New applications of the Fourier restriction norm method to wellposedness problems for nonlinear evolution equations

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0 Introduction

In this thesis we are mainly concerned with local wellposedness (LWP) problems for nonlinear evolution equations, two global results will then be a direct consequence of conservation laws. A standard scheme to prove LWP is the application of the contraction mapping principle to the corresponding integral equation in a suitable Banach function space, usually of the type $C_t(I, H^s_x) \cap Z_s$, where the choice of Z_s is determined by the knowledge of certain space time estimates for the solutions of the corresponding linear equation. In this context the use of a two parameter scale of function spaces closely adapted to the linear equation was introduced by Bourgain in [B93]. The use of these spaces not only benefits of the above mentioned space time estimates, but also exploits certain structural properties of the nonlinearity, thus improving in many cases the results previously known. The idea was picked up by many authors, further developed and simplyfied, and is meanwhile known as the "Fourier restriction norm method".

This thesis is divided into two parts, the first of them being devoted to the description of this method, starting with definitions and elementary properties, continuing with a general local existence theorem, which reduces the wellposedness problem to nonlinear estimates, explaining how to insert the space time estimates into the framework of the method and finally discussing two strategies to tackle the crucial nonlinear estimates. It also contains, in a slightly modified form, some of the Strichartz type estimates for the Schrödinger equation in the periodic case due to Bourgain. This descriptive part is - of course - based on Bourgain's work [B93], but even more on the survey article by Ginibre [G96] and the second section of [GTV97]. We have tried to reach a high degree of selfcontainedness in this exposition.

The second part contains the new research results, which we have obtained by the method. Here we are concerned with a certain class of derivative nonlinear Schrödinger equations, with solutions of nonlinear Schrödinger equations in Sobolev spaces of negative index and, finally, with the generalized Korteweg-deVries equation of order three. For a detailed summary we refer to the beginning of part II.

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Part I
Description of the Fourier restriction norm method

1 The framework: Reduction of wellposedness problems to nonlinear estimates

In this section we introduce the function spaces $X_{s,b}(\phi)$ for arbitrary measurable phase functions ϕ of at most polynomial growth and the corresponding restriction norm spaces. Elementary properties - such as duality, interpolation, embedding with respect to the time variable and behaviour under time reversion respectively complex conjugation - are studied. In order to cover a limiting case we also introduce the auxiliary spaces $Y_s(\phi)$. The basic estimates for the solutions of the homogeneous and inhomogeneous linear evolution equations are shown. Finally we state and prove a general local existence theorem for nonlinear evolution equations, which reduces the problem of local wellposedness - that is existence, uniqueness, persistence property and continuous dependance on the data - to nonlinear estimates. We include some remