/p^{k-1} & \xrightarrow{\ \pi\ } & P_\infty\mathbb{C} \end{array}$$

The map $Bpr_*: K_1(B\mathbb{Z}/p^k) \rightarrow K_1(B\mathbb{Z}/p^{k-1})$ is calculated in [22]. We have $Bpr_*(\rho^i) = o$ if $i \not\equiv o(p)$ and $Bpr_*(\rho^{ip}) = \rho^i$, which immediately gives $Bpr_*(x(i,1)) = x(i-1,1)$. Using the diagram above it follows $m_{p*}(x_i) = x_{i-1}$.

The idea in computing the number of summands is to find the maximal value of i for which βx_i can be in $A_{2n-1}(P_\infty\mathbb{C})$.

<u>Lemma 1.12</u>: The number of cyclic summands in $A_{2n-1}(P_\infty\mathbb{C})$ is

$$\max\{i \mid \beta(x_i) \text{ lies in } im(A_{2n-1}(P_\infty\mathbb{C}) \rightarrow Ad_{2n-1}(P_\infty\mathbb{C}))\} = a(n)$$

<u>Proof</u>: Let U (resp.U') be the subgroup of elements of order p in $\tilde{A}_{2n-1}(P_\infty\mathbb{C})$ (resp. in $\tilde{A}_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$. The kernel of $\beta|_{U'}$ is $\mathbb{Z}/p\cdot x_o$, so $a(n) = \dim_{\mathbb{Z}/p} U = \dim_{\mathbb{Z}/p} U' - 1$. We know that every element in $A_{2n-1}(P_\infty\mathbb{C})$ is a linear combination of the elements $\beta(\pi_*(x(i,j)))$ $(i \geq o)$, so every element in U' is a sum of x_i $(i \geq o)$. Let e be the maximal index of an x_i appearing in a linear combination for an element in U', say $y = \sum_{m=o}^{e} a_m x_m$, $a_e \neq o$. We can assume $a_m = 1$. Clearly $a+1 \leq e+1$. By (1.11) we have $m_{p^e*}(y) = x_o$, so $x_o \in U'$, $m_{p^{e-1}*}(y) = a_{e-1}x_o + x_1$, so $x_1 \in U'$. Inductively it follows that $x_o, \ldots, x_e \in U'$. The equation $z = \sum_{i=o}^{r} a_i \cdot x_i = o$ with $a_r \neq o$ implies $m_{p^r*}(z) = a_r x_o = o$ so the $x_o, \ldots, x_e$ are linear independent and we have $e+1 \geq a+1$.

Remark:

In general that is for elements not of order p it may happen that the maximal value of i for which $x(i,j)$ appears in an element $y \in A_{2n-1}(P_\infty \mathbb{C})$ is greater than the number of summands: For example $y = x(1,2)+x(2,1) \in Ad_{13}(B\mathbb{Z}/9)$ is in $A_{13}(B\mathbb{Z}/9)$ and $\pi_*(y)$ generates $A_{13}(P_\infty \mathbb{C}) \cong \mathbb{Z}/9$ $(p=3)$, but $\pi_*(x(2,1)) \notin A_{13}(P_\infty \mathbb{C})$.

Determining $a(n) = \max\{i \mid \beta(x_i) \in \tilde{A}_{2n-1}(P_\infty \mathbb{C})\}$ is by (1.7) equivalent to finding the maximal i satisfying $a_j^{(i)}$ is divisible by p for $j>n$.

Theorem 1.13: Let p be an odd prime. The number of cyclic summands in $\tilde{A}_{2n-1}(P_\infty \mathbb{C})$ is given by

$$\left[\frac{\log \frac{n+1}{s+1}}{\log p}\right]$$

where $n = t(p-1)+s$ and $o<s\leqslant p-1$

Proof: The element x_r is the image of $\sum_{j=1}^{p-1} j^n \cdot \rho^{j \cdot p^r}/p$ under the map $Ad_{2n}(B\mathbb{Z}/p^{r+1};\mathbb{Q}/\mathbb{Z}) \longrightarrow Ad_{2n}(P_\infty \mathbb{C};\mathbb{Q}/\mathbb{Z})$. Let $x_r = \sum_m a_m^{(r)} \cdot b_m/p$ with $a_m^{(r)}$ defined as in (1.7). The first m with non vanishing $a_m^{(r)}$ is $m=p^r$. We set $q=p^{r+1}$ then $\pi^*(L-1)^q = \pi^*(L^q-1)$ mod p and $\pi^*(L^q) = 1$ implies $\pi^*(L-1)^m$ is divisible by p for $m \geqslant q$, so

$$a_m^{(r)} = <\pi^*(L-1)^m,\ x(r,1)> = o \quad \text{if} \quad m \geqslant q$$

and we are left with $a_m^{(r)}$ with $p^r \leqslant m < p^{r+1}$.

Let $\sum a_k p^k$ be the p-adic expansion of m. By the formula for the mod p value of $\binom{a}{b}$ we have

$$a_m^{(r)} = \sum_{j=1}^{p-1} \binom{m}{p^r \cdot j} \cdot j^n \cdot (-1)^{j+m} \equiv \sum_{j=1}^{p-1} \binom{a_r}{j} \cdot j^n \cdot (-1)^{j+m} \quad \text{mod } p$$

Let $n = t(p-1)+s$ with $o < s \leq p-1$, then $j^n \equiv j^s$ mod p because $j^n - j^s = j^s(j^{t \cdot (p-1)} - 1)$ is divisible by p if $j \not\equiv o(p)$, so

$$a_m^{(r)} = (-1)^{a_r+m} \cdot \sum_{j=1}^{p-1} \binom{a_r}{j} \cdot j^s \cdot (-1)^{s+a_r} = (-1)^{a_r+m} \cdot a_r! \cdot S(s, a_r)$$

where $S(a,b)$ is a Stirling number of the second kind. We have $a_m^{(r)} \equiv o(p)$ iff $S(s, a_r) \equiv o(p)$. Now $S(a,a) = 1$ and $S(a,b) = o$ for $b > a$ implies $a_m^{(r)} \equiv o(p)$ for $m \geq (s+1) \cdot p^r$. We fix s with $o < s \leq p-1$ and consider now only such n satisfying $n \equiv s \mod (p-1)$. The above calculations show $x_r \in Ad_{2n}(P_n\mathbb{C}; Q/\mathbb{Z})$ if $n \geq (s+1) \cdot p^r - 1$ and $x_r \in Ad_{2n}(P_n\mathbb{C}; Q/\mathbb{Z})$ for $n < (s+1) \cdot p^r - 1$. So for all n (with $n \equiv s(p-1)$) satisfying $(s+1) \cdot p^r \leq n+1 < (s+1)p^{r+1}$ we have $\max \{ i \mid x_i \in Ad_{2n}(P_n\mathbb{C}; Q/\mathbb{Z}) = r = a(n)$.
The function log is monotonic, so

$$(s+1) \cdot p^r \leq n+1 < (s+1) \cdot p^{r+1} \Leftrightarrow \log((s+1) \cdot p^r) \leq \log(n+1) < \log((s+1) \cdot p^{r+1})$$

$$\Leftrightarrow \quad r \leq \frac{\log \frac{n+1}{s+1}}{\log p} < r+1$$

$$\Leftrightarrow \quad r = \left[\frac{\log \frac{n+1}{s+1}}{\log p} \right]$$

where $[x]$ denotes the greatest integer not exceeding x.

Another application of (1.11) is the following lemma which allows us to calculate the order of an element $x \in A_{2n-1}(P_\infty \mathbb{C})$ given as a rational linear combination of the elements b_1^i in $K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ in a simple way.

Lemma 1.14: Let z be an element in $Ad_{2n}(P_n\mathbb{C};\mathbb{Q}/\mathbb{Z})\subset K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ with order not exceeding p^n. Let $m_p: P_\infty\mathbb{C} \to P_\infty\mathbb{C}$ be the map defined by multiplication with p in the H-space structure of $P_\infty\mathbb{C}$. Then the order of $m_{p*}(z)$ in $Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ is exactly the order of $\beta(z)$ in $A_{2n-1}(P_\infty\mathbb{C})$.

Proof: The kernel of the Bockstein map consists of multiples of b_1^n So we can write z as a linear combination of elements coming from $Ad_*(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z})$, say

$$z = \sum_{r=1} c_r x(r,k_r) + a\cdot b_1^n$$

with $p^n a=o$. Then by (1.11) we have $m_{p*}(z) = \sum_{r=1} c_r x(r-1,k_r)$ because $m_{p*}(b_1^n)=p^n b_1^n$. The order of $m_{p*}(z)$ is thus the order of $z-a\cdot b_1^n$, which is the order of $\beta(z)$.

If $x\in Ad_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ is given as $x = \sum a_i b_1^i$ $K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ then the order of $\beta(x)$ is easy to calculate because $m_{p*}(b_1^i) = p^i b_1^i$ and then there are usually very few terms left.

Example: p=3. The element $z = 4\cdot b_1/27 + b_1^3/9$ is in $\ker(\psi^k-k^7)$ By (1.14) $\beta(z)$ has order 9. Direct calculation gives

$$z = \frac{1}{9}(b_3+5b_4+b_5+3b_7) \bmod b_1^7$$

$$= \pi_*(x(1,2) + x(2,1))$$

This also allows one to work out the module structure of $A_*(P_\infty\mathbb{C})$

Proposition 1.15: Let α_t denote the generator of $A_{2t(p-1)-1}$ (*)

then the order of $b_1^n \cdot \alpha_t$ in $A_{2t(p-1)+2n-1}(P_\infty \mathbb{C})$ is $\max\{1+\nu_p(t)-n, o\}$.

Proof: The order of α_t is $p^{1+\nu_p(t)}$ and the order of $\beta(b_1^n/p^{1+\nu(t)})$ follows from (1.14).

The number of cyclic summands in $J(P_n\mathbb{C})$

There is a close connection between the groups $Ad^*(X)$ and $J(X)$, the group of stable vector bundles on X under the relation of stable fibre homotopy equivalence.

Adams [1] has defined a group $J''(X)$ which serves as an upper bound for $J(X)$. By the solution of the Adams conjecture $J''(X)$ is actually $J(X)$. Stated only for the p-primary part (p odd) $J''(X)_{(p)}$ is the quotient of $\tilde{K}^o(X)$ by the subgroup generated by the elements $(\psi^k-1)(x)$ for $k\neq o(p)$ and $x \in K^o(X)$. By definition of $Ad^1(X)$ we get surjective map from $im(j: \tilde{K}^o(X) \to Ad^1(X))$ onto $J''(X)_{(p)}$

Lemma 1.15: For an odd prime p $\widetilde{Ad}^1(P_n\mathbb{C}) \cong J(P_n\mathbb{C})_{(p)}$

Proof: The exact sequence (1.1) shows

$$\nu_p(|Ad^1(P_n\mathbb{C})|) = \sum_{i=1}^{m} 1+\nu_p(i)$$

whereas the results of [5] (Lemma 5.2) imply

$$\nu_p(|J(P_n\mathbb{C})|) = \sum_{i=1}^{m} 1+\nu_p(i) \quad \text{with} \quad m = \left[\frac{n}{p-1}\right]$$

As preparation we need.

Lemma 1.16: Let $n = t(p-1)+s$ with $o\leq s\leq p-2$, then

$$Ad^1(P_n\mathbb{C}) \cong Ad^1(P_{t\cdot(p-1)}\mathbb{C})$$

Proof: The map induced by inclusion is surjective. Because the determinant of (ψ^k-1) does not change if n is altered to n-s, $Ad^1(P_n\mathbb{C})$ and $Ad^1(P_{t\cdot(p-1)}\mathbb{C})$ must have the same order.

<u>Theorem 1.17</u>: The number of cyclic summands in $J(P_n\mathbb{C})_{(p)}$ is given by $\left[\frac{\log\,(n+1)}{\log\,p}\right]$

<u>Proof</u>: The theory of elementary divisors shows that

$Ad^{2i+1}(P_n\mathbb{C})\otimes\mathbb{Z}/p$ is the cokernel of $K^o(P_n\mathbb{C})\otimes\mathbb{Z}/p \xrightarrow{\psi^k-k^i} K^o(P_n\mathbb{C})\otimes\mathbb{Z}/p$

Therefore $Ad^{2i+1}(P_n\mathbb{C})\otimes\mathbb{Z}/p \cong Ad^{2j+1}(P_n\mathbb{C})\otimes\mathbb{Z}/p$ if $k^i\equiv k^j \bmod p$ which can be seen as a remnant of Bott periodicity. Let $n = t(p-1)$, then $k^n-1\equiv o(p)$, so $Ad^1(P_n\mathbb{C})\otimes\mathbb{Z}/p \cong Ad^{1+2n}(P_n\mathbb{C})\otimes\mathbb{Z}/p$. The Kronecker product $<,>: K^o(P_n\mathbb{C})\otimes K_o(P_n\mathbb{C};Q/\mathbb{Z})\to Q/\mathbb{Z}$ induces a pairing

$$1:\ Ad^{2m+1}(P_n\mathbb{C})\times Ad_{2m}(P_n\mathbb{C};Q/\mathbb{Z})\to Q/\mathbb{Z}$$

by $1(x,y) = <\bar{x},k(y)>$ where $j(\bar{x}) = x$ and $j:K^{2m}(P_n\mathbb{C})\to Ad^{2m+1}(P_n\mathbb{C})$ and $k: Ad_{2m}(P_n\mathbb{C};Q/\mathbb{Z})\to K_{2m}(P_n\mathbb{C};Q/\mathbb{Z})$ are the natural maps. The pairing 1 induces a map $\bar{1}: Ad_{2m}(P_n\mathbb{C};Q/\mathbb{Z})\to Hom(Ad^{2m+1}(P_n\mathbb{C});Q/Z)$. Because $Hom(-,Q/Z)$ is exact and the map

$$K_o(P_n\mathbb{C};Q/\mathbb{Z})\to Hom(K^o(P_n\mathbb{C}),Q/Z)$$

corresponding to $\bar{1}$ is an isomorphism, the 5-Lemma implies that $\bar{1}$ is bijective. So the number of summands in $Ad^1(P_n\mathbb{C})$ and $Ad_{2n-1}(P_n\mathbb{C};Q/\mathbb{Z})$ are the same. By (1.13) it follows that there are $[\log\,(t(p-1)+1)/\log\,p]$ summands in $A_{2t(p-1)-1}(P_\infty\mathbb{C})$. In $Ad_{2n-1}(P_n\mathbb{C};Q/\mathbb{Z})$ there is one additional summand $\ker\beta$, so the number of summands in $\widetilde{Ad}^1(P_n\mathbb{C})$ is $[\log(t(p-1)+1)/\log p]$. It is easy to see, that for $o\leq s\leq p-2$ $[\log(t(p-1)+1+s)/\log p] = [\log(t(p-1)+1)/\log p]$, so the theorem is proved.

<u>Remark</u>:

In [37] an upper bound is given for the number of cyclic summands in $J(P_{2n}\mathbb{C})_{(p)}$ to be $[\log n/\log p]+1$ and for p=2 it is proved, that the number of summands in $J(P_{2n}\mathbb{C})_{(2)}$ is $[\log(2n)/\log 2]$

§ 2 Transfer maps

In this chapter we dicuss the transfer maps

$$t^k : \pi_*^s(P_\infty \mathbb{C}^{k\tilde{H}}) \longrightarrow \pi_*^s(S^o)$$

and establish some of their properties. These transfer maps can be induced by stable maps $\tau^k \in \pi_s^{-1}(P_n \mathbb{C}^{k\tilde{H}})$. We show that τ^o is in the image of the J-homomorphism $J: K^{-2}(P_\infty \mathbb{C}) \rightarrow \pi_s^{-1}(P_\infty \mathbb{C}^+)$ and describe the connection between transfer maps and generalized J-homomorphisms. We identify the cofibre spectrum of τ^k with a Thom spectrum and give a formula relating the images of the various t^k. We close with a description of $t^{(-1)}$ in terms of t^o.

The S^1-transfer establishes a close connection between $\pi_*^S(P_\infty\mathbb{C}^+)$ and $\pi_*^S(S^0)$. In particular, the part of $\pi_*^S(P_\infty\mathbb{C})$ detected by the e-invariant is related to elements in $\pi_*^S(S^0)$ coming from the $\mathrm{Ext}^{2,*}$-term of the BP-Adams spectral sequence.

The simplest definition of the S^1-transfer

$$t^o\colon \Omega_n^{fr}(P_\infty\mathbb{C}) \longrightarrow \Omega_{n+1}^{fr}$$

is by identifying $\Omega_n^{fr}(P_\infty\mathbb{C})$ with the bordism group of equivariantly framed free S^1-manifolds $\Omega_{n+1}^{fr}(S^1;\text{free})$ and then defining t^o as the forgetful map to framed bordism. Or more explicitly:
Given an element $[M,\phi,f]$ in $\Omega_n^{fr}(P_\infty\mathbb{C})$, by pulling the universal S^1-bundle back via f we get an induced S^1-bundle $(\tilde{M},\pi,M)$ over M. The tangent bundle of $\tilde{M}$ splits as a direct sum of π^*TM and T_F, the tangent bundle along the fibres. Let H denote the Hopf bundle over $P_\infty\mathbb{C}$, then $T_F\oplus 1$ is isomorphic to $f^*\pi^*H$ and thus canonically trivialized. We can frame $T\tilde{M}$ by putting the framings of $\pi^*\phi$ and T_F together to get a framed manifold $(\tilde{M},\tilde{\phi})$. Different splittings of $T\tilde{M}$ do not alter the bordism class of $(\tilde{M},\tilde{\phi})$ and we define $t^o[M,\phi,f] = [\tilde{M},\tilde{\phi}]$.

We first recall what is known about t^o (see [26],[35],[7]). Let $\sigma\in\pi_2^S(P_\infty\mathbb{C}^+)$ denote the element defined by the Hopf bundle $S^3\longrightarrow S^2$. Then it is well known, that

$$\pi_*^S(P_\infty\mathbb{C})/\mathrm{tor} \cong \mathbb{Z}[\sigma]$$

under the Pontrjagin product. A computation of $e_{\mathbb{C}}t(\sigma^n)$ can be found in [26],[35]. Using this and the fact that t^o raises the degree in the filtration associated to the BP-Adams spectral sequence by 1, it follows that the elements μ_r constructed by Adams in $\pi_*^s(S^o)_{(2)}$ are not in the image of t^o. Similarly it follows that the generator of $\mathrm{im}(J)_{(2)}$ in $\pi_{8n-1}^s(S^o)_{(2)}$ is not in the image of t^o (see also [7]). For the odd primes we have $\mathrm{im}(J) \subset \mathrm{im}(t^o)$.

Denote by U the infinite unitary group. Then we have an S-map $J': U \longrightarrow S^o$ defined as the image of the identity in $[U,U] \cong K^{-1}(U)$ under the J-homomorphism $J: K^{-1}(U) \longrightarrow \pi_s^o(U)$. The map induced by J' in stable homotopy is called the bistable J-homomorphism. Let w be the complex reflection map

$$w: S^1 \wedge P_n\mathbb{C}^+ \longrightarrow U(n+1) \tag{2.1}$$

defined as follows: To the pair $(\lambda, x) \in S^1 \wedge P_n\mathbb{C}$ we associate the unitary map A which is the identity on the orthogonal complement of the linie x in $\mathbb{C}^{n+1}$ defined by x and maps v in x to $\lambda \cdot v$. The transfer t^o can be represented by a S-map $\tau: S^1 \wedge P_\infty\mathbb{C}^+ \longrightarrow S^o$ see [23] or the discussion below, then t^o is related to J' as follows:

<u>Theorem 2.2</u>: Up to a minus sign $w \circ J'$ is homotopic to τ .

We first sate some corollaries of (2.2).

<u>Corollary 2.3</u>: The following diagram commutes

$$\begin{array}{ccc} \pi_n^s(P_\infty\mathbb{C}^+) & \xrightarrow{\;t\;} & \pi_{n+1}^s(S^o) \\ \downarrow w_* & \nearrow -J' & \\ \pi_{n+1}^s(U) & & \end{array} \tag{2.4}$$

This solves a problem posed in [26].

Now $w^*: K^{-1}(U) \longrightarrow K^{-1}(SP_\infty\mathbb{C}^+) \cong \tilde{K}^{-2}(P_\infty\mathbb{C}^+)$ maps id_u onto the element $\pm\beta(H)$ where $\beta\colon \tilde{K}^{-2}(X) \cong \tilde{K}(X)$ is Bott periodicity and H the Hopf line bundle over $P_\infty\mathbb{C}$. Therefore we have:

<u>Corollary 2.5</u>: Under the J-homomorphism $J\colon K^{-2}(P_\infty\mathbb{C}) \to \pi_s^{-1}(P_\infty\mathbb{C}^+)$ the element corresponding to the Hopf line bundle is mapped onto the element τ which represents t^o as an S-map.

<u>Remark</u>:

If $\pi\colon \tilde{M} \to M$ is a principal S^1-bundle and E a generalized multiplicative cohomology theory, we have a transfer map $t_\pi\colon E^i(\tilde{M}) \to E^{i-1}(M)$ with transfer index $t_\pi(1) \in E^{-1}(M)$. It is then clear how (2.5) can be used to determine $t_\pi(1)$.

All maps in the diagram (2.4) can be interpreted geometrically. The description of J' is as follows: An element in $\pi_n^s(U) = \tilde{\Omega}_n^{fr}(U)$ is a triple (M,ϕ,f) with $[M,\phi] = 0$ and $f\colon M \to U$ a map. Then $J'([M,\phi,f]) = [M,\phi^f]$ where ϕ^f is the new framing of M got by twisting the framing ϕ by f (that is, we compose ϕ by the automorphism of the trivial bundle over M defined by f). Starting with an element $[M,\phi,f] \in \Omega^{fr}(P_\infty\mathbb{C})$ we get under t^o the manifold $\tilde{M}$, which is the total space of the S^1-bundle defined by f. The other way around we arrive at the manifold $S^1 \times M$. Now the twist is not in the manifold but in the framing.

My original proof for (2.2) was by looking directly at the construction of an S-map for τ, observing that τ is homotopic to a map which is in the image of the J-homomorphism $J\colon K^{-2}(P_n\mathbb{C}) \to \pi_s^{-1}(P_n\mathbb{C}^+)$ and then comparing e-invariants. It seems nicer to see an explicit framed bordism between $\tilde{M}$ and $M \times S^1$. This is done in [41], where also a simpler proof of (2.2) is given. For a proof of (2.2) we therefore refer to [41]. A simple proof of (2.2)can also deduced from the results of [10].

We now come to the generalization of the S^1-transfer. Suppose given an n-dimensional manifold M and a map $f: M \to P_\infty\mathbb{C}$ defining a complex line bundle ξ over M. Suppose further that there is given an isomorphism ϕ of the stable normal bundle ν of M with the k-fold Whitney sum of ξ ($k \in \mathbb{Z}$). By the Thom-Pontrjagin construction the triple (M,ϕ,f) defines an element in $\Omega_n^{fr}(P_\infty\mathbb{C}^{k\tilde{H}})$ and all bordism classes can be so described. Here we write $P_\infty\mathbb{C}^{\alpha}$ for the Thom space of a bundle α and because of $X^{\alpha\oplus\mathbb{R}^n} \cong S^nX^{\alpha}$ we are allowed to consider Thom spaces of virtual bundles (e.g. $\tilde{H} = H-1$).

As in the case $k=o$ the total space $\tilde{M}$ of the S^1-principal bundle of ξ has a canonical framing: We have $T\tilde{M} \cong \pi^*TM \oplus T_F$ and $T_F \cong \pi^*(\xi)$ is canonically framed. The isomorphism ϕ allows us to identify π^*TM with -k copies of $\pi^*(\xi)$. Putting these isomorphisms together defines a framing ϕ^k of M. As in the case k=o one easily checks that $[\tilde{M},\phi^k]$ is well defined up to framed bordism. So we have defined

$$t^k: \Omega_n^{fr}(P_\infty\mathbb{C}^{k\tilde{H}}) \longrightarrow \Omega_{n+1}^{fr} \qquad (2.6)$$

<u>Remark</u>:

As in the case k=o one can describe t^k as a forgetful map from equivariant bordism to bordism. Instead of considering equivariantly framed free S^1-manifolds , one has to use free S^1-manifolds, where the stable normal bundle is equivariantly isomorphic to the representation $k\cdot(\mathbb{C},\rho)$, where $(\mathbb{C},\rho)$ denotes $\mathbb{C}$ with the standard S^1-action.

The homomorphisms t^k can be induced by stable maps at least on finite skeletons. The way how to do this is essentially in [8] and now well known. Therefore we only sketch it. Suppose given a map $f: X \longrightarrow Y$ between compact C^∞-manifolds. Approximate f by a smooth embedding $f^1: X \longrightarrow \mathbb{R}^m \times Y$ with normal bundle ν. The Thom-Pontrjagin construction defines then a map

$$T_\alpha(f^1): Y^\alpha \longrightarrow X^{\nu-m+f^*(\alpha)} \tag{2.7}$$

for every virtual bundle α over Y. Applied to the projection map $\pi: S^{2n+1} \longrightarrow P_n\mathbb{C}$ and $\alpha=0$ this yields an S-map

$$\tau: S^1 \wedge (P_n\mathbb{C}^+) \longrightarrow (S^{2n+1})^+$$

which upon composing with the projection $pr: (S^{2n+1})^+ \longrightarrow S^o$ induces the transfer t^o. Applied to π with $\alpha = k\cdot\tilde{H}$ this gives a stable map

$$\tau^k: S^1 \wedge P_n^{k\tilde{H}} \longrightarrow S^o \tag{2.8}$$

which represents t^k. The details for a proof are in [8].

The main consequence of this description of the transfer t^k is the possibility of working out the cofibre of τ^k and so embedding t^k in an exact sequence.

<u>Theorem 2.9</u>: The cofibre of τ^k is the Thom space $P_\infty\mathbb{C}^{(k-1)\tilde{H}}$, that is

$$\longrightarrow P_{n+1}\mathbb{C}^{(k-1)\tilde{H}} \xrightarrow{j!} S^2P_n\mathbb{C}^{k\tilde{H}} \xrightarrow{\tau^k} S^1 \longrightarrow$$

is a cofibre sequence.

<u>Proof</u>: The Thom Pontrjagin construction provides us with the maps

$$T_\alpha(i): P_{n+1}\mathbb{C}^{(k-1)\tilde{H}} \longrightarrow S^2P_n\mathbb{C}^{k\tilde{H}}$$

$$\alpha = (k-1)\tilde{H} \quad \text{and} \quad T_\alpha(\pi): S^2P_n\mathbb{C}^{k\tilde{H}} \longrightarrow (S^{2n+1})^{\mathbb{R}}$$

$(\alpha = k\tilde{H})$ where $i: P_n\mathbb{C} \to P_{n+1}\mathbb{C}$ is the inclusion. The naturality

properties of the construction T_α (see [8]) imply that the composition $T_\alpha(\pi)\circ T_\alpha(i)$ factorizes over
$T_\alpha(g)\colon P_{n+1}\mathbb{C}^{(k-1)\tilde{H}} \longrightarrow (*)^{2n+2+(k-1)\tilde{H}} = S^{2n+2}$ where $g\colon * \to P_{n+1}\mathbb{C}$
is the inclusion (because $i\circ\pi$ is constant). So
$pr\circ T_\alpha(\pi)T_\alpha(i) = \tau^k\circ T_\alpha(i)$ must be null homotopic, where
$pr\colon (S^{2n+1})^{\mathbb{R}} \longrightarrow S^1$ denotes the projection.

Let C_k be the cofibre of τ^k. Because $\tau^k\circ T_\alpha(i) \simeq 0$ we get a map
$f\colon S^1 P_{n+1}\mathbb{C}^{(k-1)\tilde{H}} \longrightarrow C_k$ and a commutative diagram

$$\begin{array}{ccccccccc}
P_n\mathbb{C}^{k\tilde{H}+\mathbb{C}} & \xrightarrow{\tau^k} & S^1 & \longrightarrow & S^1\wedge P_{n+1}\mathbb{C}^{(k-1)\tilde{H}} & \longrightarrow & S^3\wedge P_n\mathbb{C}^{k\tilde{H}} & \longrightarrow \\
\| & & \| & & \downarrow f & & \| & \\
P_n\mathbb{C}^{k\tilde{H}+\mathbb{C}} & \xrightarrow{\tau^k} & S^1 & \longrightarrow & C_k & \longrightarrow & S^3\wedge P_n\mathbb{C}^{k\tilde{H}} & \longrightarrow
\end{array}$$

The corresponding diagram in homology has exact rows, because $T_\alpha(i)$ is Poincaré-dual to the inclusion. So f_* is an isomorphism by the 5-Lemma and f must be an S-homotopy-equivalence. The map induced by $T_\alpha(i)$ we denote by $j^!$.

So for example we have embedded t^o in the following exact sequence:

$$\pi_n^s(P_\infty\mathbb{C}^{-\tilde{H}}) \xrightarrow{j^!} \pi_{n-2}^s(P_\infty\mathbb{C}^+) \xrightarrow{t^o} \pi_{n-1}^s(S^o) \xrightarrow{i_*} \pi_{n-1}^s(P_\infty\mathbb{C}^{-\tilde{H}}) \rightarrow \qquad (2.1o)$$

<u>Remark</u>:

The map $j^!$ induced by $T_\alpha(i)$ can be interpreted geometrically as follows: Let $(M,f,\phi)\in\Omega_n^{fr}(P_\infty\mathbb{C}^{k\tilde{H}})$ be given. We can from the submanifold $N\subset M$ dual to $c_1(f^*H)$ which then has $(k+1)\cdot f^*(H)$ as stable normal bundle and defines $j^![M,f,\phi]$. In bordism theory this is usually called the Smith homomorphism.

The main application of (2.9) is to the computation of t^k on elements in $\pi_n^s(P_\infty \mathbb{C}^{k\tilde{H}})$ which are detected by the e-invariant, see § 3 and § 4.
The generalization of (2.2) and (2.3) to the case of t^k will be discussed in § 7.

As next we discuss the relation of t^k to transfer maps of sphere bundles.

Given a m-dimensional vector bundle V and virtual bundles W,A on a space X we have the following cofibre sequence (see e.g. [18]) where S(V) is the sphere bundle of V

$$\longrightarrow S(V)^{\pi^* W} \longrightarrow X^W \longrightarrow X^{V\oplus W} \xrightarrow{\partial} S^1_\wedge S(V)^{\pi^* W} \longrightarrow \qquad (2.11)$$

Applying the Thom Pontrjagin construction to the map $S(V) \xrightarrow{\pi} X$ we get the transfer map of the sphere bundle S(V) with coefficients in A ($\tilde{V}$=V-m):

$$T_A(\pi):\ X^{A+m} \longrightarrow S^1 \wedge S(V)^{\pi^*(A-\tilde{V})}$$

Letting $W=A-\tilde{V}$ we have the boundary map of the cofibre sequence (2.11)

$$\partial:\ X^{A+m} \longrightarrow S^1 \wedge S(V)^{\pi^*(A-V)}$$

<u>Lemma 2.12:</u> The stable maps ∂ and $T_A(\pi)$ are homotopic up to sign.

<u>Proof:</u> Both constructions can be made fibrewise i.e. in the category of fibre bundles over X. One proves $T_\alpha(\pi) = \pm\partial$ in case where X is a point directly, being careful that all constructions are compatible with the structure groups of the fibre bundles involved. The proof is finished by passing to the associated fibre bundles and maps and proceeding as in [46].

The identification of the Thom space $P_r\mathbb{C}^{mH}$ with the stunted projective space $P_{r+m}\mathbb{C}/P_{m-1}\mathbb{C}$ is induced by the map

$$D(mH_r) = S^{2r+1}\times_{S^1} D(\mathbb{C}^m) \longrightarrow S^{2r+1+2m}/S^1$$

defined by $\quad [x,y] \longmapsto [y,(1-||y||^2)^{\frac{1}{2}}\cdot x]$

If we identify $P_n\mathbb{C}^{rH}$ with a stunted projective space, we can write the usual cofibre sequence of j: $P_{n+r}\mathbb{C}/P_{r-2}\mathbb{C} \longrightarrow P_{n+r}\mathbb{C}/P_{r-1}\mathbb{C}$ as follows:

$$P_{n+1}\mathbb{C}^{(r-1)H} \xrightarrow{j} P_n\mathbb{C}^{rH} \xrightarrow{\partial} S^1\wedge *^{(r-1)H} \qquad (2.13)$$

The map $M(i)$: $P_n\mathbb{C}^{(r-1)H} \longrightarrow P_{n+1}\mathbb{C}^{(r-1)H}$ induced by the inclusion followed by j is easily seen to be homotopic to the natural map $P_n\mathbb{C}^{(r-1)H} \longrightarrow P_n\mathbb{C}^{rH}$ appearing in (2.11) for $X=P_n\mathbb{C}$. We therefore get a commutative diagram

$$\begin{array}{ccccc} P_{n+1}\mathbb{C}^{(r-1)H} & \xrightarrow{j} & P_n\mathbb{C}^{rH} & \xrightarrow{\partial} & S^1\wedge *^{(r-1)H} \\ \uparrow{\scriptstyle M(i)} & & \| & & \uparrow \\ P_n\mathbb{C}^{(r-1)H} & \longrightarrow & P_n\mathbb{C}^{rH} & \xrightarrow{\partial_{rH}} & S^1\wedge S(H)^{(r-1)H} \end{array}$$

By (2.12) we can use instead of ∂_{rH} the transfer map of $S(H) \longrightarrow P_n\mathbb{C}$, so we have proved:

<u>Proposition 2.14</u>: For $k>0$ the transfer map t^k: $S^1 P_n\mathbb{C}^{k\tilde{H}} \longrightarrow S^0$ is stably homotopic to the boundary map in the cofibre sequence of stunted projective spaces in (2.13).

Let M be a multiple of the order of $J(H_n)$ in $J(P_n\mathbb{C})$, then we have a relative Thom isomorphism

$$\phi\colon h^*(P_{n+1}\mathbb{C}^A) \longrightarrow h^*(P_{n+1}\mathbb{C}^{A+M\cdot\tilde{H}})$$

for each cohomology theory h, which is a module theory over π_s^*. By choosing $h=\{P_{n+1}\mathbb{C}^A,-\}$ we see that ϕ can be represented by a stable map

$$\phi\colon P_{n+1}\mathbb{C}^{A+M\cdot\tilde{H}} \longrightarrow P_{n+1}\mathbb{C}^A \qquad (2.15)$$

By 4.1o of [8] ϕ commutes with the maps $j^!$ of (2.9) giving a commutative diagram of cofibre sequences

$$\begin{array}{ccccc}
P_{n+1}\mathbb{C}^{(r-1)H} & \xrightarrow{j^!} & P_n\mathbb{C}^{rH} & \xrightarrow{\tau^r} & S^1\wedge *^{(r-1)H} \\
\uparrow \phi & & \uparrow \phi & B & \| \\
P_n\mathbb{C}^{(r-1)H+M\tilde{H}} & \xrightarrow{j^!} & P_n\mathbb{C}^{rH+M\cdot\tilde{H}} & \xrightarrow{\tau^{r+MH}} & S^1\wedge *^{(r-1)H}
\end{array}$$

The commutativity of the square B shows

<u>Proposition 2.16</u>: $\phi(\tau^r) = \tau^{r+MH}$, that is on $P_n\mathbb{C}$ the transfer maps are perodic with period $|J(H_{n+1})|$.

<u>Remark</u>:

If M is only a multiple of $J(H_n)$ such that $M\cdot J(H_{n+1})\neq 0$, then the square corresponding to B need not be commutative.

The H-space structure of $P_\infty\mathbb{C}$ allows us to relate various transfer maps t^k. Let

$$\mu_{n,m}\colon P_n\mathbb{C}\times P_m\mathbb{C} \rightarrow P_{n+m}\mathbb{C}$$

be the restriction of the H-space multiplication. This can explicitly defined by interpreting point in $P_n\mathbb{C}$ as complex polynomials of degree n (up to a nonzero factor) and then using polynomial multiplication. We apply the Hopf construction to the map

$\mu_{1,n}: S^2 \times P_n\mathbb{C} \to P_{n+1}\mathbb{C}$ to get maps $S^3 \wedge P_n\mathbb{C} \to S^1 \wedge P_{n+1}\mathbb{C}$. These maps are compatible with the inclusions between projective spaces, so we get induced maps

$$S^3 \wedge P_n\mathbb{C}/P_m\mathbb{C} \longrightarrow S^1 \wedge P_{n+1}\mathbb{C}/P_{m+1}\mathbb{C} \tag{2.17}$$

or written as a stable map in terms of Thom spaces of virtual bundles $(k \geqslant o)$:

$$\omega: P_s\mathbb{C}^{k\tilde{H}} \longrightarrow P_s\mathbb{C}^{(k+1)\tilde{H}} \tag{2.18}$$

Let $[P_i\mathbb{C}]$ denote the fundamental class of $P_i\mathbb{C}$ in $H_*(P_\infty\mathbb{C};\mathbb{Z})$. Because the Pontrjagin multiplication in homology is given by

$$[P_i\mathbb{C}]\cdot[P_j\mathbb{C}] = \binom{i+j}{i}[P_{i+j}\mathbb{C}]$$

we see that the degree of ω on the bottom cell is $k+1$ and $k+s+1$ on the top cell.

Because ω commutes with inclusions we have a commutative diagram

$$\begin{array}{ccc}
P_{r+1}\mathbb{C}^{sH} & \xrightarrow{\ j\ } & P_r\mathbb{C}^{(s+1)H} \\
\uparrow{\scriptstyle\omega} & & \uparrow{\scriptstyle\omega} \\
S^2P_{r+1}\mathbb{C}^{(s-1)H} & \xrightarrow{\ j\ } & S^2P_r\mathbb{C}^{sH}
\end{array}$$

where the maps j are defined by natural maps

$P_{r+1+s}\mathbb{C}/P_{s-1}\mathbb{C} \longrightarrow P_{r+1+s}\mathbb{C}/P_s\mathbb{C}$ as in (2.13).

By (2.14) we extend this square to a diagram of cofibre sequences getting:

$$\begin{array}{ccccc}
P_{r+1}\mathbb{C}^{s\tilde{H}} & \xrightarrow{\ j\ } & S^2 \wedge P_r\mathbb{C}^{(s+1)\tilde{H}} & \xrightarrow{\ \tau^{s+1}\ } & S^1 \\
\uparrow{\scriptstyle\omega} & & \uparrow{\scriptstyle\omega} & & \uparrow{\scriptstyle f_{(s)}} \\
P_{r+1}\mathbb{C}^{(s-1)\tilde{H}} & \xrightarrow{\ j\ } & S^2 \wedge P_r C^{s\tilde{H}} & \xrightarrow{\ \tau^{s}\ } & S^1
\end{array} \tag{2.19}$$

where $f_{(s)}$ denotes a map of degree s.

Thus we have proved for $s \geq 0$:

Theorem 2.2o: Let r be fixed. Then there exist stable maps $\omega^1 : P_r\mathbb{C}^{s\tilde{H}} \longrightarrow P_r\mathbb{C}^{(s+1)\tilde{H}}$ for each $s \in \mathbb{Z}$, such that

$$t^{s+1} \circ \omega^1 = s \cdot t^s$$

.

Proof: To settle the case $s < 0$ we apply periodicity (2.16) of τ^k and use $\tilde{\omega}$, the map ω conjugated with the Thom isomorphisms (2.15). We then get $t^{s+1} \circ \tilde{\omega} = (s+M) \cdot t^s$ where M is a multiple of $|J(H_r)|$. But if we take M large enough then $(s+M)t^s = st^s$ because t^s is an S-map of finite order in $\pi_s^{-1}(P_r\mathbb{C}^{s\tilde{H}})$.

Corollary 2.21: $\quad s \cdot \mathrm{im}\,(t^s) \subset \mathrm{im}\,(t^{s+1})$

For later use we consider the case $k=-1$.

We denote $\mu_{1,n-1} : S^2 \times P_{n-1}\mathbb{C} \longrightarrow P_n\mathbb{C}$ by μ.

Stably we have $S^2 \times P_{n-1}\mathbb{C} \simeq S^2 \wedge (P_n\mathbb{C} \vee S^o) \vee P_n\mathbb{C}$ and we define $f : S^2 \wedge (P_n\mathbb{C} \vee S^o) \longrightarrow (S^2 \times P_n\mathbb{C})^+$ by the inclusion.

Theorem 2.22: The sequence of maps

$$\tilde{\Omega}_n^{fr}(P_\infty\mathbb{C}^{-\tilde{H}}) \xrightarrow{j^!} \Omega_{n-2}^{fr}(P_\infty\mathbb{C}) \xrightarrow{f_*} \Omega_n^{fr}(S^2 \times P_n\mathbb{C}) \xrightarrow{\mu_*} \Omega_n^{fr}(P_\infty\mathbb{C}) \xrightarrow{t^o} \Omega_{n+1}^{fr}$$

is up to a minus sign the transfer $t^{(-1)}$.

This means in terms of framed manifolds: For a given element $z = [M,f,\emptyset] \in \tilde{\Omega}_n^{fr}(P_\infty\mathbb{C}^{-\tilde{H}})$ we can form the submanifold N^{n-2} dual to $c_1(f^*H)$. The isomorphisms $\nu(N,M) \cong f^*H$ and $\emptyset$ put a framing $\emptyset^1$ on N. We take $S^2 \times N$ with the product framing and the sphere bundle of $H \hat{\otimes} f^*(H)$ as an S^1-bundle over $S^2 \times N$ with the induced framing. Then $t^{(-1)}(Z) = - [\, S(H\hat{\otimes} f^*(H), \pi^*\phi^1 \,]$.

<u>Proof of (2.22)</u>: We start with the commutative diagram of S^1-bundles

$$\begin{array}{ccc} S(H\hat{\otimes}H) & \xrightarrow{\tilde{\mu}} & S(H) \\ \downarrow & & \downarrow \\ S^2 \times P_{n-1}\mathbb{C} & \xrightarrow{\mu} & P_n\mathbb{C} \end{array}$$

Because transfer maps are natural for induced bundles we get a commuting square of transfer maps

$$\begin{array}{ccc} S(H\hat{\otimes}H)^{-\mathbb{R}} & \xrightarrow{\tilde{\mu}} & (S^{2n+1})^{-\mathbb{R}} \\ \uparrow t_{\otimes} & & \uparrow t_H \\ (S^2 \times P_{n-1}\mathbb{C})^+ & \xrightarrow{\mu} & P_n\mathbb{C}^t \end{array}$$

Composing with the projections $pr: X^t \longrightarrow S^o$ we get $pr \cdot t_{\otimes} \simeq t \circ \mu$.

Now the sphere bundle $S(H^*\hat{\otimes} H)$ is the same as $S(2H)$ and we have the commuting square

$$\begin{array}{ccc} S(H^*\hat{\otimes} H) & = & S(2H) \\ \downarrow & & \downarrow \\ S^2 \times P_{n-1}C & \xrightarrow{q} & P_{n-1}C \end{array}$$

The use of $S(H\hat{\otimes}H)$ instead of $S(H^*\hat{\otimes}H)$ gives only a change in orientation and so causes a minus sign.

By naturality of the construction T_α we arrive at a commuting diagram

$$\begin{array}{ccc} S(H\hat{\otimes}H)^{-\mathbb{R}} & = & S(2H)^{-2\tilde{H}-\mathbb{R}} \\ \uparrow t_{\otimes} & & \uparrow t_{2H} \\ (S^2 x P_{n-1}\mathbb{C})^+ & \xleftarrow{D(q)} & S^2(P_{n-1}\mathbb{C} \vee S^o) \end{array}$$

We see that $D(q)$ is the map f defined by stable inclusion (observe that $2H$ on $S(2H)$ is trivial).

The natural map $P_n\mathbb{C}^{-2H} \longrightarrow P_n\mathbb{C}^{-2H+2H} = P_n\mathbb{C}^{+}$ appearing in (2.11) can be factored as

$$P_n\mathbb{C}^{-2H} \longrightarrow P_n\mathbb{C}^{-2H+H} \xrightarrow{h} P_n\mathbb{C}^{+}$$

so we have the commutative diagram A below. Because these natural maps together with the corresponding transfers fit into cofibre sequences (2.12) we get a map $\bar{h}$ relating t_H and t_{2H}:

$$\begin{array}{ccccc}
P_n\mathbb{C}^{-2H} & \longrightarrow & P_n\mathbb{C}^{-H} & \xrightarrow{t_H} & S^1 \wedge S(H)^{-2H} \\
\| & A & \downarrow h & & \downarrow \bar{h} \\
P_n\mathbb{C}^{-2H} & \longrightarrow & P_n\mathbb{C}^{+} & \xrightarrow{t_{2H}} & S^1 \wedge S(2H)^{-2H}
\end{array}$$

The transfers t_H and t^{-1} are related by the commutative diagram ($i \circ j^! \simeq h$ is easily seen by looking at the construction of $j^!$):

$$\begin{array}{ccccc}
P_n\mathbb{C}^{-H} & \xrightarrow{h} & P_n\mathbb{C}^{+} & \longrightarrow & S^1 \wedge S(H)^{-H} \\
\| & & \uparrow i & & \uparrow \\
P_n\mathbb{C}^{-H} & \xrightarrow{j^!} & P_{n-1}\mathbb{C}^{+} & \longrightarrow & S^1 \wedge *^{-H}
\end{array}$$

So $\bar{h}t_H \simeq t_{2H}h \simeq h_{2H} \circ i \circ j^!$. If we compose all transfers with the appropriate projections $X^A \longrightarrow *^A$ and use the fact that t_{2H} on $P_n\mathbb{C}^{+}$ restricts to $P_{n-1}\mathbb{C}^{+}$ via $i: P_{n-1}\mathbb{C}^{+} \longrightarrow P_n\mathbb{C}^{+}$ we have found

$$t^{-1} \simeq pr \circ t_{2H} \circ j^!$$

If we put together, we get

$$-t^{-1} \simeq t^{o} \circ \mu \circ f \circ j^! \qquad (2.23)$$

Remarks:

(2.22) means that the stable map τ^{-1} lies almost in the im(J); more precisely by (2.5) we have $\tau^{o} \subseteq$ im(J), therefore $(\mu\circ f)^{*}\,\tau^{o} \subseteq$ im(J) where $(\mu\circ f)^{*}\,\tau^{o} \in \pi_S^{-3}(P_\infty\mathbb{C}^{+})$.
Under the map $j^{!}:\pi_S^{-3}(P_\infty\mathbb{C}^{+}) \longrightarrow \pi_S^{-1}(P_\infty\mathbb{C}^{-\tilde{H}})$ the element $(\mu\cdot f)^{*}\,\tau^{o}$ gives τ^{-1} by (2.23), but at odd primes $j^{!}$ is an isomorphism.

Let p be an odd prime and $n \equiv -2 \mod 2(p-1)$.
Then the maps of (2.22) are onto modulo torsion ($\mu_{*}\cdot f_{*}$ is multiplication by $\sigma \in \Omega_2^{fr}(P_\infty\mathbb{C})$ and for $j^{!}$ see proof of 7.35). Because im(J) $\subset$ im (t^{o}) we have: im(t^{-1}) contains elements with maximal e-invariant.

§ 3 The Whitehead conjecture

The J-homomorphism $J: \pi_n(SO) \longrightarrow \pi_n^s$ can be factored through the so-called bistable J-homomorphism $J': \pi_n^s(SO) \longrightarrow \pi_n^s$. The conjecture of G.W. Whitehead is that this map is onto in positive dimensions. For the 2-primary component J' is known to be surjective. The complex analogue of J' is closely related to the transfer map $t^o: \pi_n^s(P_\infty\mathbb{C}) \longrightarrow \pi_{n+1}^s(S^o)$; in particular they have the same image. Using the cofibre sequence of the representing S-map τ for t^o we construct a map $f: S^2P_\infty\mathbb{C} \longrightarrow MU/S^o$ which allows us to compare. t^o with the boundary map of the cofibre sequence $S^o \longrightarrow MU \longrightarrow MU/S^o \longrightarrow S^1$. By this procedure the problem of computing t from filtration 1 to filtration 2 is lifted to a pure filtration 1 problem, which can be treated by means of the e-invariant. This then allows one to find a counterexample to the conjecture of Whitehead for primes larger than three.

The usual stable J-homomorphism J: $\pi_n(SO) \longrightarrow \pi_n^s(S^o)$ is induced by a map

$$SO \longrightarrow \Omega^\infty S^\infty = Q(S^o)$$

Its adjoint J' is a stable map which induces the bistable J homomorphism

$$J': \pi_n^s(SO) \longrightarrow \pi_n^s(S^o)$$

G.W. Whitehead conjectured that J' is surjective for $n>o$. The reflection map $P_\infty\mathbb{R} \longrightarrow O$ followed by a fixied reflection defines a map $P_\infty\mathbb{R} \xrightarrow{i} SO$ and the composition $J' \circ i_*$ is a surjection onto the 2-primary part by the Kahn-Priddy theorem. So J' is onto for the prime 2. In this chapter we shall prove that J' is not onto for the odd primes larger than 3.

For the odd primary components we can equally well use the complex analogue of J'

$$J_U': \pi_n^s(U) \longrightarrow \pi_n^s(S^o)$$

This is the same as the map mentioned in § 2, where its geometric interpretation in terms of twisting framings was also given. By (2.4) we already know $im(t^o) \subset im(J_U')$. Besides the relation between t^o and J' given by the complex reflection map w, there is onother one: Let F_{S^1} be the limit of the spaces $F_{S^1}(S^n)$ of S^1-equivariant selfmaps of sheres with free S^1-action. In [7] Becker and Schultz constructed a weak homotopy equivalence

$$\lambda: F_{S^1} \longrightarrow Q(S^1 \wedge P_\infty\mathbb{C}^+) \tag{3.1}$$

Now because unitary maps are S^1-equivariant we get a forgetful map

$$\Theta: U \longrightarrow F_{S^1} = Q(S^1 \wedge P_\infty\mathbb{C}^+) \tag{3.2}$$

Let Θ' be the adjoint map of Θ. Using the results of [7] it is easy to prove that the following diagram commutes up to a minus sign:

$$\begin{array}{ccc} \pi_n^s(P_\infty\mathbb{C}^+) & \xrightarrow{t^o} & \pi_{n+1}^s(S^o) \\ \uparrow{\scriptstyle \Theta'} & \nearrow{\scriptstyle J'_U} & \\ \pi_{n+1}^s(U) & & \end{array} \qquad (3.3)$$

<u>Remark:</u>

Another approach to the map Θ' via relative Euler class and difference class in equivariant stable homotopy is given by M.C. Crabb in [10], where also a proof for the commutativity of (3.3) can be found.

By (3.3) and (2.4) we have

<u>Proposition 3.4</u>: $\mathrm{im}(J'_U) = \mathrm{im}(t^o)$

On the stable homotopy groups of a space X there exists a filtration

$$\pi_n^s(X) = F^o \supset F^1 \supset F^2 \supset \dots \supset F^k$$

associated to the Adams spectral sequence for any reasonable homology theory [3]. Here we shall always take the filtration belonging to the BP- or MU-Adams spectral sequence. If it should be necessary to distinguish between skeleton filtration and Adams spectral sequence filtration, we shall call the latter Adams filtration.

This filtration is a good measure for the complexity of elements in stable homotopy. If the space X has torsion-free homology, the elements of Adams filtration 0 can be detected by rational homology. The same is true for filtration 1 using K-theory (see (3.7) below).

On the known part of π_*^s the behaviour of Adams filtration is known. Up to some exceptions at the prime 2 the image of the J-homomorphism gives all of filtration 1. In the case $p \neq 2$ F^1/F^2 is a direct summand and F^2 is called cok (J). There exist infinite families of elements in coker (J) e.g. the β_t and ε_t families in F^2 $(p>3)$ and the γ_t-familie in F^3 $(p>5)$. Besides these families only very few indecomposable elements (e.g. $\varepsilon', \lambda', \mu, \phi, \kappa_s$ and λ_s, $1 \leq s \leq p-3$) are known. Up to now only an upper bound for F^2/F^3 is known, namely the $\mathrm{Ext}^{2,*}$-term of the BP-Adams spectral sequence [29].

The bistable J-homomorphism and the transfer maps t^s have the property that they raise the degree in the Adams filtration by at least 1 . This is a simple consequence of the vanishing of the complex cobordism groups $\widetilde{MU}^o(U)$ and $\widetilde{MU}^{-1}(P_\infty\mathbb{C}^{k\tilde{H}})$ and the multiplicative properties of the filtration (see e.g. [23]). So if $x \in F^2\pi^s_{n+1}(S^o)$ is in the image of t^o, it must come from $F^1\pi^s_n(P_\infty\mathbb{C}^+)$. Because t^o is an $\pi_*^s(S^o)$-module map the lowest dimensional element in cok (t^o) must be indecomposable.

The knowledge of the cofibre of the transfer map t^s allows us to compute

$$t^s: F^1\pi^s_n(P_\infty\mathbb{C}^{s\tilde{H}}) \longrightarrow F^2\pi^s_{n+1}(S^o)$$

modulo higher filtration.

Let $\bar{f}^s: P_\infty\mathbb{C} \to BU$ be the classifying map of the vector bundle s H and $f^s: P_\infty\mathbb{C}^{s\tilde{H}} \longrightarrow MU$ the induced map of Thom spaces. We map the cofibre sequence of t^s into the cofibre sequence

$$S^o \xrightarrow{h} MU \longrightarrow \overline{MU} \qquad (3.5)$$

where h is the Hurewicz map and $\overline{MU}$ is the relative Thom spectrum

MU/S^0 classifying bordism of stably almost complex manifolds with framed boundaries. We get the following commuting diagram:

$$\begin{array}{ccccccc} S^0 & \xrightarrow{i} & P_\infty\mathbb{C}^{(s-1)\cdot\tilde{H}} & \xrightarrow{j!} & S^2P_\infty\mathbb{C}^{s\tilde{H}} & \xrightarrow{t^s} & S^1 \\ \| & & \downarrow{\scriptstyle \bar{\bar{f}}^{(s-1)}} & & \downarrow{\scriptstyle f^s} & & \vdots \\ S^0 & \xrightarrow{h} & MU & \longrightarrow & \overline{MU} & \xrightarrow{\partial} & S^1 \end{array} \qquad (3.6)$$

Here f^s is the map induced by $\bar{\bar{f}}$ and id_{S^0} .

Now the stable homotopy of the lower sequence is the well known bordism sequence

$$\pi_n^s(S^0) \longrightarrow \Omega_n^U(*) \longrightarrow \Omega_n^{U,fr}(*) \xrightarrow{\partial} \pi_{n-1}^s(S^0)$$

Because $\Omega_{2n-1}^U = 0$ the boundary map

$$\partial\colon \pi_{2n-1}^s(\overline{MU}) \longrightarrow \pi_{2n-2}(S^0)$$

is bijective, so we have described t^s by f^s. The point is now that ∂ also raises filtration by 1 in the Adams spectral sequence, simply by definition of this filtration. Thus elements in $F^2\pi_*^s(S^0)$ must come from $F_*^1\pi^s(\overline{MU})$ under ∂ and we have reduced the problem of computing

$$t^s\colon F^1\pi_*^s(P_\infty\mathbb{C}^{s\tilde{H}}) \longrightarrow F^2\pi_*^s(S^0)$$

to a problem in filtration 1 .

To use the K-theory e-invariant instead of the bordism e-invariant we need the following lemma:

Lemma 3.7: Let X be a space with torsion-free homology then the K-theory e-invariant is injective on $F^1\pi_*^s(X)/F^2\pi_*^s(X)$, where the filtration is associated to the MU-Adams spectral sequence.

Proof: This is a well known consequence of the Hattori-Stong-theorem. Under the assumption of torsion-free homology the bordism spectral sequence collapses for the space $X\wedge M(\mathbb{Q}/\mathbb{Z})$, where $M(\mathbb{Q}/\mathbb{Z})$ is a Moore space for $\mathbb{Q}/\mathbb{Z}$. This implies that the Hurewicz map

$$\mu\colon \Omega_*^U(X;\mathbb{Q}/\mathbb{Z}) \longrightarrow K_*(X\wedge MU;\mathbb{Q}/\mathbb{Z})$$

is monomorphic. But μ maps the image of framed bordism into the subgroup $K_*(X;\mathbb{Q}/\mathbb{Z})$. So on the subgroup of sherical classes in $\Omega_*^U(X;\mathbb{Q}/\mathbb{Z})$ the natural map $\Omega_*^U(X;\mathbb{Q}/\mathbb{Z}) \longrightarrow K_*(X;\mathbb{Q}/\mathbb{Z})$ is injective. This immediately gives the conclusion.

Because the spectra $\overline{MU}$ and $P_\infty\mathbb{C}^{s\tilde{H}}$ have torsion-free homology we can apply (3.7) in these cases. Thus we have proved:

Theorem 3.8: Let $x\in\pi_{2n-1}^s(P_\infty\mathbb{C}^{s\tilde{H}})_{(p)}$. If $e(f_*^s(x))$ in $A_{2n+1}(\overline{MU})$ is zero, then $t^s(x)$ is in $F^{2+2(p-1)}\pi_{2n}^s(S^o)_{(p)}$.

Remark:

Because $\partial\colon \pi_*^s(\overline{MU}) \longrightarrow \pi_*^s(S^o)$ factorizes through $\pi_*^s(\overline{BP})$ we can use $\overline{BP}$ instead of $\overline{MU}$ in (3.8).

For later use we reformulate this:

Corollary 3.9: Let $x\in F^1\pi_{2n-1}^s(P_\infty\mathbb{C}^{s\tilde{H}})_{(p)}$ and $e\,f_*^s(x) \neq 0$ in $A_{2n+1}(\overline{BP})$ then $t^s(x)\neq 0$ in $F^2\pi_{2n}^s(S^o)_{(p)}$

We apply this now to the case of the element β_{p+1} in $F^2\ \pi^s_{2n}(S^o)_{(p)}$ $(p>3)$.

Set $n = (p-1)\cdot((p+1)^2-1)-1$. From (1.4) we get

$$\nu_p|\tilde{A}_{2n-1}(P_\infty\mathbb{C})| = p+3$$

and from (1.13) that $A_{2n-1}(P_\infty\mathbb{C})$ has two summands.

An easy calculation with (1.7) shows that the second summand must have order p and is generated by

$$\pi_*(X(2,1)) = \pi_*(\rho^{p^2} + 2^n\rho^{2p^2} + \cdots\quad)/p$$

so

$$A_{2n-1}(P_\infty\mathbb{C}) = \mathbb{Z}/p^{p+2} \oplus \mathbb{Z}/p \tag{3.1o}$$

The proof of the following two propositions is defered to § 6.

<u>Proposition 3.11</u>: The generator of the $\mathbb{Z}/p^{p+2}$-summand in $A_{2n-1}(P_\infty\mathbb{C})$ $(n = (p-1)\cdot((p+1)^2-1)-1)$ is not in the image of the e-invariant.

<u>Proposition 3.12</u>: $f^{(o)}_*(\mathrm{im}(e)) = 0$ in $A_{2n+1}(\overline{BP})$

<u>Remark:</u>

We shall actually construct the elements in im(e) in § 4, so $\mathrm{im}(e) = \mathbb{Z}/p^{p+1} \oplus \mathbb{Z}/p$ for $p \geqslant 3$.

As a corollary of (3.12), (3.11), (3.8) and (3.4) we find

<u>Theorem 3.13</u>: For $p \geqslant 5$ the element β_{p+1} is not in the image of the bistable J-homomorphism.

<u>Remarks:</u>

1. The formulas decribing f^o_* are of the same type for all odd primes, so that the behaviour of t^o on im(e) seems to be

the same for all odd primes. But for p=3 it is known that the element β_4 does not exist.

2. For a counterexample for p=3 one needs more information about the existence of the β_i elements. It is known that $\beta_1, \beta_2, \beta_3, \beta_5$ and β_6 exist, but then one leaves the known part of $\pi_*^s(S^o)_{(3)}$. In § 4 we shall show that the above β_t are in $\mathrm{im}(t^o)$.
3. In the course of the calculations for proving (3.11) and (3.12) we will see that the next β_t which has a chance of not being in $\mathrm{im}(t^o)$ is β_{2t+1} or β_{2t+2}.

Using the Gysin sequence of the sphere bundle of the p^r-th power of the Hopf line bundle one can relate $\pi_n^s(B\mathbb{Z}/p^{r+})$ and $\pi_n^s(P_\infty\mathbb{C}^+)$ by transfer maps (for fixed n and r large the bundle H^{p^r} on $P_n\mathbb{C}$ becomes orientable for stable homotopy). This allows us also to conclude that the transfer maps

$$t_{(r)}: \pi_*^s(B\mathbb{Z}/p^r) \longrightarrow \pi_*^s(S^o)_{(p)}$$

cannot be onto for $r \geqslant 2$, whereas $t_{(1)}$ is onto by the Kahn-Priddy-theorem.

§ 4 Representing stable homotopy by framed manifolds

In this chapter we show how one can use the transfer to construct framed manifolds representing elements in cok (J). Applying the principle that the transfer raises Adams filtration by at least 1 we use the transfer twice, that is we construct elements of filtration 0 in $\pi_*^s(P_\infty\mathbb{C} \times P_\infty\mathbb{C})$ and map then by a double transfer to $\pi_*^s(S^o)$ to get elements of filtration 2 . The transfer $t: \pi_{2n}^s(P_\infty\mathbb{C} \times P_\infty\mathbb{C}^+) \longrightarrow \pi_{2n+1}^s(P_\infty\mathbb{C}^+)$ from filtration 0 to filtration 1 can be described by a simple power series and the transfer from $F^1\pi_{2n+1}^s(P_\infty\mathbb{C})$ to $F^2\pi_*^s(S^o)$ can be computed using (3.9) .

One can use the fact that the transfer raises filtration by 1 not only but several times. There is also a transfer map

$$t\colon \pi_n^s(P_\infty\mathbb{C} \times P_\infty\mathbb{C}^+) \longrightarrow \pi_{n+1}^s(P_\infty\mathbb{C}^+) \tag{4.1}$$

One can define t as induced by the stable map $\tau\wedge 1$, where τ represents t^o or as the transfer of the S^1-bundle $pr^*(ES^1 \to BS^1)$ where $pr\colon P_\infty\mathbb{C} \times P_\infty\mathbb{C} \to P_\infty\mathbb{C}$ is the projection. If one identifies $\pi_n^s(P_\infty\mathbb{C} \times P_\infty\mathbb{C}^+)$ with the equivariant bordism group of free equivariantly framed T^2-manifolds $\Omega_{n+2}^{fr}(T^2;\text{free})$, then t is given by restricting the T^2-action to an S^1-action. Similarly one defines

$$t\colon \pi_m^s(BT^n) \longrightarrow \pi_{m+1}^s(BT^{n-1})$$

If we start in $\pi_{2n}^s(BT^2)$ with elements of filtration o we arrive in $\pi_{2n+2}^s(S^o)$ by applying the transfer twice in filtration at least 2 . How one can compute t from filtration 1 to filtration 2 is solved by (3.9). The necessary information for computing $t\colon \pi_{2n}^s(BT^2) \longrightarrow \pi_{2n+1}^s(P_\infty\mathbb{C})$ from filtration 0 to filtration 1 is given by the next proposition. To state this we need some more notation.

Let y be the element in $K^{-2}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ defined by reducing the power series $1/x - 1/\log(x+1)$ in $K^{-2}(P_\infty\mathbb{C};\mathbb{Q}) = \mathbb{Q}[[x]]$ modulo $\mathbb{Z}$.

Define a map

$$\hat{t}\colon K_o(BT^n) \longrightarrow K_o(BT^{n-1};\mathbb{Q}/\mathbb{Z}) \tag{4.2}$$

by the slant product $\backslash\colon K^{-2}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})\otimes K_o(BT^n) \to K_o(BT^{n-1};\mathbb{Q}/\mathbb{Z})$ with the element y.

Proposition 4.3: For $x \in \pi_{2m}^s(BT^n)$, the value of $e_{\mathbb{C}}t(x)$ in $K_o(BT^{n-1};\mathbb{Q}/\mathbb{Z})/H_{2m}(BT^{n-1};\mathbb{Q})$ is given by $\hat{t}\circ h(e)$, where $h\colon \pi_{2n}^s(BT^n) \longrightarrow K_o(BT^n)$ is the Hurewicz map.

The proof of (4.3) can be found in [24].

If we write $1/x - 1/\log(x+1) = \sum_{m=o} C_{m+1}x^m$, then the coefficients C_n can be calculated recursively by (see [20]):

$$\sum_{m=o}^{n} (-1)^m C_m/(n-m) = 0$$

Elements in $K_o(BT^2)$ invariant under $c\psi^k - k^n$ can be written as a rational linear combination of $b_1^{n-i} \times b_1^i$. We then have

$$t(a \cdot b_1^{n-i} \times b_1^i) = -a \cdot \frac{B_{i+1}}{i+1} b_1^{n-i} \tag{4.4}$$

where B_i is a Bernoulli number (defined by $z/(e^z-1) = \sum B_i z^i/i!$).
This is easily seen by well known formulas relating the c_i to the B_i [20] or using the Chern character.

To calculate $e \circ t$ we need to know the image of the Hurewicz map $h: \pi_{2n}^s(BT^2) \longrightarrow K_o(BT^2)$ or equivalently
$im(h: \pi_{2n}^s(BT^2) \longrightarrow H_{2n}(BT^2))$ or $im(h: \pi_{2n}^s(BT^2) \longrightarrow A_{2n}(BT^2))$. But this is not known in general. In a certain range of dimensions A_* is a good approximation to π_*^s, so one expects
$h: \pi_{2n}^s(BT^2) \longrightarrow A_{2n}(BT^2)$ to be onto at least for low values of n .
That this is the case can easily be seen by comparing the Atiyah-Hirzebruch spectral sequences for $\pi_*^s(X)$ and $A_*(X)$ or by using the vanishing line for the higher Ext-groups in the Adams spectral sequence:

<u>Lemma 4.5</u>: Let p be an odd prime and X an s-connected CW complex with cells only in dimensions which differ by multiples of 2(p-1). Then $h: \pi_{2n}^s(X) \longrightarrow A_{2n}(X)$ is onto as long as $n \leq s+2(p-1)^2 \cdot p$

<u>Proof</u>: If we compare the Atiyah-Hirzebruch spectral sequences for $\pi^s_{2n}(X)$ and $A_{2n}(X)$ using the Hurewicz map we see that h fails to be onto only if more nonzero differentials start on $H_{2n}(X;\pi^s_o)$ than on $H_{2n}(X;A_o)$ (there are no boundaries). By the assumption on X only d_r with $r \equiv 0 \mod 2(p-1)$ can be nonzero. So for the case in which h may fail to be onto, there must be an element of cok (J) in $\pi^s_r(S^o)$ with $r+1 \equiv 0 \mod 2(p-1)$. The first time this happens is $r = \dim \alpha_1 \beta_1^{p-1} = 2(p-1)^2 p-1$. So as long as $2n \leq 2(p-1)^2 \cdot p+s$ the Hurewicz map must be onto and im (h) can be calculated via K-theory.

We can apply this lemma to the case of BT^n and $P_\infty \mathbb{C}^{k\tilde{H}}$ because the p- localizations of suspensions of those spaces split into wedgs of spaces which have cells only in dimensions differing by multiples of $2(p-1)$ (for a proof see [30]).

Obviously the sum

$$Z = \sum_{i=o}^{n} a_i b_1^{n-i} \times b_1^i \qquad (4.6)$$

with $a_i \in \mathbb{Q}$ in $K_o(BT^2;\mathbb{Q})$ is in the eigenspace of $c\psi^k$ for the eigenvalue k^n and the only problem is the integrality of Z . This can be decided by expanding Z in terms of the usual basis $b_i \times b_j$.

<u>Example</u>: Define $x_o = (b_1 \times b_1^p - b_1^p \times b_1)/p$ in $K_o(BT^2;\mathbb{Q})$ then it is easy to see that $x_o \in K_o(BT^2;\mathbb{Z}_{(p)})$ and so $x_o \in A_{2(p+1)}(BT^2)$. We denote a preimage of x_o in $\pi^s_{2(p+1)}(BT^2)$ simply by x_o too.

This way of finding the linear combinations Z as in (4.6) which are in $K_o(BT^2;\mathbb{Z}_{(p)})$ by expanding them in terms of $b_i \times b_j$ is only a reasonable method in lowest dimensions.

We therefore proceed as follows:

<u>Theorem 4.7</u>: Let z be an element in $A_{2r}(P_\infty\mathbb{C}\times P_\infty\mathbb{C})$. Then

$$Q(z):=\frac{1}{p}(z^p-z\circ(b_1^{r\cdot(p-1)}\times 1+1\times b_1^{r\cdot(p-1)}-b_1^{(r-1)\cdot(p-1)}\times b_1^{p-1}))$$

is a well defined class in $A_{2rp}(P_\infty\mathbb{C}\times P_\infty\mathbb{C})$ and $z\mapsto Q(z)$ defines an operator

$$Q : A_{2r}(P_\infty\mathbb{C}\times P_\infty\mathbb{C}) \longrightarrow A_{2pr}(P_\infty\mathbb{C}\times P_\infty\mathbb{C}).$$

Here z^p denotes the p-th power of z in the Pontrjagin product on $A_*(P_\infty\mathbb{C}\times P_\infty\mathbb{C})$.

<u>Proof</u>: Clearly $Q(z)\in K_o(P_\infty\mathbb{C}\times P_\infty\mathbb{C};\mathbb{Q})$ is in $\ker(c\psi^k-k^{rp})$. We must show that Q(z) is in $K_o(P_\infty\mathbb{C}\times P_\infty\mathbb{C};\mathbb{Z}_{(p)})$, that is

$$<x^i\times x^j,Q(z)>\in \mathbb{Z}_{(p)} \text{ for all } i,j\geq o.$$

This is equivalent to

$$<(1+x)^m\times(1+x)^n,Q(z)>\in\mathbb{Z}_{(p)} \text{ for all } n,m\geq o.$$

Set $z_{m,n}=<(1+x)^m\times(1+x)^n,z>\in\mathbb{Z}_{(p)}$. Because

$$<(1+x)^m\times(1+x)^n,a\circ b> = <(1+x)^m\times(1+x)^n,a>\cdot<(1+x)^m\times(1+x)^n,b>$$

we have

$$<(1+x)^m\times(1+x)^n,Q(z)> = \frac{1}{p}\,(z_{m,n}^p-z_{m,n}\circ(m^{r\cdot(p-1)}+n^{r\cdot(p-1)}-m^{(r-1)\cdot(p-1)}n^{p-1})) \tag{4.8}$$

If n or m is prime to p, then $m^{r(p-1)}+n^{r(p-1)}-m^{(r-1)(p-1)}n^{p-1}\equiv 1$ mod p and $z_{m,n}^p=z_{m,n}$ mod p and the right hand side of (4.8) is an integer. If n and m are both divisible by p, then on the right hand side there remains $z_{m,n}^p/p$. But in this case we have $z_{m,n}\equiv 0$ mod p because $(1+x)^{\bar{m}p}\times(1+x)^{\bar{n}p}=\psi^p((1+x)^{\bar{m}}\times(1+x)^{\bar{n}})$ and $c\psi^p(z)=p^r\cdot z$. So $z_{m,n}^p/p\in\mathbb{Z}_{(p)}$.

Remarks:

1. If $z \in A_{2r}(BT^2)$ comes from stable homotopy, then so does $p \cdot Q(z)$, because $b_1^i \times b_1^j$ is the image of $\sigma^i \times \sigma^j$ where $\sigma \in \pi_2^s(P_\infty \mathbb{C}^+)$ is given by the Hopf bundle over S^2. So the only problem in establishing Q as an operator in stable homotopy is the divisibility by p.

2. One has other choices for the form of the second summand $z \circ (1 \times b_1^a + b_1^a \times 1 - b_1^{p-1} \times b_1^{a-p+1})$ in the definition of Q, e.g. for $a = \dim(z)/2$ even:

$$Q(z) = \frac{1}{p}(z^p - z \circ (1 \times b_1^{a \cdot (p-1)} + b_1^{a \cdot (p-1)} \times 1 - (b_1 \times b_1)^{(a \cdot (p-1)/2)}))$$

As one can easily see this does not affect the property of Q(z) of being in the image of $h: \pi_{2ap}^s(BT^2) \longrightarrow A_{2ap}(BT^2)$.

3. There are additional operators for $p=2$ using the fact that $\nu_2(k^t - 1) = 1$ if $t \not\equiv 0$ (2) and $\nu_2(k^t - 1) = 2 + \nu_2(t)$ if $t \equiv 0$ (2).

For example $Q_3(z) = \frac{1}{8}(z^3 - z(1 \times b_1^{2a} + b_1^{2a} \times 1 - b_1^2 \times b_1^{2a-2}))$ and $\dim(z) = a \geq 6$.

4. One can define similar operators on $A_{2n}(BT^m)$.
Example: $m=3$, $n \geq 2$. Set $b := p-1$, $a := nb$, $c := (n-1) \cdot b$, $d := (n-2)b$ and

$$S(n) = \sigma^a \times 1 \times 1 + 1 \times \sigma^a \times 1 + 1 \times 1 \times \sigma^a - \sigma^c \times \sigma^b \times 1 - \sigma^c \times 1 \times \sigma^b - 1 \times \sigma^c \times \sigma^b + \sigma^b \times \sigma^b \times \sigma^d$$

Then $z \longmapsto (z^p - z \circ h(S(n)))/p$ defines an operator on $A_{2n}(BT^3)$.

The operator Q on $A_{2*}(BT^2)$ has the following properties:

Proposition 4.8: a) $Q(x+y) = Q(x) + Q(y) + \frac{1}{p} \sum_{i=1}^{p-1} x^i \circ y^{p-i} \cdot \binom{p}{i}$

b) $Q(x \circ y) \equiv Q(x) \cdot y^p + Q(y) \cdot x^p \mod p$ ($A_{2n}(BT^2)$ is torsion free).

The proof of (4.8) follows directly from the definition and some straigthforward calculations.

Property b) shows that Q is most interesting on indecomposable elements of $A_{2*}(BT^2)$. We define inductively

$$x_i := Q\,(x_{i-1}) \tag{4.9}$$

where $x_o = (b_1 \times b_1^p - b_1^p \times b_1)/p.$

<u>Example</u>: p=3

$$x_1 = 1/9 \cdot b_1^{11} \times b_1 - 1o/81 \cdot b_1^9 \times b_1^3 - 2/27 \cdot b_1^7 \times b_1^5 + 2/27 \cdot b_1^5 \times b_1^7 + 1o/81 \cdot b_1^3 \times b_1^9 - 1/9 b_1 \times b_1^{11}$$

(We have used the symmetric form of Q as in remark 2 above).

By (4.5) it is then clear that x_o and x_1 come from stable homotopy. With a little work and some information on $\pi_*^s(P_\infty \mathbb{C})_{(p)}$ (see e.g. § 6) one can prove that x_2 is in the image of $h: \pi_*^s(BT^2) \longrightarrow A_*(BT^2)$ too. This is most interesting for the prime 3 as we shall see later. Using now the Pontrjagin product on $\pi_*^s(BT^2)$ we can construct out of the elements x_o, x_1 and σ a lot of further elements going far beyond the dimension limit where we know $h: \pi_{2n}^s(BT^2) \longrightarrow A_{2n}(BT^2)$ to be onto.

Suppose given a compact Lie group G with maximal torus T^n. In [23] it is shown how G/T^n defines an element $(G/T^n, f, L)$ in $\pi_*^s(BT^n)$ and how to calculate the image of this element in $H_*(BT^n)$ using the root system. This can be used to express the elements x_o in terms of homogeneous spaces for low primes. For p=2 the element x_o is given by $SU(3)/T^2$ with the canonical T^2-bundle on it. It turns out that $x_1 = Q(x_o)$ for p=2 is essentially the element given by G_2/T^2, the first exeptional Lie group. Similarly one finds: For p=3 x_o is given by $Sp(2)/T^2$ and for p=5 by G_2/T^2 .

It is an unsolved problem to give an explicit construction for x_o and x_1 or a description in terms of known manifolds for the other primes. Even more interesting would be a geometric construction describing the operator Q in stable homotopy (if it exist).

We proceed to calculate $t(x_o^i x_1^j)$ and $t \cdot t(x_o^i x_o^j)$ for some i,j . As already mentioned, the p-localization of the suspension of $P_\infty\mathbb{C}$ splits into (p-1) pieces

$$SP_\infty\mathbb{C}_{(p)} \simeq X_1 \vee X_2 \vee \dots \vee X_{(p-1)} \qquad (4.10)$$

The e-invariant calculation of [24] for the stable map τ representing t^o and (2.5) shows that τ restricts to zero on X_j with $j \neq (p-2)$. So only the space X_{p-2} is relevant for the transfer t^o.

Set
$$y_i := x_1^{i-1} \cdot x_o^{p-i-1} \qquad \text{for} \quad 1 \leq i \leq p-1$$

$$y_p := x_1^{p-2} \circ x_o^{p-1}$$

$$y_{p+1} := x_1^{p-1} \circ x_o^{p-2}$$

or in general

$$y_t := \Pi x_o^{\alpha_o} \circ x_1^{\alpha_1} \cdots \circ x_r^{\alpha_r}$$

where $t(p-1)-1 = \sum_{i=o}^{r} \alpha_i p^i$ $(o \leq \alpha_i < p)$. We then know that y_i for $i \leq p+1$ come from stable homotopy and we denote a inverse image of y_i by the same symbol. The y_i are all supported by the component $X_{p-2} \wedge X_{p-2} \subset S^2 \wedge BT^2_{(p)}$.

<u>Theorem 4.11</u>: Let p be an odd prime and $\beta_i \in \pi_r^s(S^o)_{(p)}$ the i-th element

of the β_t-family $(r=2(p-1)\cdot(i(p+1)-1)-2)$.

Then $\quad t\circ t(y_i) \doteq \beta_i \quad$ for $i\leq p$

and $\quad t\circ t(y_{p+1}) = 0$

Remark:

If we had proved that x_2 comes from stable homotopy then the same method would show $t\circ t(y_i) \doteq \beta_i$ for $2+p\leq i\leq 2p$.

(4.11) will be proved at the end of § 5. As preparation for this we collect some properties of $t(y_i)$:

Proposition 4.12: a) The order of $t(y_i)$ is as follows:

$$|t(y_1)| = p, \quad |t(y_i)| = p^{i+1} \quad \text{for } 1<i<p,$$

$$|t(y_p)| = p^p \text{ and } |t(y_{p+1})| = p^{p+1}$$

b) The elements $t(y_i)$ are of the form

$$t(y_i) = \frac{1}{p}\, b_{p(i(p-1)-1)} + \text{lower terms } (i\neq p)$$

$$t(y_p) = \frac{1}{p^2}\, b_{p^3-p^2-p} + \text{lower terms}$$

Proof: We choose the symmetric form of the operator Q as in remark 2 above. If we interchange the two factors of BT^2 then x_i goes over into $-x_i$. We recall

$$x_o = \frac{1}{p}(b_1\times b_1^p - b_1^p\times b_1) \tag{4.13}$$

$$x_1 = -\frac{1}{p^2}\cdot b_1\times b_1^{p^2-p+1} + \frac{1+p^{p-1}}{p^{p+1}}\cdot b_1^p\times b_1^{p^2} + \frac{a}{p^p}\cdot b_1^{2p-1}\times b_1^{p^2-p+1} + \cdots\cdots$$

a) We first determine the order of $t(y_i)$ by applying lemma (1.14) and formula (4.4) for the transfer. We need only consider summands $a\cdot b_1^i\times b_1^j/p^k$ in which k is larger than i-2, because the transfer can introduce only a factor b/p^2 when $i\leq p-1$.

1. The case of y_1 is handled separately : $t(y_1)$ has order p
2. The element $x_1^i x_o^{p-2-i}$ $(i>o)$ starts with the terms

$$\pm\binom{i}{j}(1+p^{p-1})^{(i-j)}\ b_1^{j+(i-j)\cdot p} \times b_1^{*} / p^{j+(i-j)\cdot p+i} \quad (o\leq j\leq i) \tag{4.14}$$

where in the notation $b_1^i \times b_1^*$ the star means a number with $i+*$ = dimension of the element in consideration.
Only for $j=1$ do we find $*+1\equiv o(p)$, so the transfer gives for $j=1$ the factor $1/p^2$, whereas for $j\neq 1$ we get a factor $1/p$. In the sum $\sum_k c_k b_1^k$ which we get after applying m_{p^*} we thus have $|c_{1+(i-1)p}| = p^{i+2}$ and $|c_r| = p^s$ with $s<i+2$ for $r\neq (i-1)p+1$. So no cancellation in the contributions of the different terms to b_1 can take place and $t(y_i)$ must have order p^{i+2} .
3. In the same way one finds the order of $t(y_p)$.
4. One readily checks that one has only to consider the same terms in (4.14) for $x_1^{p-1} \circ x_o^{p-2}$. As in 2. we find the maximal denominator for $j=1$. Then $|t(y_{p+1})| = p^{p+1}$.

b) The next task is to find the maximal filtration of $t(y_i)$
1. The skeleton filtration of $t(y_1)$ is $2p(p-2)$, that is to say

$$t(y_1) = \frac{1}{p}\, b_{p(p-2)} + \text{terms of lower filtration.}$$

2. The case $x_1^i x_o^{p-2-i}$ for $i\leq p-2$

We look at the same terms as in (4.14) but interchange the factors of BT^2 . Putting $j=o$ we find a term

$$\frac{1}{p^{p-2+ip}} \cdot b_1^{p\cdot(p-2+i(p-1))} \times b_1^{(p-2)+i\cdot(p-1)} \tag{4.15}$$

We have $\nu_p(\, p(p-2+i(p-1))!) = p-2+pi$ and $b_1^n = n!b_n +$ lower terms. So by applying the transfer we get from (4.15)

$$\frac{1}{p}\; b_{p(p-2+i(p-1))} + \text{lower terms}$$

Terms of filtration k with $k \geq p(p-2+i(p-1))$ can only come from $b_1^{p-2+(i-r)\cdot(p-1)} \times b_1^{*}$ with $o<r \leq i$. But the denominator of a product $a\cdot b_1^k \times b_1^e$ is always less than or equal to $p^{\min(k,e)+i}$ by construction of $x_1^i \cdot x_o^{p-2-i}$. Thus in those terms the denominator is strictly less than $p^{(p-2+pi)}$ even after applying the transfer. The maximal filtration is therefore $p(p-2+i(p-1))$.

An easy calculation shows that by adding a multiple of b_1^n with $2n-1 = \dim t(y_i)$ (= kernel of the Bockstein map) we cannot reduce the filtration; but for what follows this is not needed.

3. $t(y_p)$ has the form $\frac{1}{p^2}\, b_{p^3-p^2-p} +$ lower terms
4. As in 2. one finds a term $\frac{1}{p}\, b_{p(p^2-2)}$ for $j=o$ and all other terms give lower filtration.

As a corollary of (4.11) we can state:

Corollary 4.16: Let p be an odd prime. Then up to dimension $n = 2(p-1)\cdot(p^2+1)-4$ all elements in the p-component of $\pi_i^s(S^o)$ can be constructed from the three manifolds σ, x_o and x_1 using only product and Pontrjagin product.

The reason for stopping with dimension $2(p^2+1)\cdot(p-1)-4$ is the element ε_1. The elements ε_i are still in the image of the transfer t^o. One can show this by the same method as above, but with some alterations. Let $\bar{\varepsilon}_i$ denote an inverse image of ε_i in $\pi_*^s(P_\infty\mathbb{C}^+)$, then $e(\bar{\varepsilon}_i)$ generates the second summand in $A_{2m-1}(P_\infty\mathbb{C})$, that is

$$e(\bar{\varepsilon}_i) = \pi_*(x(2,1)) \qquad \text{(see § 1)} .$$

So the appearance of ε_i goes parallel to the appearance of the second summand of the transfer component in $A_*(P_\infty\mathbb{C})$. Besides the inverse image $t(y_p)$ already constructed, the element $\beta_p = \varepsilon_{p-1}$ has an inverse image of order p in $\pi_*^s(P_\infty\mathbb{C})$. But the elements $\bar{\varepsilon}_i$ $(i< p-1)$ are not in the image of

$$t\colon \pi_{2n}^s(BT^2) \longrightarrow \pi_{2n+1}^s(P_\infty\mathbb{C})$$

as some elementary calculations show.
To get a lift to filtration o one could try to use the other transfers t^s instead of t^o, but I have not done this.

The fact that β_{p+1} is not in the image of t^o does not give an insurmountable obstruction in a programme of representing stable homotopy elements by free T^n-manifolds. One must only use instead of t^o one of the generalized transfers: For example β_{p+1} lies in the image of $t^{p(p-1)}$ for $p>3$.

For p=3 there exist only parts of the β_t-family. Using the fact that x_2 is in the image of stable homotopy we can construct β_9 which seemed to be unknown up to now:

<u>Proposition 4.17</u>: In the 3-component of $\pi_*^s(S^o)$ we get from x_o, x_1, x_2 the elements $\beta_1, \beta_2, \beta_3, \beta_5, \beta_6, \beta_9$

The method of proof is the same as for (4.11). We therefore omit the calculations.

The elements x_o, x_1 can also be used to construct generators for $\pi_*^s(P_\infty\mathbb{C})$. Up to the dimension $n = 2(p-1)\cdot(p^2+1)-3$ nearly all elements of $e(\pi_*^s(P_\infty\mathbb{C})_{(p)})$ can be constructed from x_o, x_1 and σ using Pontrjagin products and the transfer; see § 6.

At present, for the odd primes, no element in $\pi_*^s(S^o)_{(p)}$ is known which is in the cokernel of all t^s.
For the prime 2, the situation is different; see § 7.
For the construction of the μ_t-family of Adams [2] using the transfer t^s, see § 7.

§ 5 Some computations with $A_*(BP)$

This chapter contains the computations which are needed in § 3, § 4 and § 6. First the elements of order p in $A_{2n-1}(BP)$ are worked out and then it is shown that one can compute in $A_{2n-1}(BP)$ modulo elements of certain filtration. This allows us to use the formulas for the map $f: S^2P_\infty\mathbb{C} \longrightarrow \overline{BP}$ which is needed to compute the transfer, in usual homology instead in K-theory.
We then compute the values of f_* mod p and mod $t_i (i > 1)$, and close with the proof of (4.11).

In actual computations it seems to be better to use BP-homology instead of complex bordism. We therefore recall some well known facts about BP; for proofs and notation see [3].
The Brown-Peterson spectrum for the fixed prime p is a ring spectrum which fits into the commuting diagram

$$\begin{array}{ccc} MU\mathbb{Z}_{(p)} & \xrightarrow{\ \varepsilon\ } & MU\mathbb{Z}_{(p)} \\ & {}_{\pi}\searrow \quad \nearrow_{i} & \\ & BP & \end{array}$$

with $\pi \circ i = id_{BP}$. We have $H_*(BP;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[q_1,\ldots q_i]$ and $\pi_*^s(BP) = BP_* = \mathbb{Z}_{(p)}[v_1,\ldots,v_s\ldots]$ where q_s and v_s both have degree $2\cdot(p^s-1)$. The Hurewicz map $h: BP_* \to H_*(BP;\mathbb{Z}_{(p)})$ is a monomorphism, so we can identify BP_* with the subring $h(BP_*)$ of $H_*(BP;\mathbb{Z}_{(p)})$. The generators v_s are defined inductively

$$h(v_s) = p\cdot q_s - \sum_{j=i}^{s-1} q_j h(v_{s-j})^{p^j} \tag{5.1}$$

In $H_*(BP;\mathbb{Z}_{(p)})$ $h(v_s)$ is divisible by p but not by p^2. We have $BP_*(BP) = \pi_*^s(BP)\,[t_1,t_2\ldots.]$ with $t_i \in BP_{2(p^i-1)}(BP)$. To describe the Hurewicz map $\eta_R: BP \longrightarrow BP_*BP$ we identify $BPQ_*(BP)$ with $H_*(BP;Q)[t_1,\ldots t_n]$. Then

$$\eta_R(q_k) = \sum_{i+j=k} q_i t_j^{p^i} \tag{5.2}$$

We define elements $t_i \in K_o(BP;\mathbb{Z}_{(p)})$ by mapping $t_i \in BP_*BP$ into $MU_*(BP)$ using i and then by the Todd map to $K_*(BP)$

<u>Lemma 5.3</u>: $K_o(BP;\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[t_1,\ldots.]$

<u>Proof</u>: Clearly $M = \mathbb{Z}_{(p)}[t_1,\dots t_i] \subset K_o(BP;\mathbb{Z}_{(p)})$. We know that $K_o(BP;\mathbb{Z}_{(p)})$ is torsion-free and that rationally $M \otimes \mathbb{Q} \cong K_o(BP;\mathbb{Q})$. Let G denote the multiplicative part in the splitting of $K(-,\mathbb{Z}_{(p)})$ into (p-1)-homology theories [4], then $BP_*BP \to G_*(BP)$ is surjective [6] with image $\mathbb{Z}_{(p)}[t_1,\dots t_r,\dots]$. But $G_*(X)$ is a direct summand of $K_*(X;\mathbb{Z}_{(p)})$ so M must be a direct summand and we are done.

The Hurewicz map η_R followed by i and then by the Todd map is the Hurewicz map h_K: $BP_* \to K_*(BP;\mathbb{Z}_{(p)})$. Because $q_n = [P_{p^n-1}\mathbb{C}]/p^n$ in $\pi_*^s(MU)\otimes\mathbb{Q}$, the Todd map sends q_n to $1/p^n$ in $K_o(BP;\mathbb{Q})$. Therefore it follows from (5.2):

$$h_K(q_k) = \sum_{i+j=k} \frac{1}{p^i} \cdot t_j^{p^i} \tag{5.4}$$

Because $h\otimes\mathbb{Q}$ is the inverse map to the Chern character (5.4) allows us to compute $ch(t_j)$.

We have $A_{2n}(MU;\mathbb{Q}/\mathbb{Z}) \subset K_o(MU;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[b_1,\dots,b_k,.]$, set $\deg(b_i)=i$ and extend this in the obvious way to all elements in $K_o(MU;\mathbb{Q}/\mathbb{Z})$. Similarly for $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ where we set $\deg(t_i)=(p^i-1)/(p-1)$.

<u>Proposition 5.5</u>: Let $x \in A_{2n}(MU;\mathbb{Q}/\mathbb{Z})$ be an element of degree r. If n is sufficiently large compared with r, then $\beta(x) \in A_{2n-1}(MU)$ is zero.

<u>Proof</u>: We have $A_{2n}(MU;\mathbb{Q}/\mathbb{Z}) = A_{2n+2N}(MU(N);\mathbb{Q}/\mathbb{Z})$ for N large. Let $BU(N,q)$ denote the q-skeleton of $BU(N)$ and $MU(N,q)$ the Thom space of the restricted universal bundle $EU(N)$. Because there exists a q-dimensional bundle E, stably isomorphic to $EU(N)$ restricted to $BU(N,q)$, we have $MU(N,q) = S^{2N-q} \wedge BU(N,q)^E$. The assumption on x implies that we can find an element $\bar{x} \in A_{2n+q}(BU(N,q)^E;\mathbb{Q}/\mathbb{Z})$ with

$q \leq r+1$ which maps to x under suspension and the map induced by inclusion. Now $X = BU(N,q)^E$ is a 2q-dimensional CW-complex, which can therefore be embedded in S^{4q+2}. Let D(X) be the 4a+2 -Spanier-Whitehead dual of X. Then

$$A_{2n+2q}(X;\mathbb{Q}/\mathbb{Z}) \cong A^{3q+1-2n}(D(X);\mathbb{Q}/\mathbb{Z})$$

If r is sufficiently small compared with n, we can choose q such that $3q+1-2n$ is still negative. But in the negativ range the e-invariant

$$\pi_s^i(X) \longrightarrow A^i(X)$$

is surjective [43]. This means that $\bar{x}$ is in the image of e on $\pi^s_{2n+q}(BU(N,q)^E)$ and so the same is true for x. But if $x \in A_{2n}(MU;\mathbb{Q}/\mathbb{Z})$ comes from stable homotopy, then $\beta(x) = o$ because $\pi^s_{2n-1}(MU) = 0$.

<u>Corollary 5.6</u>: Let $x \in A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ be an element of degree $\leq r$. If n is sufficiently large compared with r, then $\beta(x) \in A_{2n-1}(BP)$ is zero.

<u>Proof</u>: Because $\pi: A_{2n}(MU;\mathbb{Q}/\mathbb{Z}) \longrightarrow A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ is onto we can find a polynomial in the b_i which maps to x. In an inverse image of t_i we only need b_j with $j \leq (p^i-1)$, thus we can find an inverse image y for x with degree $y \leq r(p-1)$ and can apply (5.5).

Let $x \in A_{2n}(BP;\mathbb{Q}/\mathbb{Z}) \subset K_o(BP;\mathbb{Q}/\mathbb{Z})$ be an element of order p, then x also lies in $A_{2m}(BP;\mathbb{Q}/\mathbb{Z})$ with $m=n+t(p-1)$, $t \geq 0$, because $(c\psi^k-k^n)(x)=0$ implies $(c\psi^k-k^m)(x)=0$ in this case.

For some large t (5.6) implies that the element x is in the kernel of β, or what is the same, it is of the form y/p where $y \in A_{2n+2t(p-1)}(BP)$. By the Hattori-Stong theorem h: $\pi^s_{2m}(BP) \to A_{2m}(BP)$

is an isomorphism, so y is a polynomial in the elements $h(v_i)$.

Example: We have $\psi^k(t_1) = k^{(p-1)}\cdot t_1 + \frac{k^{(p-1)}-1}{p} \equiv t_1 + a \bmod p$

So $\psi^k(t_1^p - t_1) = t_1^p - t_1 \bmod p$

Therefore $x = (t_1^p - t_1)/p$ defines an element in $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ with $n = p\cdot(p-1) + t\cdot(p-1)$. For t=0 we have $\beta(x) \neq 0$ because $\ker(\beta)$ consists of the multiples of $h(v_1^p) = (pt_1+1)^p$. The element $\beta(x)$ (t=0) is the first nonzero element in $A_i(BP)$ with i odd. Suppose now $t > o$. Because $\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \bmod p$ in $BP_* BP$ [34] we have $h(v_2) \equiv t_1^p - t_1 \bmod p$ in $K_o(BP;\mathbb{Z}_{(p)})$. So $x = h(v_2 v_1^{t-1})/p$ and therefore $\beta(x) = o$ in this case. Similarly $(c\psi^k - k^n)(t_1^p - t_1)^k \equiv 0 \bmod p$ $(n \equiv 0(p-1))$ and $(t_1^p - t_1)^k/p$ defines an element in $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ with $n = kp(p-1) + s(p-1)$ $(k \geq 1, s \geq o)$. For $k \leq p$ it is easy to see, that this element has nonzero Bockstein as long as $s < k$. We denote this element by $h(v_2^k)/p$.

As an immediate corollary of (5.6) we get

Corollary 5.7: The elements of order p in $A_{2n(p-1)}(BP;\mathbb{Q}/\mathbb{Z})$ are of the form y/p where y is a polynomial in $h(v_i)$ and $\deg(y) \leq n$.

Using the Bockstein sequence

$$A_{2n}(BP) \xrightarrow{p} A_{2n}(BP) \longrightarrow A_{2n}(BP;\mathbb{Z}/p) \xrightarrow{\beta} A_{2n-1}(BP)$$

one can easily work out which of these polynomials have nonzero Bockstein.

The order of $A_{2n-1}(BP)$ can be computed by a similar method to that used for $P_\infty\mathbb{C}$. We know the Hurewicz map $A_{2n}(BP) \to H_{2n}(BP)$

and therefore the order of the differentials in the Atiyah-Hirzebruch spectral sequence on the bottom line $H_*(BP;A_o)$. Because there are no other differentials for dimensional reasons, and we know the number of elements in the E_2-term, we can compute the number of elements in the E_∞-term.

<u>Lemma 5.8</u>: For $n < p(p^2-1)$ the group $A_{2n-1}(BP;\mathbb{Q}/\mathbb{Z})$ is cyclic.

<u>Proof</u>: Because $h(v_3) \equiv t_2^p - t_2 +$ terms in t_1 mod p (see [34]) has degree $p(p+1)$, elements of order p in dimensions less than $2p(p^2-1)$ can be expressed using only $h(v_1)$ and $h(v_2)$. One easily sees that the only elements not in ker (β) can be given by $h(v_2^k)/p$ for some k, that is by the elements discussed above. If there are two values of k such that $h(v_2^k)/p$ is in $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$, then there is only one with nontrivial image under β.

For later use we work out $A_{2m}(BP;\mathbb{Z}/p)$ for $m = \deg(v_3 v_2) = p(p+1)+p$.

<u>Lemma 5.9</u>: The elements $h(v_3v_2)/p, h(v_2^{p+1})/p$ and $h(v_2^{p+2})/p$ are in $A_{2m}(BP;\mathbb{Z}/p)$ and have nonzero image under β. The group $A_{2m-1}(BP)$ has 3 cyclic summands.

The proof of (5.9) is an easy calculation which we omit.

<u>Remarks</u>:

1. The cofibre sequence $S^o \longrightarrow BP \longrightarrow \overline{BP}$ shows

$$A_{2m-1}(\overline{BP}) \cong A_{2m-1}(BP) \qquad (5.1o)$$

2. A cofibre sequence which induces short exact sequences in BP-homology induces a long exact sequence of Ext-groups. In this way one can map

$Ext^{o,*}_{BP_*BP}(BP_*, BP_*(\overline{BP};\mathbb{Q}/\mathbb{Z}))$ surjectively onto $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$

using the two boundary maps

$$\mathrm{Ext}^{0,*}(BP_*, BP_*(\overline{BP};\mathbb{Q}/\mathbb{Z})) \longrightarrow \mathrm{Ext}^{1,*}(BP_*, BP_*(\overline{BP})) \longrightarrow \mathrm{Ext}^{2,*}(BP_*, BP_*)$$

But $\mathrm{Ext}^{0,*}(BP_*, BP_*(\overline{BP};\mathbb{Q}/\mathbb{Z}))$ = set of primitive elements in $BP_*(\overline{BP};\mathbb{Q}/\mathbb{Z})$ can be mapped into $A_{2*}(\overline{BP};\mathbb{Q}/\mathbb{Z})$ and then using the Bockstein map into $A_{2*-1}(\overline{BP})$. This defines an injective map

$$\mathrm{Ext}^{2,2n+1}_{BP_*BP}(BP_*, BP_*) \longrightarrow A_{2n-1}(BP) .$$

The element β_s of the β_t-family is described in $\mathrm{Ext}^{0,*}(BP_*, BP_*(\overline{BP}_*;\mathbb{Q}/\mathbb{Z}))$ by $(\eta_R(v_2)^s - v_2^s)/pv_1$ and maps to $h(v_2)^s/p$ in $A_{2n-1}(\overline{BP})$. So we know the e-invariant of an inverse image of β_s under the boundary map $\partial: \pi_*^s(\overline{BP}) \longrightarrow \pi_*^s(S^0)$.
The formulas for $\eta_R(v_2)^s$ and $h(v_2)^s$ show why $A_{2n-1}(\overline{BP})$ is much larger than $\mathrm{Ext}^{1,2n}_{BP_*BP}(BP_*, BP_*(\overline{BP}))$.

For connected spectra like BP there exists an Atiyah-Hirzebruch spectral sequence. There is a filtration of BP, denoted by BP^r whose associated graded groups are the E^∞-terms of the E^2-terms $E^2_{s,t} = H_s(BP;A_t(*))$. In the group $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$ this filtration is the one given by degree (up to the factor $2(p-1)$). The statement of (5.7) means then that nonzero elements in $A_{2n-1}(BP)$ must have high filtration. We can use this observation to simplify some calculations. For late use we restrict to the following dimensions $m = 2(p-1)\cdot(p^k+p-1)$ and $k \leq p+1$, although the statement of the following proposition seems to be true in general.

<u>Proposition 5.11</u>: The canonical map $j_*: A_{2m-1}(BP) \longrightarrow A_{2m-1}(BP/BP^r)$ with $r = 2(p-1)(pk-2)$ and $m = (p-1)\cdot(pk+k-1)$ $k \leq p+1$ is injective.

(5.11) means that in $A_{2m}(BP;\mathbb{Q}/\mathbb{Z})$ we can calculate modulo terms of degree $\leq(pk-2)$ without losing $A_{2m-1}(BP)$. We identify $H_*(BP;\mathbb{Q})$ with its image in $K_o(BP;\mathbb{Q}/\mathbb{Z})$. For the proof of (5.11) we require the following lemma:

<u>Lemma 5.12</u>: If we have in $K_o(BP;\mathbb{Q})$ the following equation

$$\sum_{i=o}^{p} a(i)\cdot q_2^i q_1^{a-i(p+1)} + a(p+1)q_3q_1^{(p-1)} \equiv 0$$

mod $\mathbb{Z}_{(p)}$ and mod degree $\leq p^2+p-2$ where $a = p^2+2p$, then $a(i) \in \mathbb{Z}_{(p)}$ and moreover $a(i) \equiv 0 \bmod p^{p-2}$.

<u>Proof:</u> The statement $a(i) \in \mathbb{Z}_{(p)}$ follows from the fact that the terms of highest degree in $q_2^i q_1^{a-i(p+1)}$ are linearly independent.

Then by an easy calculation one sees that the coefficient of $t_2^j t_1^{p^2+p-j(p+1)}$ in $q_2^k q_1^{p^2+2p-j(p+1)}$ for $k>j$ has denominator at most p^p. The number $a(p+1)$ must be divisible by $p^{(p-1)}$ because of the term t_3/p^{p-1} in degree p^2+p+1 of $q_3q_1^{(p-1)}$. In degree p^2+p the term t_2^p appears in $q_2^p q_1^{p^2+2p-(p+1)p} = q_2^p q_1^p$ with denominator p^p and in $a(p+1)q_3q_1^{(p-1)}$ with denominator $\leq p$. So $a(p) \equiv o \ (p^{p-1})$. The term $t_2^{p-1}t_1^{p+1}$ appears in $q_2^p q_1^p$ and in $q_2^{p-1}q_1^{2p+1}$. Its denominator in $q_1^{p-1}q_2^{2p+1}$ is p^p and in $a(p)\cdot q_2^p q_1^p$ at most p. So at least $a(p-1) \equiv o(p^{p-1})$. Proceeding inductively the conclusion follows. One factor p is lost because the denominator of the coefficient of $t_2t_1^{p^2-1}$ in $q_2q_1^{p^2+p-1}$ is only p^{p-1}.

Proof of (5.11):

1. We first treat the case $k \leq p$. Then $A_{2m-1}(BP)$ is cyclic and the element of order p is given by $\beta h(v_2{}^k)/p$. In $A_{2m}(BP/BP^r;\mathbb{Q}/\mathbb{Z})$ we have

$$h(v_2^k)/p = \frac{1}{p}\cdot(t_1^p - t_1)^k \equiv \frac{1}{p}\cdot t_1^{pk} \qquad (5.13)$$

Suppose that t_1^{pk}/p lies in the kernel of β: $A_{2m}(BP/BP^r;\mathbb{Q}/\mathbb{Z}) \longrightarrow A_{2m-1}(BP/BP^r)$, that is in the image of $A_{2m}(BP/BP^r;\mathbb{Q}) \to A_{2m}(BP/BP^r;\mathbb{Q}/\mathbb{Z})$; then we can write

$$\frac{1}{p}\cdot t_1^{pk} = \sum_{i=o}^{k-1} a_i\cdot q_1^{(p+1)(k-i)-1}\cdot q_2^i \qquad (5.14)$$

But if we look at the coefficients of t_1^{pk} and t_1^{pk-1} in $q_1^{(k-i)(p+1)-1}q_2^i$ we see that they are equal. So an equation like (5.14) cannot exist and $\beta(t_1^{pk}/p) \neq 0$ in $A_{2m-1}(BP/BP^r)$.

Remark: If we try to compare a given element $z \in A_{2m}(BP;\mathbb{Q}/\mathbb{Z})$ with $h(v_2^k)/p$ we may not only calculate mod degree $\leq pk-2$ but can also introduce the additional relation $t_2 = 0$. Then $h(v_2^k)/p$ is still nonzero modulo those relations, which then all reduce to $t_1^{pt}/p = -t_1^{pt-1}/p$.

2. The case $k=p+1$.

As for $k \leq p$ the coefficients of $t_1^{p^2+p}$ and $t_1^{p^2+p-1}$ are the same in $q_2^i\, q_1^{p^2+2p-i\cdot(p+1)}$ with the one exception $i=1$, where we have

the coefficient of $t_1^{p^2+p}$ in $q_2\cdot q_1^{p^2+p-1}$ is $\binom{p^2+p-1}{p-1}/p^p$

the coefficient of $t_1^{p^2+p-1}$ in $q_2\cdot q_1^{p^2+p-1}$ is $\binom{p^2+p-1}{p-1}/p^p + \frac{1}{p^2}$

The group $A_{2m-1}(BP)$ is a sum of 3 cyclic groups and the elements of order p are $\beta(h(v_2^{p+2})/p)$, $\beta(h(v_2^{p+1})/p)$ and $\beta(h(v_2 v_3)/p)$.

Now let

$$a\cdot\frac{h(v_2^{p+2})}{p}+b\cdot\frac{h(v_2^{p+1})}{p}+c\cdot\frac{h(v_2v_3)}{p} = \sum_{i=o}^{p} a(i)\cdot q_2^i\cdot q_1^{p^2+2p-i(p+1)}+a(p+1)q_3q_1^{p-1}$$

mod degree $\leq p^2+p-2$ in $A_{2m}(BP/BP^r;\mathbb{Q}/\mathbb{Z})$. Then by (5.12) we have $a(i) \equiv 0\ (p^{p-3})$. Looking at the coefficients of $t_1^{p^2+p-1}$ and $t_1^{p^2+p}$ mod $\mathbb{Z}$ we see that they are the same; the additional summand $t_1^{p^2+p-1}\cdot a(1)/p^2$ vanishes (for $p>3$, for $p=3$ one calculates directly). We have

$$h(v_2^{p+2})/p \equiv \frac{1}{p}\cdot t_1^{p^2+2p} - \frac{2}{p}\cdot t_1^{p^2+p+1}$$

$$h(v_2v_3)/p \equiv \frac{1}{p}\cdot(t_2^p\cdot t_1^p - t_1^{p^2+p+1} - t_1\cdot t_2^p)$$

$$h(v_2^{p+1})/p \equiv \frac{1}{p}\cdot t_1^{p^2+p}$$

mod degree $\leq p^2+p-2$.

On the left hand-side of (5.15) the coefficient of $t_1^{p^2+p}$ is b/p and that of $t_1^{p^2+p-1}$ is zero, but on the right-hand side they are equal, so $b \equiv 0\ (p)$. Because $a(i) \equiv 0\ (p)$ we find $a \equiv 0$ and $c \equiv 0$ mod p by looking at the terms of highest degree.

The usual generators for $H_*(MU)$ are the elements $b_i^H = \phi(b_i)$ where $\phi: H_*(BU) \longrightarrow H_*(MU)$ is the Thom isomorphism, whereas $H_*(BP;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[q_1,\dots,q_i,\dots]$. Let $\pi: MU \longrightarrow BP$ be the projection. We need some information about $\pi_*(b_i^H)$ in terms of the q_i. The values of $\pi_*(m_i^H)$ (notation as in [3]) are known and there are formulas for computing b_i^H in terms of m_i^H (7.5 in [3]). However for computing $\pi_*(b_i^H)$ mod p we proceed as follows:

<u>Proposition 5.16:</u> $\pi_*(b_j^H) \equiv 0$ mod p if $j \neq p^i-1$

$$\pi_*(b^H_{p^i-1}) \equiv -\sum_{k=1}^{i} q_i\cdot\pi_*(b_{p^{i-k}-1})^{p^k} \text{ mod } p$$

<u>Proof:</u> We have a commutative diagram

$$\begin{array}{ccccc}
H_*(MU) & \xrightarrow{\pi_*} & H_*(BP) & \xleftarrow{\quad T_{BP} \quad} & \\
\uparrow T_{MU} & & \uparrow T_{MU} & & \\
MU_*(MU) & \xrightarrow{\pi_*} & MU_*(BP) & \xrightarrow{\pi} & BP_*(BP)
\end{array}$$

with $T_{MU}(b_1^{MU}) = b_1^H$. In $BP\wedge BPQ^*(P_\infty C)$ the following equation is true (see [3] §16):

$$\sum_f \eta_L(q_f)\cdot(x^L)^{p^f} = \sum_f \eta_R(q_f)\left(\sum_{j\geq 0} \pi_*(b_j^{MU})(x^L)^{j+1}\right)^{p^f} \tag{5.17}$$

Applying T_{BP} gives

$$q_0 = \sum_f q_f\cdot\left(\sum_{j>0} \pi_*(b_j^H)\, x^{j+1}\right)^{p^f}$$

Calculating mod p we arrive at

$$\sum_f \sum_j \pi_*(b_j^H)^{p^f}\cdot x^{(j+1)p^f}\cdot q_f \equiv q_0 \mod p$$

Inductively it follows that $\pi_*(b_j^H) \equiv 0\ (p)$ if $j \neq p^s-1$ and for $i>0$

$$\sum_{k=0}^{i} q_k\, \pi_*(b_{p^{i-k}-1})^{p^k} = 0$$

<u>Remark:</u> A similar argument gives a corresponding formula in K-theory.

<u>Examples:</u>

$$\pi(b_{p-1}) \equiv -q_1$$

$$\pi(b_{p^2-1}) \equiv q_1^{p+1} - q_2 \tag{5.18}$$

$$\pi(b_{p^3-1}) \equiv -q_1^{p^2+p+1} + q_1^{p^2} q_2 + q_1 q_2^p - q_3 \mod p$$

<u>Corollary 5.19</u>: $\pi_*(b_{p^i-1}) \equiv (-1)^i \cdot q_1^{(p^i-1)/(p-1)}$ mod p, q_2, q_3,.

For computing the transfer t^s in filtration 1 we need to calculate the map $S^2P_\infty\mathbb{C}^{s\tilde{H}} \xrightarrow{f^s} \overline{MU} \xrightarrow{\pi} \overline{BP}$. Recall that the map f^s was defined by the diagram

$$\begin{array}{ccccccc}
S^o & \longrightarrow & P_\infty\mathbb{C}^{(s-1)\tilde{H}} & \xrightarrow{j^!} & S^2P_\infty\mathbb{C}^{s\tilde{H}} & \xrightarrow{t^s} & S^1 \\
\| & & \big\downarrow \bar{\bar{f}}^{s-1} & & \big\downarrow f^s & & \| \\
S^o & \longrightarrow & MU & \longrightarrow & \overline{MU} & \xrightarrow{\partial} & S^1
\end{array}$$

If we define the generators b_i^s of $H_*(P_\infty\mathbb{C}^{s\tilde{H}})$ using the Thom isomorphism ϕ: $H_*(P_\infty\mathbb{C}) \longrightarrow \tilde{H}_*(P_\infty\mathbb{C}^{s\tilde{H}})$ we can use the commutative diagram

$$\begin{array}{ccc}
\tilde{H}_*(P_\infty\mathbb{C}^{s\tilde{H}}) & \xrightarrow{\bar{\bar{f}}^s} & \tilde{H}_*(MU) \\
\big\downarrow \phi & & \big\downarrow \phi \\
H_*(P_\infty\mathbb{C}) & \xrightarrow{f^s} & H_*(BU)
\end{array}$$

Note that $H_*(MU)$ and $H_*(BU)$ are both rings and the Thom isomorphism is a ring homomorphism. But $\bar{f}_*^s(1+b_1+b_2+\ldots) = (1+b_1+b_2+\ldots)^s$ in $H_*(BU)$, so

$$\bar{\bar{f}}_*^s\ (b_i^s) = (b_o^H+b_1^H+\ldots\ldots)^s_{(i)} \tag{5.20}$$

where (i) means term of degree i. Because $j^!:\tilde{H}_*(P_\infty\mathbb{C}^{(s-1)\tilde{H}}) \to \tilde{H}_*(P_\infty\mathbb{C}^{s\tilde{H}})$ maps b_i^{s-1} to b_{i-1}^s we arrive at

$$f_*^s\ (b_{i-1}^s) = (b_o^H+b_1^H+b_2^H+\ldots)^{s-1}_{(i)} \tag{5.21}$$

The analogous formulas are true in K-theory.

For calculating with the transfer t^o we need therefore

$$(\pi_*(b_o)+\pi_*(b_1)+\pi_*(b_2)+\ldots)^{-1}$$

Modulo p and q_i for $i>1$ this is easy to calculate:

<u>Proposition 5.22</u>: The coefficient c_t of q_1^t in $(\pi_* b_o^H + \pi_* b_1^H + \pi_* b_2^H + \ldots)^{-1}$ has the following mod p value: write $t(p-1) = \sum_{i=0}^{r} \alpha_i p^i$ with $0 \leqslant \alpha_i < p$, then $c_t \equiv 0$ if $\sum \alpha_i > p-1$ and $c_t \equiv (-1)^t / \alpha_o! \alpha_1! \alpha_2! \ldots \alpha_r!$ if $\sum \alpha_i = p-1$.

<u>Proof</u>: We calculate $(1+y+y^{p+1}+\ldots)^{-1}$ and set $y = x^{p-1}$, then

$$(1+x^{p-1}+x^{p^2-1}+\ldots)^{-1} = \frac{1}{x}\left(\frac{x}{1+x^{p-1}+x^{p^2-1}+\ldots}\right)$$

$$= 1 - \frac{x^p + x^{p^2} + x^{p^3} + \ldots}{x + x^p + x^{p^2} + \ldots\ldots}$$

$$= 1 - \frac{(x + x^p + x^{p^2} + \ldots)^p}{(x + x^p + x^{p^2} + \ldots)}$$

$$= 1 - (x + x^p + x^{p^2} + \ldots)^{p-1}$$

$$= 1 - \sum_{\substack{\alpha_o, \alpha_1, \ldots, \alpha_r \geqslant 0 \\ \alpha_o + \alpha_1 + \alpha_2 + \ldots \alpha_r = p-1}} \frac{(p-1)!}{\alpha_o! \alpha_1! \alpha_2! \ldots \alpha_r!} x^{\alpha_o + \alpha_1 p + \ldots}$$

The desired identity follows on comparing the coefficients of $x^{t(p-1)}$ on both sides. (5.19) completes the proof.

Similarly, one can deduce formulas for $\pi_*(b_o+b_1+b_2+\ldots)^s$ mod $p, q_2, \ldots$

<u>Proof of (4.11)</u>:

a) We want to show $tt(y_i) \neq 0$ for $i \leqslant p-1$. Because $\pi_*^s(S^o)_{(p)}$ in this dimension is generated by β_i we must have $tt(y_i) = a\beta_i$ with $a \not\equiv 0(p)$.

<u>Remark</u>: Because $\mathrm{Ext}^{2,2n+3}_{BP_*BP}(BP_*, BP_*)$ can be mapped monomorphically into $A_{2n+1}(\overline{BP})$ one can also compute the value of a and so does not actually need the information that $\pi_*^s(S^o)_{(p)}$ is generated by β_i. One would then get $tt(y_i) = a\beta_i$ mod higher filtration.

We set $f = \pi f^{o}$. By (3.9) we must show that $\beta f_* t(y_i)$ is nonzero in $A_*(\overline{BP})$ and by (5.11) we are allowed to compute mod terms of degree $\leq (pi-2)$. We know $t(y_i) = (1/p)\cdot b^K_{p((p-1)i-1)} + z$ with $\deg(z)<p((p-1)i-1)$. So at once $f_*(z) \equiv 0$ and we need to know only the term of top degree in $f_*(b^K_{p((p-1)i-1)})$. This of course can be calculated via homology and we can use (5.22) to obtain

$$f_* t(y_i) = \frac{1}{p}\, t_1^{ip-1} \quad \text{mod } t_2 \text{ and degree} \leq (pi-2)$$

As remarked in the proof of (5.11), this suffices to show $\beta f_* t(y_i) \neq 0$ because $h(v_2^i)/p \equiv t_1^{ip}/p \equiv -t_1^{ip-1}/p \mod q_1^{ip-1+i}$.

b) The proof for $tt(y_p) \doteq \beta_p$ runs along similar, but more complicated, lines.

c) To prove $tt(y_{p+1}) = 0$ we must show that $\beta f_* t(y_{p+1})$ goes to zero in $A_*(\overline{BP})$. By the same filtration argument as in a) we need only look at the term in degree $(p+1)p-1$.

The following two statements are easily proved by direct calculation:

1. The coefficient of $\pi_*(b_o^H + b_1^H + b_2^H + \ldots)^{-1}$ in degree $(p+1)p-1$ is $t_1^p \cdot t_2^{p-1}$ mod p (using (5.18)).

We have already observed in (5.22) that this is zero mod t_2, t_3.

2. We have the relations

$$p^{p-1}\cdot q_3 q_1^{p-1} \equiv \frac{1}{p}\cdot t_2^p$$

$$p^{p-1}\cdot q_2^p q_1^p \equiv \frac{1}{p}\cdot\left(t_2^p + a_1 t_2^{p-1} t_1^p + a_2 t_1^{p^2+p} + a_2 t_1^{p^2+p-1}\right)$$

$$p^{p-1}\cdot q_1^{p^2+2p} \equiv \frac{1}{p}\cdot\left(t_1^{p^2+p} + t_1^{p^2+p-1} \right)$$

mod $\leq$ degree p^2+p-2 with $a_1, a_2 \not\equiv 0$ (p).

Statement 1. proves $f_*(t(y_{p+1})) \equiv \frac{1}{p}\, t_1^p \cdot t_2^{p-1}$ mod degree $\leq p^2+p-2$ and

2. shows that this goes to zero under the Bockstein map.

As an example we include the explicit formula for $f_*t(y_{p+1})$ in the case p=3 :

$$f_*(t(y_{p+1})) = \frac{1}{3}\cdot t_2^3 + \frac{1}{3}\cdot t_1^6 t_2 + \frac{1}{3}t_2^3 + \frac{1}{3}\cdot t_1^4 + \frac{8}{9}\cdot t_1^3 t_2 + \frac{8}{9}\cdot t_1^2 + \frac{19}{27}\cdot t_1^9 +$$

$$\frac{7}{27}\cdot t_2 + \frac{13}{27}\cdot t_1^6 + \frac{11}{27}\cdot t_1 + \frac{79}{81}\cdot t_1^3 .$$

§ 6 On the image of the e-invariant

In this chapter we discuss the problem of computing the values which the e-invariant can take on $\pi_*^s(P_\infty\mathbb{C})$. The computation of the algebraic K-groups for $P_\infty\mathbb{C}$ in § 1 gives an upper bound for the image of the e-invariant. In contrast to the case of stable cohomotopy in negative dimensions, where the existence of the J-homomorphism implies surjectivity of the e-invariant, the e-invariant need not be surjective. Indeed the deviation from surjectivity is connected with other interesting phenomena such as behaviour of the transfer maps $t^s:\pi_{2n-1}^s(P_\infty\mathbb{C}^{k\tilde{H}})\to\pi_{2n}^s(S^o)$ from Adams filtration 1 to filtration 2. If the e-invariant were surjective, then all t^s would have to map $F^1\pi_{2n-1}^s(P_\infty\mathbb{C}^{k\tilde{H}})$ to higher filtration than F^2. Every element of $F^2\pi_{2n}^s(S^o)$ which is in the image of some t^s causes an infinite series of elements in coker (e). We shall use this to find elements in the cokernel of $e: \pi_{2n-1}^s(P_\infty\mathbb{C}) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$.

A further upper bound for im(e) is given by the groups

$$\mathrm{Ext}^{1,2n}_{BP_*BP}(BP_*,BP_*(P_\infty\mathbb{C}))\subset A_{2n-1}(P_\infty\mathbb{C})$$

It is certainly difficult but not impossible to calculate these groups using the methods of [29]. But even this would not describe im(e) completely, because there are nonzero differentials in the Adams spectral sequence on these ext-groups. Via the transfer map those differentials are related to differentials on $\mathrm{Ext}^2_{BP_*BP}(BP_*,BP_*)$ and so a complete computation of im(e) would give information on $F^2\pi_*^s(S^o)_{(p)}$.

For lower bounds for im(e), the situation is not even as good as for upper bounds. There are several ways of giving lower bounds for the image of e. First, we can use S-duality and the J-homomorphism to show that elements of low skeletal filtration in $A_{2n-1}(P_\infty\mathbb{C})$ are always in the image of e. Second, we can use the transfer map $\pi_*^s(BT^2) \longrightarrow \pi_*^s(P_\infty\mathbb{C})$ to produce elements in im(e). The first method only gives a coarse lower bound in all dimensions. The second can only be used effectively in a certain range of dimensions because the image of the Hurewicz map $h\colon \pi_{2n}^s(BT^2) \longrightarrow H_{2n}(BT^2)$ whose knowledge is necessary for this method is not known in general.

We start with giving lower bounds using S-duality. We then discuss a simple method to find elements in the cokernel of $e\colon \pi_{2n-1}^s(P_\infty\mathbb{C}) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$, describe the relation to the transfer maps and use this to calculate $\mathrm{im}(e) \subset A_{2n-1}(P_\infty\mathbb{C})$ for $n \leqslant (p-1)p^2-p$. This then allows us to compute $\pi_*^s(P_\infty\mathbb{C})_{(p)}$ in the same range of dimensions. We also show how one can describe and construct the elements in $\pi_*^s(P_\infty\mathbb{C})$ in this range by framed manifolds (up to some exeptions). As an application, we compute the image of the Hurewicz map $h\colon \pi_{2n-1}^s(B\mathbb{Z}/p^2) \longrightarrow K_1(B\mathbb{Z}/p^2)$ for $n \neq o(p)$.

We first discuss the way to give lower bounds by use of S-duality and the J-homomorphism.

<u>Proposition 6.1</u>: Let $x \in A_{2n-1}(P_\infty\mathbb{C})$ be in the image of the map induced by inclusion $i : A_{2n-1}(P_k\mathbb{C}) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$ and $2k+m \leq 2n-1$ where m is the geometric dimension of the stable normal bundle of $P_k\mathbb{C}$. Then x is in the image of $\pi^s_{2n-1}(P_\infty\mathbb{C}) \longrightarrow A_{2n-1}(P_\infty\mathbb{C})$.

<u>Proof</u>: If the S-dual of an element x is in the image of e, then $x \in \mathrm{im}(e)$. We have a commutative diagram:

$$\begin{array}{ccc} \pi^s_{2n-1}(P_k\mathbb{C}) & \xrightarrow[\cong]{D} & \pi_s^{1-2n+2k+m}(P_k\mathbb{C}^\nu) \\ \downarrow e & & \downarrow e \\ A_{2n-1}(P_k\mathbb{C}) & \xrightarrow{\cong} & A^{1-2n+2k+m}(P_k\mathbb{C}^\nu) \end{array}$$

where D is S-duality and $P_k\mathbb{C}^\nu$ the Thom space of the normal bundle of $P_k\mathbb{C}$. If $r = 1-2n+2k+m \leq 0$ we can look at the J-homomorphism

$$J\colon K^{r-1}(P_k\mathbb{C}^\nu) \longrightarrow \pi_s^r(P_k\mathbb{C}^\nu)$$

The map J factors as

$$K^{r-1}(P_k\mathbb{C}^\nu) \longrightarrow A^r(P_k\mathbb{C}^\nu) \xrightarrow{j} \pi_s^r(P_k\mathbb{C}^\nu)$$

and for an odd prime the composition $e \circ j$ is bijective [43]. This shows that $e\colon \pi^s_{2n-1}(P_k\mathbb{C}) \rightarrow A_{2n-1}(P_k\mathbb{C})$ is surjective if $2k+m \leq 2n-1$.

<u>Example</u>: p=3 The element $x = (b_3-b_4+b_5)/3$ in $A_{2n-1}(P_\infty\mathbb{C})$ with n odd has filtration 6 (because $x = a\cdot b_1+b\cdot b_2+c\cdot b_3$ mod b_1^n in $K_o(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$), so if $2n-1 \geq 11$, x is in the image of e.

The same argument applied to $B\mathbb{Z}/p^r$ gives

<u>Proposition 6.2</u>: Let $L^{2k+1}(p^r)$ be the $2k+1$-skeleton of $B\mathbb{Z}/p^r$ and m the geometric dimension of the stable normal bundle of $L^{2k+1}(p^r)$. Then $\mathrm{im}(A_i(L^{2k+1}(p^r)) \to A_i(B\mathbb{Z}/p^r))$ is contained in $\mathrm{im}(e)$ if $2k+1+m \leq i$.

In [38] it is shown, that $m(L^{2k+1}(p^r)) \leq 2[\frac{k}{2}]+1$ if p is odd. So we always have $\mathrm{im}(A_i(L^{2k+1}(p^r)) \to A_i(B\mathbb{Z}/p^r)) \subset \mathrm{im}(e)$ if $k+1+[\frac{k}{2}] \leq i$.

<u>Example</u>: $p=3$ $z = \dfrac{\rho^3+2^n\rho^6}{3}$, n odd is in $A_{2n-1}(L^{11}(9)) \to A_{2n-1}(B\mathbb{Z}/9)$ so $z \in \mathrm{im}(h)$ if $n \geq 9$.

<u>Remark</u>:

By the surjectivity of $\pi^s_{2n-1}(B\mathbb{Z}/p^r) \to \pi^s_{2n-1}(P_\infty\mathbb{C})_{(p)}$ for large r (6.2) also gives information for $P_\infty\mathbb{C}$, which is sometimes better than that of (6.1). Given $x \in A_{2i-1}(B\mathbb{Z}/p^r)$, the value of m with $x \in \mathrm{im}(A_{2i-1}(L^{2m-1}(p^r)) \to A_{2i-1}(B\mathbb{Z}/p^r))$ can be calculated as in (1.8) .

To find elements in the cokernel of $e: \pi^s_{2n-1}(P_\infty\mathbb{C}) \to A_{2n-1}(P_\infty\mathbb{C})$ we can look at spaces X where the image of e in $A_*(X)$ is already known, and then use maps $f: P_\infty\mathbb{C} \to X$ to compare the cokernels of e in both groups.

If $x \in A_{2n-1}(P_\infty\mathbb{C})$ is mapped by f_* onto a nonzero element in $A_{2n-1}(X)/\mathrm{im}(e)$, then x cannot come from stable homotopy.

Spaces or spectra for which the image of the e-invariant is known are MU and BP. Because $\pi^s_{2n-1}(MU) = \Omega^U_{2n-1} = 0$ and $\pi^s_{2n-1}(BP) = BP_{2n-1}(*) = 0$ we have $\mathrm{im}(e) = 0$ but nevertheless $A_{2n-1}(MU) \neq 0$.

Example: $p=3$ We have $A_9(P_\infty\mathbb{C}) \cong \mathbb{Z}/3$ generated by an element z_9 with

$$z_9 = \frac{1}{3}(b_3 - b_4 + b_5) \equiv \frac{2}{9}b_1 - \frac{1}{9}b_1^3 \equiv \frac{1}{9}b_1 + \frac{1}{3}b_2 + \frac{1}{3}b_3 \quad \text{mod } b_1^5$$

Using (5.17) and (5.4) one finds that z_9 is mapped under

$$\pi \circ f^o : S^2P_\infty\mathbb{C} \longrightarrow \overline{BP}$$

to $ch^{-1}(\frac{2}{9}q_1 + \frac{4}{3}q_1^2) = \frac{2}{9}t_1 + \frac{1}{3}t_1^2 \in A_{12}(\overline{BP};\mathbb{Q}/\mathbb{Z}) \subset K_o(\overline{BP};\mathbb{Q}/\mathbb{Z})$.

The kernel of the Bockstein map is $q_1^3 = (t_1 + 1/3)^3 = t_1^3 + t_1^2 + t_1/3$, so $\pi_* f^o_*(z_9) \equiv -t_1^3/3$.

But $A_9(P_\infty\mathbb{C}) \cong A_{10}(X_1)$ where $SP_\infty\mathbb{C}_{(3)} \simeq X_1 \vee X_2$ and the cofibre spectrum of $X_1 \longrightarrow \overline{BP}$ is 15-connected. Because $A_{11}(\overline{BP}) \cong \pi^s_{11}(\overline{BP}) \cong \mathbb{Z}/3$ it then follows that z_9 is in the image of e (what we already know, because $z_9 = e(t(x_o))$).

Using this method we shall prove:

Proposition 3.11: Let x be an element of order p^{p-2} in

$$A_{2m-1}(P_\infty\mathbb{C}) = \mathbb{Z}/p^{p+2} \oplus \mathbb{Z}/p \quad (m = (p-1)\cdot((p+1)^2-1)-1)$$

then x is not in the image of the e-invariant.

Proof: One can use the formulas of § 1 to show that $\bar{x} = \pi_*(x(1,p+2))$ in $A_{2m}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ has Bockstein $\beta(\bar{x}) = x$ of order p^{p+2} and then compute the coefficients of the highest terms of $\bar{x}$. Then by subtracting a suitable multiple of b_1^m one can lower the filtration of x so that

$$\bar{x} = \frac{1}{p}\, b_{m-(p-1)} + \text{ terms of lower degree} \tag{6.3}$$

in $A_{2m}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$.

To avoid such calculations one can argue as follows: We look at the Atiyah-Hirzebruch spectral sequence for $A_*(P_\infty\mathbb{C})$. The first differential is the same as the first differential of the spectral sequence for $\pi_*^s(P_\infty\mathbb{C})$ and it is well known and easy to see that

$$d^1 x = \bar{\mathcal{P}}^1(x)\cdot\alpha_1 \tag{5.4}$$

where $\bar{\mathcal{P}}^1$ is dual to the first mod p Steenrod power. (We write $\alpha_1 \in \pi^s_{2p-3}(S^o)_{(p)}$ for the generators of both $A_{2p-3}(*)$ and $\pi^s_{2p-3}(S^o)_{(p)}$).

In the case where $n+1\equiv o(p)$ we then have that

$$d^1 : H_{2n}(P_\infty\mathbb{C};A_o) \longrightarrow H_{2n-2(p-1)}(P_\infty\mathbb{C};A_{2p-3})$$

is zero and for dimensional reasons the generator of $H_{2n-2(p-1)}(P_\infty C;A_{2p-3})$ is a permanent cycle.

This implies that there is an element y in $A_{2m-1}(P_\infty\mathbb{C})$ which has skeleton-filtration $2(m-p+1)$. By comparing the spectral sequences for $A_*(P_\infty\mathbb{C})$ and $A_*(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$, it follows that in $A_{2m}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$ there is an element y with the same filtration and nonzero Bockstein. It is easy to see that $z = \beta\pi_*(x(2,p))$ is a generator for the second summand, and from § 1 we then know

$$\bar{z} = \pi_*(x(2,p)) = \frac{1}{p}\left(b^K_{p^2(p-1)-1} - b^K_{p^2(p-1)-2} + b^K_{p^2(p-1)-3} - \cdots\cdots\right) \tag{6.5}$$

Because we know the filtration of $px = t(y_{p+1})$ (see § 4)
and that of z, $\bar{y}$ must be $\bar{x}$ and (6.4) is proved.

We now map $P_\infty\mathbb{C}$ into BP as follows:

$$g : P_\infty\mathbb{C} \xrightarrow{pr} P_\infty\mathbb{C}/P_{k-1}\mathbb{C} = P_\infty\mathbb{C}^{kH} \xrightarrow{\bar{\bar{f}}^k} S^{2k}MU \xrightarrow{\pi} S^{2k}BP \tag{6.6}$$

Here pr is the natural projection, $\bar{\bar{f}}^k$ the map induced by the classifying map of $k\cdot H$ and $k = m-(p-1)p = (p-1)(p^2+p)-1$. In $A_{2p(p-1)}(BP;\mathbb{Q}/\mathbb{Z})$ we can calculate modulo terms of degree $<(p-1)$.

and so proceed as in the proof of (4.11): We get

$$g_*\left(\frac{1}{p}\cdot b_{m-(p-1)}+\ldots\right) \equiv \text{term of degree } (p-1) \text{ in } \frac{1}{p}\left(1-t_1+t_1^{p^2+1}-t_2+\ldots\right)^k$$

which is $\pm\binom{k}{p-1}/p\cdot t_1^{p-1} = \pm\frac{1}{p}\,t_1^{p-1}$.

By (5.11), this is not in the kernel of

$\beta\colon A_{2p(p-1)}(BP;Q/Z) \longrightarrow A_{2p(p-1)-1}(BP)$.

We can now prove proposition (3.12) of § 3

__Proposition 3.12__: The image of e in $A_{2m-1}(P_\infty\mathbb{C})$, where $m=(p-1)((p+1)^2-1)-1$, is mapped to zero in $A_{2m+1}(\overline{BP})$ under $\pi_* f_*^{\circ}$.

__Proof__: By (4.11) we have proved that $px=t(y_{p+1})$ goes to zero under $\pi_* f_*^{(o)}$. But, as observed in the proof of (3.11), the generator of the second summand z has filtration $\leqslant p^2$ and in $A_{2m}(BP;\mathbb{Q}/\mathbb{Z})$ we are allowed to compute mod terms of degree $\leqslant(p^2+p-2)$ so $\pi_* f_*^{(o)}(z)=0$.

We now discuss the relation between the elements in coker (e) and the behaviour of the transfer-maps on Adams filtration 1. For simplicity we shall restrict to dimensions n with $n\leqslant \dim \beta_1^p$. Define $C_*(X)$ to be the cofibre homology theory of the Hurewicz map $\pi_*^s \longrightarrow A_*$, so that for every space we have an exact sequence

$$C_n(X) \xrightarrow{\;i\;} \pi_n^s(X)_{(p)} \xrightarrow{\;h\;} \widetilde{A}_n(X) \xrightarrow{\;\Delta\;} C_{n-1}(X) \qquad (6.7)$$

For an odd prime we see $C_*(S^o) \cong \text{coker}\,(J) \subset \pi_*^s(S^o)_{(p)}$. The only way I know of to compute $C_*(X)$ is by means of some spectral sequence. If we use the Atiyah-Hirzebruch spectral sequence, then in the dimension range $n\leqslant \dim \beta_1^p$ and for spaces like $P_\infty\mathbb{C}$ this is easy. One needs only the knowledge of two differentials $d_{2(p-1)}$ and $d_{2(p-1)^2}$. The values of d_r for C_* are deduced from those of the π_*^s-spectral sequence. There we have

<u>Lemma 6.8</u>: $d_r^{2n,k}: H_{2n}(P_\infty\mathbb{C};\pi_k^s) \longrightarrow H_{2n-r}(P_\infty\mathbb{C};\pi_{k+r-1}^s)$ for $r=2(p-1)$ and $2(p-1)^2$ is given by

$$d_{2(p-1)}^{2n,k}([P_n\mathbb{C}]\otimes z) = (n-p+1)\cdot[P_{n-p+1}\mathbb{C}]\otimes z\cdot\alpha_1$$

$$d_{2(p-1)^2}^{2n,k}([P_n\mathbb{C}]\otimes z\cdot\alpha_1) = \binom{n-(p-1)^2}{p-1}\cdot[P_{n-(p-1)^2}\mathbb{C}]\otimes z\cdot\beta_1$$

<u>Proof</u>: This is known, see [44], [9] and can also be deduced by applying the method for finding elements in coker(e) discussed above.

Let $X = P_\infty\mathbb{C}$ or $P_\infty\mathbb{C}/P_k\mathbb{C}$.

We then see that the cycles which are left by $d_{2(p-1)}$ are killed by $d_{2(p-1)^2}$ with the following exceptions

a) the differentials map into negative dimensions or

b) $d_{2(p-1)^2}$ starts on an element $[P_i\mathbb{C}]\otimes z$ where z is indecomposable in the ring π_*^s, that is $z\in\{\beta_1,\beta_2\ldots\beta_{p-1}\}$

For $X = P_\infty\mathbb{C}$ there remain for dimensional reasons (case a))

$$[P_j\mathbb{C}]\otimes z \text{ and } [P_{jp}\mathbb{C}]\otimes\alpha_1\cdot z \text{ with } 1\leq j<(p-1)$$
$$\text{and } z\in\{\beta_1^i\beta_k \mid k\leq p-1 \text{ and } o<i+k\leq p-1\} \quad (6.9)$$

For $X = P_\infty C/P_k C$ there are left, by reason b), the cycles $[P_{tp+(p-1)}\mathbb{C}]\otimes\beta_s$. (6.1o)

The elements left for dimensional reasons in $C_{2n}(P_\infty\mathbb{C})$ map to nonzero elements in $\pi_{2n}^s(P_\infty\mathbb{C})$:

Clearly $[P_j\mathbb{C}]\otimes z$ $(j<p-1)$ represents $\sigma^i\cdot z\in\pi_*^s(P_\infty\mathbb{C})$, where $\sigma\in\pi_2^s(P_\infty\mathbb{C})$ is the generator.

But we know $t(\sigma^{p-2}\cdot z) = \alpha_1 z \neq o$ in $\pi_*^s(S^o)$

So $\sigma^i z \neq o$ for $i\leq p-2$.

Thus elements in the image of $\Delta: A_{2n-1}(P_\infty\mathbb{C}) \to C_{2n-2}(P_\infty\mathbb{C})$ are represented by cycles $[P_{tp+(p-1)}\mathbb{C}]\otimes \beta_s$ and we have proved

<u>Lemma 6.11</u>: Let $2n-1 \leqslant \dim \beta_1^p$. Then

$\operatorname{coker}\,(e\colon \pi^s_{2n-1}(P_\infty\mathbb{C}) \longrightarrow A_{2n-1}(P_\infty\mathbb{C}))$

has order at most p .

<u>Proposition 6.12</u>: Let $n \leqslant \dim \beta_1^p/2$ and $x \in A_{2n-1}(P_\infty\mathbb{C})$ be in $\operatorname{coker}\,(e)$. Then there exists a number k such that the projection of x into $A_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C})$ is in the image of e and for an element $\bar{x}$ with $e(\bar{x}) = x$ we have $t^{(k+1)}(\bar{x}) \neq 0$

<u>Proof</u>: $x \in A_{2n-1}(P_\infty\mathbb{C})$ is mapped to an element represented by some $[P_{(p-1)+tp}\mathbb{C}]\otimes\beta_s$ in $C_{2n-2}(P_\infty\mathbb{C})$. Let $k=p-1+tp$ and $pr^k\colon P_\infty\mathbb{C} \to P_\infty\mathbb{C}/P_k\mathbb{C}$ be the projection. We look at the exact sequences induced by the transfer t^{k+1} :

$$\begin{array}{ccccccc}
A_{2n-2k-1}(S^0) & \xrightarrow{k} & A_{2n-1}(P_\infty\mathbb{C}/P_{k-1}\mathbb{C}) & \xrightarrow{j^!} & A_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C}) & \xrightarrow{t^{k+1}} & A_*(S^0) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow \\
C_{2n-2k-2}(S^0) & \longrightarrow & C_{2n-2}(P_\infty\mathbb{C}/P_{k-1}\mathbb{C}) & \longrightarrow & C_{2n-2}(P_\infty\mathbb{C}/P_k\mathbb{C}) & \longrightarrow & C_*(S^0)
\end{array}$$

We have $\Delta\circ pr^k_*(x)=0$, so there exists $\bar{x} \in \pi^s_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C})$ with $e(\bar{x}) = pr^k_*(x)$. Then $t^{k+1}(\bar{x}) \neq 0$, because in case $t^{k+1}(\bar{x}) = 0$ we should have $\bar{x} = j^!(\bar{\bar{x}})$ for some $\bar{\bar{x}}$ with $e(\bar{\bar{x}}) = pr^{k-1}_*(x)+k(z)$ contradicting $\Delta pr^{k-1}_*(x) = 0$. Clearly $t^{k+1}(x) = a\cdot\beta_s$ with $a\not\equiv 0\ (p)$.

Proposition 6.13: If $x \in \pi^s_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C})$ is an element with $t^{k+1}(x) = \beta_s$, then $e(x) \in A_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C})$ has an inverse image in $A_{2n-1}(P_\infty\mathbb{C})$ which is in coker (e) .

Remark:

Because $A_{2i}(P_k\mathbb{C})$ is free, $pr^k_*: A_{2i+1}(P_\infty\mathbb{C}) \to A_{2i+1}(P_\infty\mathbb{C}/P_k\mathbb{C})$ is onto.

Proof: Because the transfer map $t^{k+1}: A_{2n-1}(P_\infty\mathbb{C}/P_k\mathbb{C}) \to A_{2n-2k-2}(S^0)$ is zero, $e(x)$ has inverse image y in $A_{2n-1}(P_\infty\mathbb{C}/P_{k-1}\mathbb{C})$. This cannot be in the image of e because the existence of an inverse image $\bar{y}$ would imply $t^{k+1}(x) \neq \beta_s$ ($j^!(\bar{y})-x$ is in Adams-filtration $F^{1+2(p-1)}$). Then the lift of $e(x)$ to $A_{2n-1}(P_\infty C)$ can be mapped onto an element in cok (e), so it cannot come from stable homotopy.

So one can describe the cokernel of e by computing the transfers t^k on filtration 1 and vice versa.

Remark:

The elements in cok (e) are always connected with values of the transfer maps, but they need not always correspond to values of t^i in Adams filtration 2. So we cannot determine them simply by computing t^k via f^k_* and $A_{2n+1}(\overline{BP})$ and the boundary map $\partial: \pi^s_{2m+1}(\overline{BP}) \longrightarrow \pi^s_{2m}(S^0)$. The appearance of such a phenomenon - connected with the relation $\alpha_1\beta_1^p=0$ in stable homotopy - in dimension $2p^2(p-1)-3$ is the reason for our dimension limit.

The method of § 3 allows us to compute t^k from filtration 1 to filtration 2 and the results of § 5 reduce this to a computation of a coefficient in a power series, namely the coefficient of t_1^{sp-1} in $(1-t_1+t_1^{p+1}-t_1^{p^2+p+1}+\ldots)^{k-1}$.

Theorem 6.14: Let s be in $\{1,\ldots,p-1\}$. If the element β_s is in the image of t^k, then $k\equiv 0(p)$.
The element β_s is in the image of t^{pt} if and only if $t-1$ mod p is in the set $\{s,\ldots,p\}$

Proof: If $\beta_s = t^k(x)$, then by (6.13) there exists an element y in $A_{2n-1}(P_\infty\mathbb{C})$ with $\Delta(y) \neq 0$ in $C_{2n-2}(P_\infty\mathbb{C})$. The element $\Delta(y)$ has a representing cycle of the form $[P_{tp+p-1}\mathbb{C}] \times\beta_s$. It follows that $2n-2 = 2tp+2(p-1) + \dim\beta_s = \dim\beta_s + 2k-2$ and thus $k\equiv 0 \bmod p$. The proof of the second statement is by the same method as in § 4. For the existence of the inverse image of β_s one uses the Atiyah-Hirzebruch spectral sequence.

From (6.14) we deduce:

Corollary 6.15: a) Let $n=s(p^2-1)+tp$ with $s\in\{1,..,p-1\}$ and $t\geq 0$. If t mod p is in $\{s,..,p\}$ then the cokernel of $e:\pi^s_{2n-1}(P_\infty\mathbb{C}^+) \rightarrow A_{2n-1}(P_\infty\mathbb{C})$ is at least $\mathbb{Z}/p$.

b) If $n \leq (p-1)\cdot p^2-p$ then a) describes coker(e) completely, that is : if $n=s(p^2-1)+tp$ with $s\in\{1,..,p-1\}$ and $t\geq 0$, then coker(e) is $\mathbb{Z}/p$ iff t mod p is in $\{s,\ldots,p\}$ and zero otherwise.

Let z_i be the generator of $\pi^s_m(P_\infty\mathbb{C})_{(p)}$ with $m=2p-3+2ip$ and $1\leq i\leq p-2$ (these are the first torsion elements occurring in the individual components of the splitting of $SP_\infty\mathbb{C}_{(p)}$ into p-1 pieces;they can be constructed from x_o using the transfer). The cycles $\alpha_1\beta_1^j\beta_s[P_{ip}\mathbb{C}]$ in $C_{2n-1}(P_\infty\mathbb{C})$, listed under (6.9), map to nonzero elements in $\pi^s_{2n-1}(P_\infty\mathbb{C})$ because $\pi^s_{2n-2}(P_\infty\mathbb{C}) \longrightarrow A_{2n-2}(P_\infty\mathbb{C})$ is onto. Clearly they

represent the elements $\beta_1^j\beta_s\cdot z_i$.
Putting the discussion above together gives us a computation of $\pi_i^s(P_\infty\mathbb{C})_{(p)}$ for $i\leq \dim \beta_1^p$:

<u>Corollary 6.16</u>: Let p be an odd prime and $n\leq \dim \beta_1^p = 2(p-1)p^2-p$. Then

$$\operatorname{tor} \pi_n^s(P_\infty\mathbb{C})_{(p)} \cong \ker(e) \oplus \operatorname{im}(e)$$

where im(e) is described by (6.15) b). The subgroup ker (e) is generated by the elements $z_i\cdot\beta_1^k\beta_s$, $\sigma_i^i\beta_1^k\beta_s$ $(1\leq i\leq p-2,\ 0\leq s\leq p-1,\ 0<k+s\leq p-1)$ and elements of order p in dimension 2n-1 with $n=s(p^2-1)+tp$ $(t\geq 0,\ s\in\{2,..,p-1\}$ and $t \bmod p\in\{1,..,s-1\}$ represented by the cycles $[P_{tp+p-1}\mathbb{C}]\otimes\beta_s$.

It is of course possible to press the calculations further, but then the results become more complicated to state than to derive.

Almost all elements in $\pi_*^s(P_\infty\mathbb{C})$ in the dimension range considered in (6.16) can be described by framed manifolds. For elements in ker (e) this is already in (6.16). It remains to describe the elements in im (e).

One only needs the three elements σ, x_o and x_1 and their Pontrjagin products to describe all elements of im (e) via the transfer $t\colon \pi_{2n}^s(BT^2) \longrightarrow \pi_{2n+1}^s(P_\infty\mathbb{C})$ with the following exeptions:
We look at each component of $SP_\infty\mathbb{C}_{(p)} \simeq X_1\vee\ldots\vee X_{p-1}$ separately. As long as $A_*(X_i)$ is cyclic we have im (e) = im (t). The generating element in the second summand of $A_{2m}(X_i)$ if it appears for the

first time is not obtained in this way. Then for $n=s\cdot(p^2-1)+tp$, $s\geq 2$, $t\geq 0$ and $t \bmod p \in\{2,\ldots,s-1\}$ a subgroup of order p is not in im(t). The proof is a simple application of (1.14) and (4.4) and will be omitted.

<u>Example</u>: $p=5$.

$A_{113}(P_\infty\mathbb{C}) \cong \mathbb{Z}/25 \oplus \mathbb{Z}/5$ generated by $t(x_1\circ(1\times\sigma^{26}))$ and $t(x_1\circ x_0^4(1\times\sigma^2))$ (e is onto).

It is also possible to describe some of the elements in im(e) by S^1-bundles over homogeneous spaces using (4.1).

Using (6.15) and arguments as above one can also compute

$$\pi_i^s(P_\infty\mathbb{C}/P_{k-1}\mathbb{C})$$

for $i \leq 2((p-1)p^2-p+k)$. The result is similar to (6.16) but slightly more complicated to state. The computation of $\pi_i^s(P_\infty\mathbb{C}/P_{k-1}\mathbb{C})$ might be more interesting than (6.16) because these groups are closely related to metastable homotopy groups of U(n), see for example [17] (the dimension range considered there is $i < 2p^2-1+2k$) and §7.

Computation of $h: \pi^s_{2n-1}(B\mathbb{Z}/p^2) \longrightarrow K_1(B\mathbb{Z}/p^2)$ for $n \not\equiv 0(p)$

Let p be an odd prime. From (1.6) we know

$$Ad_{2n-1}(B\mathbb{Z}/p^2) \cong \mathbb{Z}/p^{2+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)}$$

so for $n \not\equiv 0(p)$ we have $im(h) \subset \mathbb{Z}/p^2 \oplus \mathbb{Z}/p$. The generator of the first summand is always in the image of h because this element is the image of $\sigma^{n-1} \in \pi^s_{2n-2}(P_\infty\mathbb{C})$ under the transfer map [21]. The second summand is generated by

$$(\rho^p + 2^n\rho^{2p} + \dots (p-1)^n\rho^{(p-1)p})/p$$

Separating the congruence classes of n mod $(p-1)$ we get $(p-1)$ different classes z_s $(1 \leq s \leq p-1)$ in $K_1(B\mathbb{Z}/p^2)$. By (1.8) z_s is in $A_{2n-1}(B\mathbb{Z}/p^2)$ iff $\pi_*(z_s)$ is in $A_{2n-1}(P_\infty\mathbb{C})$. Using (1.7) we find that the lowest value of n such that $z_s \in A_{2n-1}(P_\infty\mathbb{C})$ is $n=(p-1)+sp$. So for $n\not\equiv 0(p)$, $n\equiv s \mod (p-1)$ and $n \geq (p-1)+sp$ we have $A_{2n-1}(B\mathbb{Z}/p^2) = \mathbb{Z}/p^2 \oplus \mathbb{Z}/p$. We first show that z_s is in the image of h . For $\pi_*(z_s)$ this is true by (6.15). By [25] we have $J(H^{p^2})=0$ in $J(P_m\mathbb{C})_{(p)}$ as long as $m<(p-1)p^2$. So $H^{p^2} \longrightarrow P_{(p-1)p^2-1}\mathbb{C}$ is orientable for stable homotopy and we get an exact Gysin-sequence:

$$\longrightarrow \pi^s_{2n-1}(B\mathbb{Z}/p^{2^+}) \xrightarrow{\pi_*} \pi^s_{2n-1}(P_\infty\mathbb{C}^+)_{(p)} \xrightarrow{\cap e} \pi^s_{2n-3}(P_\infty\mathbb{C}^+)_{(p)} \longrightarrow \qquad (6.17)$$

as long as $n<(p-1)p^2$.

For $n=(p-1)+sp$, $s=1,\dots,p-2$, the map $h: \pi^s_{2n-3}(P_\infty\mathbb{C}^+) \longrightarrow A_{2n-3}(P_\infty\mathbb{C})$ is injective, so $h^{-1}\pi_*(z_s)$ must come from $\pi^s_{2n-1}(B\mathbb{Z}/p^{2^+})$ because $\pi_*(z_s)$ comes from $A_{2n-1}(B\mathbb{Z}/p^2)$. For $s=p-1$ we have $\alpha_1\beta_1[x] \in \pi^s_{2n-3}(P_\infty\mathbb{C}^+)$

coming from the base point and we must check that $h^{-1}(\pi_* z_{p-1}) \cap e$ is not a multiple of $\alpha_1 \beta_1 [*]$. We compose $t: \pi_*^S(P_\infty \mathbb{C}^+) \longrightarrow \pi_*^S(B\mathbb{Z}/p^{2+})$ with the transfer $t_2: \pi_*^S(B\mathbb{Z}/p^{2+}) \longrightarrow \pi_*^S(B\mathbb{Z}/p^+)$ and by an easy spectral sequence argument we see $t_2 t(\alpha_1 \beta_1 [*]) \neq 0$, thus $\alpha_1 \beta_1 [*]$ cannot be in the image of $- \cap e$.

So for $s=1,..,p-1$ we have found an element $\bar{z}_s \in \pi_{2n-1}^S(B\mathbb{Z}/p^{2+})$ $(n=sp+p-1)$ with $h(\bar{z}_s) = z_s$ in $K_1(B\mathbb{Z}/p^2)$. Because $\bar{z}_s$ is of order p the Toda bracket $<\bar{z}_s, p, \alpha_1>$ is defined. We have $h(<\bar{z}_s, p, \alpha_1>) = h(\bar{z}_s)$. Proceeding in this way we come into the range where we can apply (6.1). Thus we have proved:

<u>Theorem 6.18</u>: Let $n \not\equiv 0$ (p). Then the image of the Hurewicz map

$$h: \pi_{2n-1}^S(B\mathbb{Z}/p^{2+}) \longrightarrow K_1(B\mathbb{Z}/p^2)$$

is $\mathbb{Z}/p^2 \oplus \mathbb{Z}/p$ if $n=t(p-1)+s$ with $s \in \{1,..,p-1\}$ and $t \geq s$ and $\mathbb{Z}/p^2$ if $t < s$.

<u>Remark</u>:

Using (6.17) and the computation of $\pi_n^S(P_\infty C)_{(p)}$ for $n \leq \dim \beta_1^p$ it is possible to work out $\pi_*^S(B\mathbb{Z}/p^k)$ for $k \geq 2$ in the same range. The case $\pi_*^S(B\mathbb{Z}/p)$ can be handled by using instead of (6.17) the cofibre sequence of the transfer map $S^1 \wedge P_\infty \mathbb{C}^+ \longrightarrow B\mathbb{Z}/p^+$.

§ 7 Some James numbers.

In this chapter we are concerned with the computation of some James numbers $\bar{U}(n,k)$, which describe the minimal degree of a composition $\pi\circ f$, where $\pi: W_{n,k} \longrightarrow S^{2n-1}$ is the projection of the Stiefel manifold to the sphere and $f: S^{2n-1} \longrightarrow W_{n,k}$ a map. We consider stable James numbers $U(n,k)$ which describe the corresponding stable problem for stunted projective spaces. The numbers $U(n,k)$ are the same as the $\bar{U}(n,k)$ in the stable range. There is a conjecture that the numbers $U(n,k)$ are determined by K-theory, that is $U(n,k)_{(p)} = U_A(n,k)$ where $U_A(n,k)$ is defined in the same way as $U(n,k)$ using algebraic K-theory A_* instead of stable homotopy. Using properties of transfer maps, we verify this conjecture for $n=1,2,k-2,k-1,k+2$ and in the case where n is divisible by a large power of p. By periodicity this determines some of the numbers $U(n,k)$.

The numbers $U(n,k)$ are closely related to the values of the transfer maps t^k on $F^o\pi_*^s(P_\infty\mathbb{C}^{k\tilde{H}})$. This allows us to show that the elements μ_r in the 2-component of $\pi_*^s(S^o)$ are in the image of t^s for $s=2$ and $s=-2$.

This chapter should be viewed only as a first attempt at the problem of computing $U(n,k)$. This study is continued in [12], where it is shown that $U(n,k)$ for $o\leq n\leq k$ is determined by K-theory.

Let $W_{n,k}$ denote the complex Stiefel manifold $U(n)/U(n-k)$.
We then have a fibration

$$W_{n-1,k-1} \longrightarrow W_{n,k} \xrightarrow{p} W_{n,1} = S^{2n-1} \tag{7.1}$$

inducing an exact sequence in homotopy

$$\pi_{2n-1}(W_{n-1,k-1}) \longrightarrow \pi_{2n-1}(W_{n,k}) \xrightarrow{p_*} \pi_{2n-1}(S^{2n-1})$$

The James number $\bar{U}(n,k)$ is defined to be the index of $p_* \pi_{2n-1}(W_{n,k})$ in $\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$.
The numbers $\bar{U}(n,k)$ have the following properties (see [18] or [19] where also a table of $\bar{U}(n,3)$ is given) :

1. $\bar{U}(n,k)$ is a multiple of $\bar{U}(n,m)$ if $m \leq k$ (7.2)
2. $\bar{U}(n,k) \cdot \bar{U}(m,k)$ is a multiple of $\bar{U}(n+m,k)$
3. If $\bar{U}(n,k) = 1$ and $m \geq 2k-1$ then $\bar{U}(m,k) = \bar{U}(n+m,k)$
4. $\bar{U}(n,k) = 1$ iff n is a multiple of M_k
   where M_k is the order of $J(H)$ in $J(P_{k-1}\mathbb{C})$

Because $p_* : H_{2n-1}(W_{n,k}) \longrightarrow H_{2n-1}(S^{2n-1})$ is onto, the commutative diagram

$$\begin{array}{ccc} \pi_{2n-1}(W_{n,k}) & \xrightarrow{p_*} & \pi_{2n-1}(S^{2n-1}) \\ \downarrow & & \\ H_{2n}(W_{n,k}) & \xrightarrow{p_*} & H_{2n-1}(S^{2n-1}) \end{array} \tag{7.3}$$

allows us to interpret $\bar{U}(n,k)$ as a property of the Hurewicz map.
The numbers $\bar{U}(n,k)$ are related to several interesting geometric problems.

Using the reflection map $w: SP_{n-1}\mathbb{C}^+ \longrightarrow U(n)$, in the stable range the fibre sequence (7.1) can be compared with the cofibre sequence of stunted projective spaces

$$P_{n-2}\mathbb{C}/P_{n-k-1}\mathbb{C} \longrightarrow P_{n-1}\mathbb{C}/P_{n-k-1}\mathbb{C} \xrightarrow{p} S^{2n-2} \qquad (7.4)$$

We therefore define $U(n,k)$ to be the index of $p_*\pi^s_{2n-2}(P_\infty\mathbb{C}/P_{n-k-1}\mathbb{C})$ in $\pi^s_{2n-2}(S^{2n-2})$. This number is the same as the order of the cokernel of $h(\pi^s_{2n-2}(P_\infty\mathbb{C}/P_{n-k-1}\mathbb{C}))$ in $H_{2n-2}(P_\infty\mathbb{C}/P_{n-k-1}\mathbb{C})$. Because the pair $(W_{n,k}, SP_{n-1}\mathbb{C}/P_{n-k-1}\mathbb{C})$ is $4(n-k)+3$ connected (see e.g. [18]) we have

$$\bar{U}(n,k) = U(n,k) \qquad \text{for } n \geqslant 2k-1 \qquad (7.5)$$

$$\pi_i(W_{n,k}) \cong \pi^s_{i-1}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) \qquad \text{for } i \leqslant 4(n-k) \qquad (7.6)$$

In any case, there is a map Θ (see [10], λ in [7])

$$\Theta: \pi_i(W_{n,k}) \longrightarrow \pi^s_{i-1}(P_{n-1}\mathbb{C}/P_{n-k-1}\mathbb{C}) \qquad (7.7)$$

so $\bar{U}(n,k)$ is always a multiple of $U(n,k)$.

The map Θ factorizes through the stable homotopy groups of $W_{n,k}$ so we get a commutative diagram

$$\begin{array}{ccc} \pi^s_{2n-1}(W_{n,k}) & \xrightarrow{p_*} & \pi^s_{2n-1}(S^{2n-1}) \\ \Theta' \downarrow \quad \uparrow w_* & & \| \\ \pi_{2n-2}(P_{n-1}C/P_{n-1-k}C) & \xrightarrow{p_*} & \pi^s_{2n-2}(S^{2n-2}) \end{array} \qquad (7.8)$$

Thus we can interpret $U(n,k)$ as a sort of stable James number.

Remark:

If we work at a fixed prime p then theorem 3.2 of [19] allows us to extend the range where (7.6) is true to $i \leq 2p(n-k+1)-4$ and

$$\tilde{U}(n,k)_{(p)} = U(n,k)_{(p)} \qquad \text{for } n \geq k-1+k/(p-1) \tag{7.9}$$

The numbers $U(n,k)$ depend only on the S-type of $P_{n-1}\mathbb{C}/P_{n-k-1}\mathbb{C}$ so we can use periodicity of stunted projective spaces to define $U(n,k)$ for all $n \in \mathbb{Z}$ and $k \geq 0$.

The numbers $U(n,k)$ have several other interpretations; an important one is the following: By S-duality $U(n,k)$ is the order of the cokernel of

$$h: \pi_s^o(P_{k-1}\mathbb{C}^{-n\tilde{H}}) \longrightarrow \tilde{H}^o(P_{k-1}\mathbb{C}^{-n\tilde{H}};\mathbb{Z}) \tag{7.1o}$$

Let $i: S^o \longrightarrow P_{k-1}\mathbb{C}^{-n\tilde{H}}$ denote the inclusion of the fibre, then the commutative diagram

$$\begin{array}{ccc} \pi_s^o(P_{k-1}\mathbb{C}^{-n\tilde{H}}) & \xrightarrow{i^*} & \pi_s^o(S^o) \\ \downarrow h & & \| \\ \tilde{H}^o(P_{k-1}\mathbb{C}^{-nH}) & \xrightarrow{\cong} & H^o(S^o) \end{array}$$

shows, that $U(n,k)$ measures the extent to which $-n\tilde{H}$ fails to be orientable for stable homotopy. More generally, for an orientable vector bundle ξ over a connected space X, we define the codegree of ξ $cd(\xi)$ to be the index of $h\pi_s^o(X^{\tilde{\xi}})$ in $\tilde{H}^o(X^{\tilde{\xi}};\mathbb{Z}) = \mathbb{Z}$. Thus $U(n,k) = cd(-nH_{k-1})$ where H_{k-1} denotes the Hopf bundle over $P_{k-1}\mathbb{C}$.

It is well known that $cd(\xi) = 1 \iff J(\xi) = 0$, even p-locally $\nu_p(cd(\xi)) = 0 \Leftrightarrow \nu_p|J(\xi)| = 0$.

So $cd(\xi)$ and $|J(\xi)|$ contain the same primes, but in general with different exponents. Whereas $|J(-\xi)| = |J(\xi)|$ we have in general $cd(\xi) \neq cd(-\xi)$.

The interpretation of $U(n,k)$ as a codegree allows simple proofs for the properties of the numbers $U(n,k)$ which we collect in the following proposition:

<u>Proposition 7.11</u>: a) 1. $U(n,k)$ is a multiple of $U(n,m)$ if m k

2. $U(n,k) \cdot U(m,k)$ is a multiple of $U(n+m, k)$

3. $U(n,k) = U(n+r \cdot M_k, k)$

4. $U(n,k) = 1$ iff n is a multiple of $M_k = |J(H_{k-1})|$

b) The statements under a) are true p-locally, that is one considers only the powers of a fixed prime in $U(i,j)$. In 3. and 4. we can use instead of M_k the p-primary part of M_k.

<u>Proof:</u>

1. Follows by diagram chasing using the map $P_{m-1}\mathbb{C}^{-nH} \to P_{k-1}\mathbb{C}^{-n\tilde{H}}$.

2. $U(m,k)$ times the Thom class U of $-mH_{k-1}$ comes from stable cohomotopy, say $h(y) = U(m,k) \cdot U$. Then multiplication by y defines a map ϕ_1 in the commutative diagram

$$\begin{array}{ccc} \pi_s^o(P_{k-1}\mathbb{C}^{-n\tilde{H}}) & \xrightarrow{\phi_1} & \pi_s^o(P_{k-1}\mathbb{C}^{-(n+m)H}) \\ \downarrow cd(-nH) & & \downarrow cd(-(n+m)H) \\ \tilde{H}^o(P_{k-1}\mathbb{C}^{-nH}) & \xrightarrow{cd(-mH)} & \tilde{H}^o(P_{k-1}\mathbb{C}^{-(n+m)H}) \end{array}$$

which proves 2.

3. Because $r\cdot M_k\cdot H_{k-1}$ is J trivial and so π^s_*-orientable we have a Thom isomorphism

$$\pi^o_s(P_{k-1}\mathbb{C}^{-n\tilde{H}}) \longrightarrow \pi^o_s(P_{k-1}\mathbb{C}^{-(n+r\cdot M_k)\cdot H})$$

this proves 3.

4. follows from the equivalence of the notions of π^s_*-orientability and J-triviality for a vector bundle

The p-primary versions of the proofs of 1. - 4. prove b)

The p-localization of $SP_n\mathbb{C}/P_{n-k}\mathbb{C}$ splits up into a wedge of $(p-1)$ spaces X_i, where each X_i has cells only in dimensions differing by $2(p-1)$ (for a proof see [3o]). This immediately implies

<u>Propositon 7.12</u>: $U(n,k)_{(p)} = U(n,k+1)_{(p)}$ if $k \not\equiv 0(p-1)$ or what is the same $U(n,1+r+t(p-1))_{(p)}$ is constant and equals $U(n,1+t(p-1))_{(p)}$ as long as $o \leq r < p-1$.

<u>Remark</u>:
By the splitting of $W_{n,k}$ [3o] the same is true for the James numbers $\bar{U}(n,k)_{(p)}$.

The number $U(n,k)$ is the index of $h\pi^s_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$ in $H_{2n-2}(P_{n-j}\mathbb{C}/P_{n-1-k}\mathbb{C})$. We define a number $U_A(n,k)$ by using $A_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$ instead of $\pi^s_*(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$. Clearly $U(n,k)_{(p)}$ is a multiple of $U_A(n,k)$. It is a conjecture that $U(n,k)_{(p)}$ is actually the same as $U_A(n,k)$ (using the right theory for p=2). This can also be stated in the form that K-theory detects the attaching maps for $P_n\mathbb{C}/P_k\mathbb{C}$ (stably).

Equivalent to the conjecture $U(n,k)_{(p)} = U_A(n,k)$ are the statements that the K-theory Hurewicz map

$$h_k: \pi^s_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) \longrightarrow K_o(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})_{(p)}$$

maps $\pi^s_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$ onto a direct summand, or that

$$h_A: \pi^s_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) \longrightarrow A_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$$

is onto.

By looking at the diagram of exact sequences

$$\begin{array}{ccccc}
\pi_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) & \longrightarrow & \pi^s_{2n-2}(S^{2n-2}) & \xrightarrow{\partial_\pi} & \pi^s_{2n-3}(P_{n-2}\mathbb{C}/P_{n-1-k}\mathbb{C}) \\
\downarrow h_A & & \| & & \downarrow e \\
A_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) & \longrightarrow & A_{2n-2}(S^{2n-2}) & \xrightarrow{\partial_A} & A_{2n-3}(P_{n-2}\mathbb{C}/P_{n-1-k}\mathbb{C})
\end{array}$$

we see the connection between the statements "h_A is onto" and $|\partial_\pi(1)| = |e\partial_\pi(1)|$.

We have $U_A(n,k) = |\partial_A(1)|$ and either coker $h_A: A_{2n}(P_n\mathbb{C}/P_k\mathbb{C}) \longrightarrow H_{2n}(P_n\mathbb{C}/P_k\mathbb{C})$ or $\partial_A(1)$ can be calculated using K-theory (see below). These numbers express the K-theory restrictions forthe existence of a stable q-section f of $P_n\mathbb{C}/P_k\mathbb{C} \xrightarrow{\pi} S^{2n}$, that is for an S-map $f: S^{2n} \longrightarrow P_n\mathbb{C}/P_k\mathbb{C}$ with $\pi{\circ}f$ of degree q. These restrictions have been found and considered by several authors, e.g. [44], [27], [16].

$U_A(n,k)$ can be calculated as follows $(p\neq 2)$: First we construct a generator of $A_{2n}(P_n\mathbb{C}/P_k\mathbb{C}) \subset K_o(P_n\mathbb{C}/P_k\mathbb{C})$. The element $b^n_1 \in K_o(P_n\mathbb{C})$ lies in $\ker(c\psi^k - k^n)$, therefore $pr_*(b^n_1) \subset K_o(P_n\mathbb{C}/P_k\mathbb{C})$ too. But

$$pr_*(b_1^n) = \sum_{j=k+1}^{n} S(n,j)j! \cdot b_j \tag{7.13}$$

can be divided by the greatest common divisor r of the numbers $S(n,j)j!$, so $pr_*(b_1^n)/r$ generates $A_{2n}(P_n\mathbb{C}/P_k\mathbb{C})$. Because $h_A(b_1^n) = n![P_n\mathbb{C}]$ we find $\nu_p(n!/a) = \nu_p(U_A(n+1,n-k))$ or

$$\partial_A(1) = \frac{1}{n!} \sum_{j=k+1}^{n-1} S(n,j)j! \cdot b_j \tag{7.14}$$

in $A_{2n-1}(P_{n-1}\mathbb{C}/P_k\mathbb{C}) \cong A_{2n}(P_{n-1}\mathbb{C}/P_k\mathbb{C};\mathbb{Q}/\mathbb{Z}) \subset K_o(P_{n-1}\mathbb{C}/P_k\mathbb{C};\mathbb{Q}/\mathbb{Z})$

It is known that

$$p^{\left[\frac{k}{p-1}\right]} \cdot \frac{S(n+k,n)n!}{(n+k)!} \in \mathbb{Z}_{(p)} \tag{7.15}$$

therefore we have

<u>Proposition 7.16</u>: $\nu_p U_A(n,k) \leq \left[\frac{k-1}{p-1}\right]$ for an odd prime p

The interpretation of $U(n,k)$ as a codegree leads to a second possibility of calculating $U_A(n,k)$:
Let ξ be a complex vector bundle on a connected finite CW-complex X. we denote by U_K and U_H the Thom classes of ξ in K-theory or homology, then it is well known that

$$chU_K = T(\xi) \cup U_H$$

where $T(L) = (e^{c_1(L)}-1)/c_1(L)$ for a line bundle L.
So $ch^{-1}U_H = (ch^{-1}T^{-1}(\xi)) \cup U_K$ in $K^o(X;\mathbb{Q})$. If $m \cdot U_H$ lies in the image of the Hurewicz map h (or h_A) , then because h factors over K-theory via ch, the class mU_H must be in $ch(K^o(X))$. This means the class $m \cdot ch^{-1}T^{-1}(\xi)$ must be integral, that is in the image of $K^o(X) \to K^o(X;\mathbb{Q})$.

For a line bundle L we have $ch^{-1}T^{-1}(L) = ch^{-1}(c_1(L)/(e^{c_1(L)}-1))$ $= \ln(x+1)/x$ where $x = L-1$. So for $X = P_{n-1}\mathbb{C}$ and $\xi = kH$:

$$ch^{-1}T^{-1}(kH_{n-1}) = \left(\frac{\ln(x+1)}{x}\right)^k / x^n = o \qquad (7.17)$$

and the denominators of this polynomial must divide $cd(kH_{n-1}) = U(-k,n)$, a well known condition on $U(-k,n)$.

For $p \neq 2$ $\tilde{A}^o(P_{n-1}\mathbb{C}^{k\tilde{H}})$ is torsion free and so a generator z of this group gives in $\tilde{K}^o(P_{n-1}\mathbb{C}^{k\tilde{H}})$ an integral class $\tilde{z}$ which is mapped by the chern character onto $\pm U_A(-k,n)\cdot U_H$. This implies that the maximal denominator in (7.17) is exactly $U_A(-k,n)$.

Using this it is easy to prove

Proposition 7.18: The $e_{\mathbb{C}}$-invariant of the stable map

$\tau^r \in \pi_s^{-1}(P_{n-1}\mathbb{C}^{r\tilde{H}})$ inducing the transfer t^r is

given by $e_{\mathbb{C}}(\tau^r) = \frac{1}{x}\left\{\left(\frac{\ln(x+1)}{x}\right)^{r-1} - 1\right\} / x^n = o \cup U_K$

in $K^o(P_{n-1}\mathbb{C};\mathbb{Q}/\mathbb{Z})$.

A proof can also be deduced by the method of [24].
This formula also allows to compute the transfer map

$$t^r:\ \pi^s_{2n}(X^+ \wedge P_\infty\mathbb{C}^{r\tilde{H}}) \longrightarrow \pi^s_{2n+1}(X^+)$$

from Adams filtration 0 to 1 by generalizing (4.3) in the obvious way.

The trick used in [47] allows us to find out for which values of n and k the numbers $U_A(n,k)$ take their maximal value:
Let ξ be a (p-1)-th root of p , that is $\xi^{p-1} = p$, then the denominators of

$$\ln(x\cdot\xi+1)/x\cdot\xi = 1 - \frac{x\xi}{2} + \ldots + x^{p-1} - p\xi\,\frac{x^p}{p+1} + \ldots$$

do not contain p . Therefore the same is true for $(\ln(x\xi+1)/x\xi)^n$ for all $n\in\mathbb{Z}$. If we write

$$\left(\frac{\ln(x+1)}{x}\right)^n = \sum_t a(n+t,n)x^t$$

$$(n\geqslant 0)$$

$$\left(\frac{\ln(x+1)}{x}\right)^{-n} = \sum_t b(n+t,n)x^t$$

we have proved $p^{\left[\frac{t}{p-1}\right]}\cdot a(n+t,n)\in\mathbb{Z}_{(p)}$ and $p^{\left[\frac{t}{p-1}\right]}\cdot b(n+t,n)\in\mathbb{Z}_{(p)}$. This proves (7.16).

The series $\ln(x\xi+1)/x\xi$ reduced mod p is simply the polynominal of degree $p-1$:

$$\ln(x\xi+1)/x\xi = 1 - \frac{x\xi}{2} + \frac{(x\xi)^2}{3} - + x^{p-1}$$

The binomial theorem then gives the mod p-value of $p^k\cdot a(n+k(p-1),n)$ and $p^k\cdot b(n+k(p-1),n)$:

$$p^k\cdot a(n+k\cdot(p-1),n) \equiv \binom{n}{k} \mod p \qquad (7.19)$$

$$(-1)^k p^k\cdot b(n+k\cdot(p-1),n) \equiv \binom{n-1+k}{k} \mod p$$

From this it follows:

<u>Proposition 7.2o:</u> $n\geqslant 0$

$$U_A(n,k(p-1)+1) = p^k \Leftrightarrow \binom{k+n-1}{k} \not\equiv 0(p)$$

$$U_A(-n,k(p-1)+1) = p^k \Leftrightarrow \binom{n}{k} \not\equiv 0(p)$$

Remarks

1. Similarly one can derive formulas for the mod p^2 value of $a(n+t,n)$ and so find conditions for $U_A(n,k(p-1)+1)=p^{k-1}$.

2. Another possibility to compute $U_A(n,r)$ in some cases is to compute the mod p value of $S(n,k)$ (see (7.14)). If $S(m,m-t(p-1)) \not\equiv 0(p)$ one gets $U_A(m+1,t(p-1)+1) = m!/(m-t(p-1))!$ In some cases it is possible (see below) to deduce a closed formula for $U_A(n,k)$, but in general this seems to be rather hard.

3. The splitting principle of [46] or calculations with the Chern character imply that for every complex vector bundle ξ on a 2n-dimensional CW-complex X the number $p^{\left[\frac{n}{p-1}\right]}$ is an upper bound for the index of $h_A: \tilde{A}^o(X^\xi) \longrightarrow \tilde{H}^o(X^\xi;\mathbb{Z})$. At present no example seems to be known, where $cd(\xi)$ is larger than $p^{\left[\frac{n}{p-1}\right]}$ $(p\neq 2)$.

4. Let f be an element in $\mathbb{Q}[x]/(x^n)$ with $f(0) = 1$. We call the minimal positive number m with $m\cdot f\in \mathbb{Z}[x]/(x^k)$ the additive order of f and the minimal number k with $f^k\in \mathbb{Z}[x]/(x^n)$ the multiplicative order of f. For $f = (\ln(x+1)/x)^r$ the multiplicative order of f is $|J(rH_n)|$ and the additive order is $U_A(-r,n+1)$. Some elementary manipulations with the polynomial f show that in case of $f = (\ln(x+1)/x)^r$ the multiplicative order of f is always a multiple of the additive order. It would be interesting to know under what conditions on X $|J(\xi)|$ is always a multiple of $cd(\xi)$ (see also (7.25)). At present it is only known that both numbers contain the same set of primes.

5. In cases where $U_A(n,k) = U(n,k)_{(p)}$, (7.16) shows that $|J(nH_{k-1})|$ can be much larger than $cd(-nH_{k-1})$.

Before we start with the proofs that $U_A(n,k) = U(n,k)_{(p)}$ for certain values of n, we discuss shortly the special case $p=2$.

Let $AR^*(\ )$ be the cohomology theory defined by using real K-theory and ψ^3-1 in the same way as A^* is defined by complex K-theory; see [36] . Then we can define numbers $U_{AR}(n,k)$ by

$$U_{AR}(n,k) = \text{index of } \ h_{AR}\colon \widetilde{AR}^0(P_{k-1}\mathbb{C}^{-n\tilde{H}}) \longrightarrow \tilde{H}^0(P_{k-1}\mathbb{C}^{-n\tilde{H}})$$

We still have the corresponding numbers $U_A(n,k)$, defined by complex K-theory for p=2. The numbers $U_A(n,k)$ for $p=2$ can be computed in the same way as $U_A(n,k)$ for $p\neq 2$.

<u>Proposition 7.21</u>: Let r be odd, then $U_{AR}(r,m) = U_A(r,m)$ unless $m\equiv 3(4)$ and $U_A(r,m+1)=2\cdot U_A(r,m)$; in this case we have $U_{AR}(r,m)=2U_A(r,m) = U_{AR}(r,m+1)$.
If r is even, then $U_{AR}(r,m) = U_A(r,m)$ unless $m\equiv 2(4)$ and $8\cdot U_A(r,m)/U_A(r,m+1) \not\equiv 0(2)$, where we have $U_{AR}(r,m) = 2U_A(r,m)$.

<u>Proof:</u> We use the commutative diagram

$$\begin{array}{ccccc}
\widetilde{AR}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}) & \xrightarrow{k_{\mathbb{R}}} & \widetilde{KO}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}) & \xrightarrow{\psi^3-1} & \\
\downarrow c_A & & \downarrow c & & \\
\tilde{A}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}) & \xrightarrow{k_{\mathbb{C}}} & \tilde{K}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}) & & \\
 & & \downarrow \delta & & \\
 & & \widetilde{KO}^{-6}(P_{m-1}\mathbb{C}^{-r\tilde{H}}) & &
\end{array}$$

where $k_{\mathbb{R}}$, $k_{\mathbb{C}}$ are the maps of the defining long exact sequences for AR and A, and the vertical sequence is part of the Bott sequence relating real and complex K-theory.

The maps $k_{\mathbb{R}}$ and c are inclusions of direct summands as long as

$\widetilde{KO}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}})$ and $\widetilde{KO}^{-6}(P_{m-1}\mathbb{C}^{-r\tilde{H}})$ are free (modulo torsion in $\widetilde{AR}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}})$). This is the case except when A) $r \equiv 1(2)$ and $m \equiv 3(4)$ or B) $r \equiv 0(2)$ and $m \equiv 2(4)$ (for $KO^*(P_n\mathbb{C})$ see [14]). In the other cases $c_A(\widetilde{AR}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}))$ must be a direct summand in $\tilde{A}^0(P_{m-1}\mathbb{C}^{-r\tilde{H}}) \cong \mathbb{Z}$ or what is the same: c_A must be onto.

Let's first treat A):

In the following commutative diagram

$$\begin{array}{ccccc}
\widetilde{AR}^0(P_{3+4s}\mathbb{C}^{-r\tilde{H}}) & \xrightarrow{j_{AR}} & \widetilde{AR}^0(P_{2+4s}\mathbb{C}^{-r\tilde{H}}) & \longrightarrow & AR_{8s+5} = 0 \\
\downarrow c_1 & & \downarrow c_2 & & \downarrow \\
\tilde{A}^0(P_{3+4s}\mathbb{C}^{-r\tilde{H}}) & \xrightarrow{j_A} & \tilde{A}^0(P_{2+4s}\mathbb{C}^{-r\tilde{H}}) & \longrightarrow & A_{8s+5} \cong \mathbb{Z}_2
\end{array}$$

j_{AR} and c_1 must be onto. The map j_A is multiplication by $\bar{a} = U_A(r,4s+4)/U_A(r,4s+3)$. So the index of c_2 must be $\bar{a}$ too. This completes the case r odd.

In case B) the problem is the torsion group $\mathbb{Z}_2$ in $\widetilde{KO}^0(P_{4k+1}\mathbb{C}^{-2r\tilde{H}}) \cong KO^0(P_{4k+1}\mathbb{C})$. Because this subgroup comes from a sphere, it is in the kernel of ψ^3-1, but it may happen that $\mathbb{Z}_2 \subset \mathrm{im}\,(\psi^3-1)$. In this case it could be that $U_{AR}(2r,4k+2) = 2U_A(2r,4k+2)$. By a^i we denote the elements in $\widetilde{KO}^0(P_j\mathbb{C}^{-2r\tilde{H}})$ which correspond under the Thom isomorphism to the generators $(r(H)-2)^i$ of $KO^0(P_j\mathbb{C})$. The torsion group of $\widetilde{KO}^0(P_{4k+1}\mathbb{C}^{-2r\tilde{H}})$ is then generated by a^{2k+1}. It is easy to see that $U_A(2r,4k+2) = U_A(2r,4k+1) = U_{AR}(2r,4k+1)$. Let x be the generator of the image of $k_{\mathbb{R}}$: $\widetilde{AR}^0(P_{4k}\mathbb{C}^{-2r\tilde{H}}) \to \widetilde{KO}^0(P_{4k}\mathbb{C}^{-2r\tilde{H}})$. We consider x also as an element of $\widetilde{KO}^0(P_{4k+i}\mathbb{C}^{-2r\tilde{H}})$ by using the same linear combination of a^i as the one defining x in $\widetilde{KO}^0(P_{4k}\mathbb{C}^{-2r\tilde{H}})$.

If $(\psi^3-1)\,x = a^{2k+1}$ in $\widetilde{KO}(P_{4k+1}\mathbb{C}^{-2r\tilde{H}})$ (because $(\psi^3-1)a^{2k+1} = 0$ this does not depend on the choice of the lifting of x from $P_{4k}\mathbb{C}^{-2r\tilde{H}}$ to $P_{4k+1}\mathbb{C}^{-2r\tilde{H}}$) then

$U_{AR}(2r,4k+2) = 2U_{AR}(2r,4k+1) = 2U_A(2r,4k+2)$; otherwise

$U_{AR}(2r,4k+2) = U_A(2r,4k+2)$.

Let $\bar{x}$ be the generator of im $(k_{\mathbb{R}}: \widetilde{AR}^o(P_{4k+2}\mathbb{C}^{-2r\tilde{H}}) \to \widetilde{KO}^o(P_{4k+2}\mathbb{C}^{-2rH}))$ and $s_1 = U_A(2r,4k+2)$, $s_2 = U_A(2r,4k+3)$. Then

$$\bar{x} = (s_2/s_1)\cdot x + b\cdot a^{2k+1}$$

and $(\psi^3-1)\bar{x} = 0$ in $\widetilde{KO}^o(P_{4k+2}\mathbb{C}^{-2r\tilde{H}})$. Because $(\psi^3-1)a^{2k+1} = (3^{4k+2}-1)a^{2k+1}$ we have

$$(\psi^3-1)x = 8(s_1/s_2)b'\cdot a^{2k+1}$$

If $\nu_2(s_2) > \nu_2(s_1)$ then b and b' are not divisible by 2. By restricting to $\widetilde{KO}^o(P_{4k+1}C^{-2r\tilde{H}})$ we see that

$$U_{AR}(2r,4k+2) = 2U_A(2r,4k+2) \text{ iff } 8s_2/s_1 \not\equiv 0(2) .$$

<u>Remark:</u>

The examples suggest that $8U_A(2r,4k+3)/U_A(2r,4k+2) \not\equiv 0(2)$ never happens.

In [44] G. Walker has shown the equation $|\partial_\pi(1)| = |e\ \partial_\pi(1)|$ to be true for p odd and $k \leq (p-1)\cdot(p^2-p+1)$ (with some restrictions on n because he worked with $W_{n,k}$) , and so he has verified the conjecture $U(n,k) = U_A(n,k)$ in this range.

We can easily reprove this: one gets nearly the same range of validity for $U_A(n,k) = U(n,k)_{(p)}$ from (4.5) applied to $P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}$. Namely as long as $k \leq (p-1)\cdot(p^2-p)$ the Hurewicz map

$$h_A: \pi^s_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C}) \longrightarrow A_{2n-2}(P_{n-1}\mathbb{C}/P_{n-1-k}\mathbb{C})$$

must be onto, so $U_A(n,k) = U(n,k)_{(p)}$ for $k \leq (p-1)(p^2-p)$.

By calculating the Atiyah-Hirzebruch spectral sequence for $C_*(P_n\mathbb{C}/P_k\mathbb{C})$ (see § 6) one can push the bound on k higher; without difficulty one gets the range $k \leq (p-1)(2p^2-2p+1)$. For $p=2$ the equation $U_{AR}(n,k) = U(n,k)_{(2)}$ is true for $k \leq 8$ by the results of [31] .

Beyond those values of k, the equality $U_A(n,k) = U(n,k)_{(p)}$ is known for $n=k$ and $n=k+1$ because $\pi^s_{2n}(P_\infty\mathbb{C}^+) \longrightarrow A_{2n}(P_\infty\mathbb{C})$ and $\pi^s_{2n}(P_\infty\mathbb{C}^H) \longrightarrow A_{2n}(P_\infty\mathbb{C}^H)$ are onto ($U(n,n) = (n-1)!$ and $U(n+1,n) = n!$). This indicates that it may be simpler to look first at those values of n for which (n,k) is not in the range where $U(n,k) = \bar{U}(n,k)$ and then deduce results on $U(n,k)$ using periodicity.

We turn back to the interpretation of $U(n,k)$ as a codegree

<u>Theorem 7.22</u>: If E is an n-dimensional vector bundle over a twofold suspension S^2X, then the codegree of E is determined by K-theory, that is

$$h_A: \pi^n_s(M(E^n)) \longrightarrow A^n(M(E^n)) \quad \text{is onto}$$

<u>Proof</u>: We write $\tilde{E}$ for E-n. It is well known that for bundles over supension spaces the inclusion of the fibre $S^o \longrightarrow M(E)$ is part of the following cofibre sequence

$$\longrightarrow SX \xrightarrow{g} S^o \xrightarrow{i} M(E) \xrightarrow{j} S^2X \longrightarrow \qquad (7.23)$$

where g is the image of E under the J-homomorphism $J: \widetilde{KO}^{-2}(X) \longrightarrow \pi^{-1}_s(X)$. We get the following commutative diagram

$$\begin{array}{ccccccc}
\pi_s^{-1}(X) & \xleftarrow{g^*} & \pi_s^o(S^o) & \longleftarrow & \pi_s^o(M(\tilde{E})) & & \\
\downarrow e & & \| & & \downarrow h_A & & \\
\tilde{A}^{-1}(X) & \longleftarrow & \tilde{A}^o(S^o) & \longleftarrow & \tilde{A}^o(M(\tilde{E})) & \longleftarrow & \tilde{A}^{-2}(X)
\end{array} \qquad (7.24)$$

The J-homomorphism $J: KO^{-2}(X) \longrightarrow \pi_s^{-1}(X)$ can be factored through a map $j: A^{-1}(X) \longrightarrow \pi_s^{-1}(X)_{(p)}$ and in [43] it is proved that ej is bijective for odd primes. Because $g^*(1) = g$ lies in the image of J, e must preserve the order of $g^*(1)$, that is

$$|g^*(1)| = |e\ g^*(1)|$$

But then h_A must be onto.

<u>Remark</u>: For $p=2$ (7.22) remains true with A_* replaced by the theory AR_* ($AR_*(*) = im(J) + \mu_r$ and $\eta\mu_r$-elements). We omit the proof.

<u>Corollary 7.25</u>: Let E be a vector bundle over a twofold suspension S^2X, then the Hurewicz map $\pi_s^n(M(E)) \longrightarrow \tilde{H}^n(M(E);\mathbb{Z}_{(p)})$ has cokernel of order $|J(E)|_{(p)}$ where $J(E)$ is the class of E in $J(S^2X)_{(p)}$.

As a first application of (7.22) we get

<u>Corollary 7.26</u>: Let $r+1 \geq m$ and $s \equiv o\ (M_{r+1})_{(p)}$ then

$$U_A(s,r+m+1) = U(s,r+m+1)_{(p)} = |J(sH_{r+m})|_{(p)}$$

<u>Proof</u>: We consider the following cofibre sequence

$$P_r\mathbb{C} \xrightarrow{i} P_{r+m}\mathbb{C} \xrightarrow{p} P_{r+m}\mathbb{C}/P_r\mathbb{C}$$

If $r+1 \geq m$ then $P_{r+m}\mathbb{C}/P_r\mathbb{C}$ is a double suspension say $P_{r+m}\mathbb{C}/P_r\mathbb{C} = S^2X$. Choose s such that $s\cdot H$ is fibre homotopy trivial on $P_r\mathbb{C}$, that is $s \equiv 0\ (M_{r+1})_{(p)}$. Because the the squence of J-groups

$$0 \longrightarrow J(P_{r+m}\mathbb{C}/P_r\mathbb{C})_{(p)} \longrightarrow J(P_{r+m}\mathbb{C})_{(p)} \longrightarrow J(P_r\mathbb{C})_{(p)} \longrightarrow 0$$

is exact, we can find a vector bundle ξ on S^2X with

$$J(p^* \xi) = J(sH)$$

We then have the following cofibre sequence of Thom spaces

$$\longrightarrow M(-s\tilde{H}) \xrightarrow{M(p)} M(-\tilde{\xi}) \longrightarrow S^1 \wedge P_r\mathbb{C} \longrightarrow S^1 \wedge M(-s\tilde{H}) \longrightarrow$$

and therefore

$$\begin{array}{ccccc} \pi_s^o(M(-\tilde{\xi})) & \longrightarrow & \pi_s^o(M(-s\tilde{H})) & \longrightarrow & \pi_s^o(P_r\mathbb{C}) \\ \downarrow h_A & & \downarrow & & \downarrow \\ \tilde{A}^o(M(-\tilde{\xi})) & \xrightarrow{M(p)_*} & \tilde{A}^o(M(-s\tilde{H})) & \longrightarrow & \tilde{A}^o(P_r\mathbb{C}) \end{array}$$

The map h_A is onto by (7.17) and $M(p)_*$ is onto because $\tilde{A}^o(P_r\mathbb{C})=0$, so $\pi_s^o(M(-s\tilde{H})) \longrightarrow \tilde{A}^o(M(-s\tilde{H}))$ must be onto.

Therefore $U_A(s,r+1+m) = U(s,r+1+m)_{(p)} = cd(-sH_{r+m})_{(p)}$ and by (7.20) this is $|J(sH_{r+m})|_{(p)}$.

With some more work the result carries over to the case $p=2$.

Remarks:

1. (7.26) is a generalization of a result of James [19], where he considers the situation $m=1$.
2. Because the codegree of bundles induced from a twofold suspension is the same as a J-order, it changes in the same manner if we look at multiples of such bundles, that is

$$cd(k \cdot f^*(\xi)) = \frac{cd(f^*(\xi))}{(k\ ,\ cd(f^*(\xi))\ )} \qquad (7.27)$$

where (a,b) denotes the greatest common divisor of a and b. By looking at examples of $U(n,m)$ and $U(r\cdot n,m)$ one sees that not all (in fact very few) multiples of H are induced from a suspension.

In using (7.22) one does not really need that the bundle in consideration is induced from a suspension, only that the Thom space is S-equivalent to the Thom space of such a bundle. This is the case in the next two applications:

Theorem 7.28: For all n we have

$$U(1,n+1) = U_A(1,n+1) = \prod_{p\geq 2} p^{\left[\frac{n}{p-1}\right]}\cdot a_n$$

where $a_n=2$ if $n\equiv 2$ (4) and $a_n=1$ if $n\not\equiv 2$ (4).

Proof: We look at the cofibre sequence of the S^1-transfer map τ (§2):

$$S^o \longrightarrow P_n\mathbb{C}^{-\tilde{H}} \longrightarrow S^2P_{n-1}\mathbb{C}^+ \xrightarrow{\tau} S^1 \longrightarrow \qquad (7.29)$$

By (2.5) we know $\tau = J(\beta(-H))$ where $\beta : \tilde{K}^o(P_{n-1}\mathbb{C}^+) \to \tilde{K}^o(S^2P_{n-1}\mathbb{C}^+)$ is Bott periodicity. Therefore we have a second cofibre sequence inducing τ:

$$S^o \longrightarrow M(\beta(-\tilde{H})) \longrightarrow S^2P_{n-1}\mathbb{C}^+ \xrightarrow{\tau} S^1 \longrightarrow \qquad (7.3o)$$

We map (7.29) into (7.3o) and get some map $P_n\mathbb{C}^{-\tilde{H}} \longrightarrow M(\beta(-\tilde{H}))$ which is a stable homotopy equivalence because the induced map in homology is an isomorphism.

So $U(1,n+1) = |\tau| = |e(\tau)| = U_A(1,n+1)$ where $\tau\in\pi_s^{-1}(P_{n-1}\mathbb{C}^+)$. The e-invariant of τ is computed in [24] and is given by

$$e(\tau) = \frac{1}{x} - \frac{1}{\log(x+1)} \qquad (7.31)$$

in $K^{-2}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[[x]]$.

Up to a dimension shift the denominators in $e(\tau)$ are the same as the denominators in $x/\log(x+1)$. Because $\binom{k}{k} \not\equiv 0(p)$ for all k it follows from (7.19) that the coefficient of $x^{t(p-1)}$ in $x/\log(x+1)$ has denominator exactly p^t. The extra factor for $p=2$ follows from (7.21).

Theorem 7.32: For all n we have $U(2,n)_{(p)} = U_A(2,n)$.

Proof: The stable map γ representing the transfer t^{-1} is an element in $\pi_s^{-1}(P_n\mathbb{C}^{-\tilde{H}})$. We look at the exact sequence induced by the cofibre sequence of t^o:

$$\pi_s^{-1}(S^o) \xleftarrow{i^*} \pi_s^{-1}(P_n\mathbb{C}^{-\tilde{H}}) \xleftarrow{j^!} \pi_s^{-3}(P_{n-1}\mathbb{C}^+) \xleftarrow{t^o} \pi_s^{-2}(S^o)$$

By (2.22) we know that there is an element of $\pi_s^{-3}(P_{n-1}\mathbb{C}^+)$ lying in the image of the J-homomorphism which maps to γ under $j^!$. But the behaviour of t^o on $\pi_s^{-2}(S^o)$ can be determined using AR^*-theory so $|\gamma| = |e(\gamma)|$ even at the prime 2 . Then the cofibre sequence of γ in cohomotopy shows $\pi_s^o(P_n\mathbb{C}^{-2\tilde{H}}) \longrightarrow \tilde{A}^o(P_n\mathbb{C}^{-2\tilde{H}})$ to be onto (for p=2 we use $\widetilde{AR}^o(P_n\mathbb{C}^{-2\tilde{H}})$) .

There is also a close relation between the transfer maps t^r und the numbers $U(n,k)$: The transfer t^r induces a cofibre sequence

$$\pi_{2n}^s(P_\infty\mathbb{C}^{(-r-1)\tilde{H}}) \xrightarrow{j!} \pi_{2n-2}^s(P_\infty\mathbb{C}^{-r\tilde{H}}) \xrightarrow{t^{(-r)}} \pi_{2n-1}^s(S^o)$$

$j^!$ induces a map from $\pi_{2n}^s(P_\infty\mathbb{C}^{(-r-1)\tilde{H}})/tor$ to $\pi_{2n-2}^s(P_\infty\mathbb{C}^{-r\tilde{H}})/tor$ which is multiplication by the number

$$a = U(n-r,n+1)/U(n-r,n)$$

or what is the same:

$$\frac{U(n-r,n+1)}{U(n-r,n)} = \min\left\{|t^{-r}(x)| \;\middle|\; x\in\pi_{2n-2}^s(P_\infty\mathbb{C}^{-r\tilde{H}}) \text{ maps to a generator in } \pi_*^s(P_\infty\mathbb{C}^{-r\tilde{H}})/tor\right\}$$

If we look at the situation in K-theory, we simply have

$$|t_A^{-r}(x)| = U_A(n-r,n+1)/U_A(n-r,n)$$

An application of this is

<u>Proposition 7.33</u>: $U(n,n+1) = U_A(n,n+1) = (n-1)!$ denominator of $\left(\frac{B_n}{n}\right)$

<u>Proof</u>: We first prove that $t^o(\sigma^n) \in im(J)$.
The map $S^3 \wedge P_n\mathbb{C} \longrightarrow S^1 \wedge P_{n+1}\mathbb{C}$ induced by restricting the H-space multiplication map to $S^2 \times P_n\mathbb{C}$ and then applying the Hopf construction (see §2) maps σ^n to σ^{n+1} in $\pi_*^s(P_\infty\mathbb{C}^+)$. The same map is used to construct a generator u_n for $\pi_{2n+1}(U)$, see [43]. This implies

$w_*(\sigma^n) = i(u_n)$ in the commutative diagram

$$\begin{array}{ccccc} \pi_{2n+1}(U) & \xrightarrow{\;i\;} & \pi^s_{2n+1}(U) & \xleftarrow{\;w_*\;} & \pi^s_{2n}(P_\infty\mathbb{C}^+) \\ & {\scriptstyle J}\searrow & \downarrow{\scriptstyle J'_U} & \swarrow{\scriptstyle -t^o} & \\ & & \pi^s_{2n+1}(S^o) & & \end{array}$$

Therefore $t(\sigma^n) \in \mathrm{im}(J)$.

For later use we remark that the map $\theta' : \pi^s_{2n+1}(U) \longrightarrow \pi^s_{2n}(P_\infty\mathbb{C}^+)$ (see (3.2)) has the property $\theta' \circ w_* = \mathrm{id}$ (see [10] or [11]); the equation $i(u_n) = w_*(\sigma^n)$ implies

$$\theta' i(u_n) = \sigma^n \tag{7.34}$$

The proof of (7.33) is finished by observing that, because $t^o(\sigma^n) \in \mathrm{im}(J)$, we have $|e(t^o(\sigma^n))| \cdot \sigma^n \in \mathrm{im}\, j^!_\pi$ and $|e(t^o(\sigma^n))| \cdot b^n_1 \in \mathrm{im}\, j^!_A$, thus h_A must be onto. The maps mentioned are from the diagram

$$\begin{array}{ccccc} \pi^s_{2n+2}(P_\infty\mathbb{C}^{-\tilde{H}}) & \xrightarrow{\;j^!\;} & \pi^s_{2n}(P_\infty\mathbb{C}^+) & \longrightarrow & \pi^s_{2n+1}(S^o) \\ \downarrow{\scriptstyle h_A} & & \downarrow & & \downarrow{\scriptstyle e} \\ \tilde{A}_{2n+2}(P_\infty\mathbb{C}^{-\tilde{H}}) & \xrightarrow{\;j^!_A\;} & A_{2n}(P_\infty\mathbb{C}) & \longrightarrow & \tilde{A}_{2n+1}(S^o) \end{array}$$

For $p=2$ one has to attend to the base point : $\eta \circ \mu_k [*]$ as well as $4 \cdot \sigma^{4k+1}$ has image $\eta^2 \cdot \mu_k$ = element of order 2 in $\mathrm{im}(J)$ under the transfer t^o.

<u>Proposition 7.35</u>: $U(n-2,n)_{(p)} = U_A(n-2,n)$

$$\text{e.g.}\quad \nu_2(U(4k+i-1,4k+i+1)) = \begin{cases} 1+\nu_2(4k!) & i=0, i=1 \\ 2+\nu_2(4k!) & i=2, i=3 \end{cases}$$

<u>Proof</u>: We first construct elements in $\pi^s_{2n}(P_\infty\mathbb{C}^{-\tilde{H}})$ which are mapped

into im(J) under the transfer t^{-1}.

It is well known that one can include the J-homomorphism in the following exact sequence

$$\pi^s_{k+1}(S^o) \longrightarrow \Omega^{ap}_{k+1} \xrightarrow{\ \Delta\ } \pi_k(SO) \xrightarrow{\ J\ } \pi^s_k(S^o) \longrightarrow \tag{7.36}$$

where Ω^{ap} is the bordism group of closed smooth almost parallelizable manifolds.

In [13] it is shown that one can map this sequence to the cofibre sequence of the bistable J-homomorphism to get a commutative diagram, of which we state the complex analogue:

$$\begin{array}{ccccccc}
\pi^s_{k+1}(S^o) & \longrightarrow & {}^U\Omega^{al}_{k+1} & \xrightarrow{\ \Delta\ } & \pi_k(U) & \xrightarrow{\ J_U\ } & \pi^s_k(S^o) \\
\| & & \downarrow s & & \downarrow i & & \| \\
\pi^s_{k+1}(S^o) & \longrightarrow & \pi^s_{k+1}(M(-id_U)) & \longrightarrow & \pi^s_k(U) & \xrightarrow{\ J'_U\ } & \pi^s_k(S^o) \\
\| & & \downarrow \bar{\theta} & & \downarrow \theta' & & \| \\
\pi^s_{k+1}(S^o) & \xrightarrow{\ i_*\ } & \pi^s_{k+1}(P_\infty\mathbb{C}^{-\tilde{H}}) & \xrightarrow{\ j^!\ } & \pi^s_{k-1}(P_\infty\mathbb{C}^+) & \longrightarrow & \pi^s_k(S^o)
\end{array} \tag{7.37}$$

Here θ' is the map of (3.2) and $\bar{\theta}$ is induced by θ' and id_{SO}. From this diagram, (7.34) and (2.22) it follows that the set $\bar{\theta}\circ s({}^U\Omega^{al}_*)$ is mapped into im(J) by t^{-1}. The rest of the proof is then as in (7.33). From the known value of $|t^O(\sigma^n)|$ and (2.22) we get the explicit formula.

As an application of (7.35) we show that the μ_r-family of Adams [2] (which is not in $\mathrm{im}(t^o)$) is in $\mathrm{im}(t^{-2})$ and can so be constructed via transfer maps. For this we need (see also (7.18)):

<u>Proposition 7.38</u>: Let $z=[M,\phi,f]$ be an element in $\pi^s_{2n}(P_\infty\mathbb{C}^{k\tilde{H}})$ and $x=c_1(f^*H)$ then

$$e_{\mathbb{C}}t^k(z) = \left\langle \frac{1}{x}\left(\left(\frac{e^x-1}{x}\right)^{k-1}-1\right), [M]\right\rangle \bmod \mathbb{Z}.$$

Proof: The proof is a simple generalization of the proof for k=0. Let ξ be the complex line bundle $f^*(H)$ on M. Then the sphere bundle $S(\xi)$ represents $t^k(z)$. To compute $e_{\mathbb{C}}t^k(z)$ we use the definition of e via Todd-genus of a bounding manifold. We take as a bounding manifold the disk-bundle $D(\xi)$ and then have to compute the Kronecker product

$$< \mathrm{Todd}(y) \ , \ [D(\xi),S(\xi)] >$$

where y is the vector bundle on $D(\xi)/S(\xi)$ got from the tangent bundle of $D(\xi)$ and the framing on $S(\xi)$. The map $j^* : \tilde{K}^0(D(\xi),S(\xi)) \longrightarrow K^0(D(\xi))$ maps y to the class of the tangent bundle of $D(\xi)$. Thus using the difference construction in K-theory we can write

$$y = d((1-k)\tilde{\xi}, \mathbb{C}^0; \phi\ )$$

But ϕ is constructed universally, so we can compute this on $P_\infty\mathbb{C}$ where j^* is injective. There the class with $j^*U = (1-k)\tilde{H}$ is $(1-k)$-times the standard K-theory Thom class $U(H)$ so

$$y = (1-k)\cdot U(\xi)$$

If ϕ is the Thom isomorphism of ξ ,then

$$\begin{aligned} e_{\mathbb{C}}t^k(z) &= < \mathrm{Todd}(y),\ [D(\xi),S(\xi)] > \\ &= < \phi^{-1}(\mathrm{Todd}(U(\xi))^{1-k}-1),\ [M] > \\ &= < \frac{1}{x}\ \left(\left(\frac{e^x-1}{x}\right)^{k-1}-1\right),\ [M] > \end{aligned}$$

Theorem 7.39: Let $z \in \pi^s_{8k}(P_\infty\mathbb{C}^{-2\tilde{H}})_{(2)}$ be an element which maps to a generator in $\pi^s_{8k}(P_\infty\mathbb{C}^{-2\tilde{H}})/\mathrm{tor}$, then

$$e_{\mathbb{C}}t^{-2}(z) = \frac{1}{2} \mod \mathbb{Z}_{(2)}$$

that is $t^{-2}(z) = \mu_k \in \pi^s_{8k+1}(S^0)_{(2)} \mod F^3$.

Proof: Denote by ϕ the Thom isomorphism of $-2H$, then by (7.35) an element z, which maps to a generator in $\pi^s_{8k}(P_\infty\mathbb{C}^{-2\tilde{H}})/\mathrm{tor}$ has Hurewicz image $h(z) = (4k)!2\cdot[P_{4k}\mathbb{C}]$ (mod odd primes)

We have to calculate ($x=c_1(H)$)

$$r = et^{-2}(z) = 4k!2\cdot < \frac{x^2}{(e^x-1)^3}, [P_{4k}\mathbb{C}] > \mod \mathbb{Z}_{(2)}$$

If we write $x/(e^x-1) = (1-x/2+a)$, then a contains only elements of even degree and we get

$$\begin{aligned} r &= 4k!\cdot 2\cdot \text{ term of degree } (4k+1) \text{ in } (1-x/2+a)^3 \\ &= 4k!\cdot 2\cdot 3\cdot (a-a^2/2)_{(4k)} \\ &\doteq 4k!\ a^2_{(4k)} \qquad \mod \mathbb{Z}_{(2)} \end{aligned}$$

The proof is finished by using the following lemma:

Lemma 7.40: $r = \sum_{i=1}^{2k-1} \binom{4k}{2i}\cdot B_{2i}\cdot B_{4k-2i} \equiv \frac{1}{2} \mod \mathbb{Z}_{(2)}$

Proof: $r = 2\cdot\sum_{i=1}^{k-1} \binom{4k}{2i}\cdot B_{2i}\cdot B_{4k-2i} + \binom{4k}{2k}\cdot B_{2k}^2$. Because $B_{2i} \equiv \frac{1}{2}$ mod $\mathbb{Z}_{(2)}$, we get $r = \frac{1}{2}\cdot\sum_{i=1}^{k-1}\binom{4k}{2i} + \frac{1}{2}\cdot\binom{4k-1}{2k-1}$. But

$$4r \equiv \sum_{i=1}^{4k-1}\binom{4k}{i} = \sum_{i=o}^{4k}\binom{4k}{i} - 2 \equiv 2^{4k}-2 \mod 4,$$

so $r \equiv \frac{1}{2} \mod \mathbb{Z}_{(2)}$.

Remarks:

1. An element z as above can be characterized as follows: It suffices to know $h(z)$ is divisible by $4k!\cdot 2$ and not by any number with larger 2 primary factor, or to be more explicit, there exists a manifold M^{8k} with a complex line bundle ξ such that the stable tangent bundle of M is isomorphic to 2ξ and such that

$$<c_1(\xi)^{4k}, [M]> = a\cdot 2\cdot 4k!$$

with $a\not\equiv 0$ (2). For all such manifolds we have : $S(\xi)$ with the induced framing represents μ_k mod Adams filtration 3.

2. One can choose an z such that $t^{-2}(z)$ is determined up to $\eta\circ\pi^s_{8k}(S^o) + \nu\circ\pi^s_{8k-2}(S^o)$. It should be possible to define an element z such that $t^{-2}(z) = \mu_k$.
3. In course of the calculation of $e_{\mathbb{C}}t^{-2}(z)$ we have seen that the only odd term in $x/(e^x-1)$ is closely related to the μ_r-family.
4. One can show that $\mu_r \notin \mathrm{im}(t^k)$ with k odd.

In (7.39) we have seen that one needs to know the numbers $U(n,s)$ in order to compute the transfer t^k from filtration 0 to filtration 1. One can now, of course, ask if every element $x\in\pi^s_n(S^o)$ lies in $\mathrm{im}(t^k)$ for some k (depending opon k). For the odd primes no counterexample seems to be known. For p=2 we have already:

For no k is $\sigma\in\pi^s_7(S^o)_{(2)}$ in $\mathrm{im}(t^k)$.

This is proved by computing the relevant $U(n,m)$. By periodicity these are only finitely many cases to check (see table 1).

We turn now to $U(n+i,n)$ with $i\geq 2$.

<u>Proposition 7.41</u>:

$$U(n+2,n)_{(p)} = U_A(n+2,n) \quad (p \text{ odd})$$
$$U(n+2,n)_{(2)} = U_{AR}(n+2,n) \quad (n\not\equiv 0\ (4))$$

$$\text{e.g.}\quad \nu_p(U(n+2,n)) = \begin{cases} \nu_p((n+1)!)-1 & \text{if } n\equiv 0\ (p-1) \\ \nu_p((n+1)!) & \text{if } n\not\equiv 0\ (p-1)\end{cases} \quad (p \text{ odd})$$

<u>Proof</u>: Using the transfer sequence

$$\tilde{A}_{2n}(P_\infty\mathbb{C}^{\tilde{H}}) \longrightarrow \tilde{A}_{2n-2}(P_\infty\mathbb{C}^{2\tilde{H}}) \xrightarrow{t^2} \tilde{A}_{2n-1}(S^o) \xrightarrow{i_*} \tilde{A}_{2n-1}(P_\infty\mathbb{C}^{\tilde{H}})$$

we see $\ker(i_*) \cong \mathbb{Z}/p$ iff $n\equiv 0\ (p-1)$, because the order of

$i_*(b_1/p^{1+\nu_p(n)})$ in $\tilde{A}_{2n-1}(P_\infty \mathbb{C}^{\tilde{H}}) \cong \tilde{A}_{2n+1}(P_\infty \mathbb{C})$ is $p^{\nu_p(n)}$ by (1.14). This gives $U_A(n+2,n) = U_A(n+2,n)/p$ if $n \equiv 0\ (p-1)$ and $U_A(n+2,n) = U_A(n+2,n)$ if $n \not\equiv 0\ (p-1)$. The following diagram shows that it suffices to prove that $\alpha_t \in \pi^s_{2t(p-1)-1}(S^0)_{(p)}$ is in $\mathrm{im}(t^2)$.

$$\begin{array}{ccccc}
\pi^s_{2n}(P_\infty \mathbb{C}^{\tilde{H}}) & \longrightarrow & \pi^s_{2n-2}(P_\infty \mathbb{C}^{2\tilde{H}}) & \xrightarrow{t^2} & \pi^s_{2n-1}(S^0) \\
\downarrow & & \downarrow h & & \downarrow e \\
\tilde{A}_{2n}(P_\infty \mathbb{C}^{\tilde{H}}) & \longrightarrow & \tilde{A}_{2n-2}(P_\infty \mathbb{C}^{2\tilde{H}}) & \longrightarrow & \tilde{A}_{2n-1}(S^0)
\end{array}$$

For this we need a generalization of (3.3):

The generalized J-homomorphism (see e.g. [40])

$$J_{n,k}: \pi_i(W_{n,k}) \longrightarrow \pi_{i+2k}(S^{2n})$$

is defined as follows: we look at $W_{n,k}$ as the space of norm-preserving maps from $\mathbb{C}^k$ into $\mathbb{C}^n$, so we get an evaluation map

$$S^{2k-1} \times W_{n,k} \longrightarrow S^{2n-1}$$

to which the Hopf construction is applied. As in the case n=k we get $J_{n,k}$ and stable versions e.g.

$$J'_{n,k} : \pi^s_i(W_{n,k}) \longrightarrow \pi^s_{i+2k-2n}(S^0)$$

The results of [10],[7] show that the following diagram commutes (d=n-k):

$$\begin{array}{ccc}
\pi_m(W_{n,k}) & \xrightarrow{J_{n,k}} & \pi_{m+2k}(S^{2n}) \\
\downarrow \theta & & \downarrow \theta \\
\pi^s_{m-2d-1}(P_{k-1}\mathbb{C}^{d\tilde{H}}) & \xrightarrow{t^d} & \pi^s_{m-2d}(S^0)
\end{array} \tag{7.42}$$

There is a commutative diagram due to James which allows one to interpret $J_{n,k}$ via boundary maps (see [33], [18]):

$$\begin{array}{ccc} \pi_m(W_{n,k}) & \xrightarrow{J_{n,k}} & \pi_{m+2k}(S^{2n}) \\ \downarrow q_* & & \uparrow S \\ \pi_m(\,U(n)/U(k)\times U(d)\,) & \xrightarrow{\partial} & \pi_{m-1}(S^{2d-1}) \end{array}$$

where q is the projection of $W_{n,k}$ to the Grassmann manifold $U(n)/U(k)\times U(d)$. We fix $d=n-k$ and let k tend to ∞ to get

$$\begin{array}{ccc} \pi_m(U/U(d)) & \xrightarrow{J'_d} & \pi_{m-2d}(S^o) \\ \downarrow q_* & & \uparrow S \\ \pi_m(BU(d)) & \xrightarrow{\partial} & \pi_{m-1}(S^{2d-1}) \end{array} \qquad (7.43)$$

Because $BU(1) \simeq K(\mathbb{Z},2)$ we see that for d=2 ∂ is onto. Clearly q_* is onto. It is known [15] that α_t has S^3 as sphere of origin, so α_t lifts to $\pi_m(U/U(2))$ and so to an element in $\pi_m^s(P_\infty\mathbb{C}^{2\tilde{H}})$ which maps to α_t under t^2. This proves (7.41).
For p=2 the same argument applies using [28], except when $n\equiv 0\ (4)$ because the sphere of origin for the element of order 2 in $im(J)\subset\pi^s_{8k-1}(S^o)_{(2)}$ is not S^3.

Because S^3 is the sphere of origin for the μ_r-family we get:

<u>Corollary 7.44</u>: $\mu_k\in im(t^2)$ for all k.

Using the splitting of the p-localizations of Lie groups [30], one can use the same sort of argument to prove

$$U(n+i,n)_{(p)} = U_A(n+i,n) \quad 0\leq i\leq p \qquad (7.45)$$

It is even possible to use (7.42) and (7.43) to deduce some results about instable homotopy groups in a simple manner:
Because of (7.42) and (7.43) $im(t^2)$ is an upper bound for the image

of the suspension map $\pi_{m-1}(S^3) \longrightarrow \pi^s_{m-4}(S^o)$.

<u>Example</u>: If $\alpha_1 \cdot x \neq 0$ for $x \in \pi^s_m(S^o)$, then one can show that $x \notin im(t^2)$ and therefore

$$x \in \pi^s_{m-4}(S^o) : \quad \alpha_1 \cdot x \neq 0 \text{ implies } x \notin im(\pi_{m-1}(S^3) \rightarrow \pi^s_{m-4}(S^o))$$

In the metastable range, we can put the complex reflection map into the diagrams (7.42) and (7.43) to get

$$\begin{array}{ccc}
\pi^s_{m-1}(P_\infty\mathbb{C}/P_{d-1}\mathbb{C}) & \xrightarrow{\quad t^d \quad} & \pi^s_{m-2d}(S^o) \\
\theta \uparrow \downarrow \; w_* & & \| \\
\pi_m(U/U(d)) & \xrightarrow{\quad J_d \quad} & \pi^s_{m-2d}(S^o) \\
\downarrow q_* & & \uparrow S \\
\pi_m(BU(d)) & \xrightarrow{\quad \partial \quad} & \pi_{m-1}(S^{2d-1})
\end{array}$$

We always have $e_{\mathbb{C}}\partial \subset im(e_{\mathbb{C}}t^d)$,whereas $e_{\mathbb{C}}\partial \supset im(e_{\mathbb{C}}t^d)$ in the metastable case ($n<4d$)). So if we know $U(\bar{m},\bar{m}-d)$, $\bar{m}=(m+1)/2$, we can compute the e-invariant of the following obstruction class:
Given an d-dimensional complex vector bundle ξ on S^m, we can look at the obstruction to the existence of a section, which is an element $\sigma(\xi) \in H^m(S^m;\pi_{m-1}(S^{2d-1})) \cong \pi_{m-1}(S^{2d-1})$, then $e_{\mathbb{C}}(\sigma(\xi)) \in im\, e_{\mathbb{C}}t^d$.

The information $U_A(i,j) = U(i,j)$ for a fixed pair (i,j) in the metastable range can be used to construct infinite families of elements $e_{m,d}$ in $\pi_{2d+2m}(U(d))$ as follows:

If $U(d+m+1,m+1) = U_A(d+m+1,m+1)$ we know that $h_A: \pi^s_{2m}(P_\infty\mathbb{C}^{d\tilde{H}}) \longrightarrow \tilde{A}_{2m}(P_\infty\mathbb{C}^{d\tilde{H}})$ is onto and we can choose an element $x_{m,d}$ such that $h_A(x_{m,d})$ generates $\tilde{A}_{2m}(P_\infty\mathbb{C}^{d\tilde{H}})/tor.$
Let $\omega: P_\infty\mathbb{C}^{d\tilde{H}} \longrightarrow P_\infty\mathbb{C}^{(d+1)\tilde{H}}$ be the map of (2.18). From $x_{m,d}$ we get a family of elements $x_{m,d+t} = \omega^t(x_{m,d})$ in $\pi^s_{2m}(P_\infty\mathbb{C}^{(d+t)\tilde{H}})$ $(t \geqslant 0)$.

We set

$$e_{m,r} = \partial w_*(x_{m,r}) \in \pi_{2m+2r}(U(r))$$

for $r \geq d$,where $\partial: \pi_{2m+2r+1}(U/U(r)) \to \pi_{2m+2r}(U(r))$ is the boundary map. The exact sequence

$$A_{2m+2d}(P_\infty\mathbb{C}) \longrightarrow A_{2m+2d}(P_\infty\mathbb{C}/P_{d-1}\mathbb{C}) \longrightarrow A_{2m+2d-1}(P_{d-1}\mathbb{C})$$

shows, that the order of $e_{\mathbb{C}}(\theta(e_{m,d}))$ is $\bar{a} = (m+d)!/U(m+d+1,m+1)$.
The commutative diagram $(r = d+t)$

$$\begin{array}{ccccc}
A_{2m+2d}(P_\infty\mathbb{C}) & \longrightarrow & \tilde{A}_{2m}(P_\infty\mathbb{C}^{d\tilde{H}}) & \longrightarrow & A_{2m+2d-1}(P_{d-1}\mathbb{C}) \\
\omega_*^t \downarrow \wr & & \omega_*^t \downarrow & & \downarrow \\
A_{2m+2r}(P_\infty\mathbb{C}) & \longrightarrow & \tilde{A}_{2m}(P_\infty\mathbb{C}^{r\tilde{H}}) & \longrightarrow & A_{2m+2r-1}(P_{r-1}\mathbb{C})
\end{array}$$

then gives $e_{\mathbb{C}}(\theta(e_{m,d+t})) = \bar{a}$.
If t is large enough, the elements $e_{m,d+t}$ are independent of the choice of $x_{m,d}$:
The reason for this is that if t is large enough ω_*^t kills all torsion in $\pi_{2m}^s(P_\infty\mathbb{C}^{d\tilde{H}})$. This can be proved by using the Atiyah-Hirzebruch spectral sequence: The induced map ω_*^p in the E_2-term is multiplication by at least p . So for every torsion element x, there exist some t such that $\omega_*^t(x) = 0$ in the E_2-term. This means that $\omega_*^t(x)$ has a skeleton filtration which is strictly less than the one of x. By repeating this, $\omega_*^s(x) = 0$ follows.

Because the order of the torsion groups in $\pi_{2m}^s(P_\infty\mathbb{C}^{t\tilde{H}})(t \in \mathbb{Z})$ is bounded for m fixed, it follows that $j_*(\sigma^{m+t})$ in $\pi_{2m}^s(P_\infty\mathbb{C}^{t\tilde{H}})$ is divisible in an unique way by some number c which increases with t(σ^k is the generator of $\pi_{2k}^s(P_\infty\mathbb{C})/tor$ and $j: P_\infty\mathbb{C} \to P_\infty\mathbb{C}/P_{k-1}\mathbb{C}$ the natural map). This implies, that for t large enough, the order of $e_{m,d+t}$ must be exactly $\bar{a} = (m+d)!/U(m+d+1,m+1)$.

We close by mentioning the **main results** of [12], which show $U_A(i,j) = U(i,j)_{(p)}$ to be true in a large number of cases.

In (7.28) the abstruction class for lifting a generator of $\pi^s_{2m}(P_\infty\mathbb{C})/tor$ to $\pi^s_{2m+2}(P_\infty\mathbb{C}^{-\tilde{H}})$ was the transfer image of σ^m. Because this element was in $im(J)$, we could conclude that $h: \pi^s_{2m+2}(P_\infty\mathbb{C}^{-\tilde{H}}) \longrightarrow A_{2m+2}(P_\infty\mathbb{C}^{-\tilde{H}})$ is onto. In [12] this is generalized as follows: The transfer of the sphere bundle $r\cdot H$ appearing in the cofibre sequence (2.11)

$$\pi^s_{2m+2r}(P_{m+r}\mathbb{C}^{-r\tilde{H}}) \longrightarrow \pi^s_{2m}(P_{m+r}\mathbb{C}^+) \xrightarrow{T_r} \pi^s_{2m+2r-1}(S(rH)^{-r\tilde{H}}) \qquad (7.46)$$

has the same property, namely $T_r(\sigma^m)$ is in the image of a J-homomorphism. This implies

$$U(r,m+r) = U_A(r,m+r) \quad \text{for all } m,r \geq 0$$

Closely related to cofibre sequence (7.46) is the following cofibre sequence of projective spaces. $(a\in\mathbb{Z}, n, r\in\mathbb{N})$:

$$P_{r-1}\mathbb{C}^{a\tilde{H}} \rightarrow P_{n+r}\mathbb{C}^{a\tilde{H}} \rightarrow P_n\mathbb{C}^{(a+r)\tilde{H}} \xrightarrow{\partial} S^1\wedge P_{r-1}\mathbb{C}^{a\tilde{H}}$$

Let $x\in\pi^s_{2n}(P_n\mathbb{C}^{(a+r)\tilde{H}})$ be an element with the property

"$\partial x\in\pi^s_*(P_{r-1}\mathbb{C}^{a\tilde{H}})$ in determined by its e-invariant" (7.47)

that is $|\partial x| = |e\partial x|$. Then we know that $|e\partial x|\cdot x$ can be lifted to $\pi^s_{2n+2r}(P_{n+r}\mathbb{C}^{a\tilde{H}})$. Suppose we have an element x in $\pi^s_{2n}(P_n\mathbb{C}^{(a+r)\tilde{H}})$ mapping to a generator of the free part of $A_{2n}(P_n\mathbb{C}^{(a+r)\tilde{H}})$ with property (7.47) then we can conclude as above, that $h_A: \pi^s_{2n+2r}(P_{n+r}\mathbb{C}^{a\tilde{H}}) \longrightarrow A_{2n+2r}(P_{n+r}\mathbb{C}^{a\tilde{H}})$ is onto.
Let M be the J-order of the Hopf bundle on $P_w\mathbb{C}$ where $w = \max(n,r-1)$.
The we have a (in general non-commuting) diagram

$$\begin{array}{ccc} \pi_*^s(P_n\mathbb{C}^{(a+r)H}) & \xrightarrow{\partial_1} & \pi_*^s(P_{r-1}\mathbb{C}^{aH}) \\ \updownarrow \phi & & \updownarrow \phi \\ \pi_*^s(P_n\mathbb{C}^{(a+r+M)\tilde{H}}) & \xrightarrow{\partial_2} & \pi_*^s(P_{r-1}\mathbb{C}^{(a+M)\tilde{H}}) \end{array} \qquad (7.48)$$

defined by the relative Thom isomorphisms of $M\cdot\tilde{H}$. It is then possible to work out the deviation from commutativity for (7.48) and to show that the element describing this deviation is in the image of a J-homomorphism. In a large number of cases this implies, that if $\partial_1 x$ is determined by its e-invariant, the same is true for $\partial_2\phi x$. This allows us to conclude in case of

$U_A(a+r+n+1, n+1) = U(a+r+n+1, n+1)$ that

$$U_A(a+n+r+1 + M\cdot s, n+r+1) = U(a+n+r+1 + M\cdot s, n+r+1)$$

The element, which describes the deviation from commutativity in (7.48) very often has a small order. This implies that $U(m+M\cdot s, n+r+1)$ is constant as a function of s, which explains for example some of the shorter periodicities in table 1.

Table 1

2-primary part of U(n,r)

$M_2=2$ U(2,2)=1 U(3,2)=2

$M_3=8$ U(3,3)=2, U(4,3)=2, U(5,3)=8, U(6,3)=4, U(7,3)=4, U(8,3)=1, U(9,3)=8, U(10,3)=4

$M_4=8$ U(n,4)=U(n,3)

$M_5=64$

U(5,5) =8		period 8
U(6,5) =8	U(14,5)=4	period 16
U(7,5) =8	U(15,5)=4	period 16
U(n·8,5)=8/(n,8)		period 64
U(9,5) =16		period 8 (U(9,5)–U(12,5))
U(10,5)=16		
U(11,5)=16		
U(12,5)=16		

$M_6=64$

U(6,6) =8	U(14,6)=4	period 16		
U(7,6) =16	U(15,6)=8	U(23,6)=16	U(31,6)=4	period 32
U(n·8,6)=8/(n,8)		period 64		
U(9,6) =32		period 8 (U(9,6)–U(12,6))		
U(10,6) =16				
U(11,6) =32				
U(12,6) =16				
U(13,6) =16	U(21,6)=8	period 16		

$M_7=128$

U(7,7) =16	U(15,7)=8	period 16		
U(n·8,7)=16/(n,16)		period 128		
U(9,7) =128		period 8 (U(9,7)–U(12,7))		
U(10,7) =64				
U(11,7) =64				
U(12,7) =16				
U(13,7) =16	U(5,7)=64	U(21,7)=64	U(29,7)=32	period 32
U(14,7) =4	U(22,7)=32	U(30,7)=16	U(38,7)=32	period 64 (both rows)
U(46,7) =8	U(54,7)=32	U(62,7)=16	U(70,7)=32	

$M_8=128$ U(n,8)=U(n,7)

$M_9=2048$ U(n,9) $9\leq n\leq 126$

$U(i,9)=2^8$ i= 17+k·16,......,24+k·16 $(k\geq 0)$

$U(i,9)=2^7$ i= 9+k·32,...,16+k·32 ; 25+k·32 $(k\geq 0)$

$U(i,9)=2^6$ i=26,..,32; i=50,59; i=90,..,96; i=122,123

$U(i,9)=2^5$ i=61,..,64; i=124,125

$U(i,9)=2^4$ i=60,126

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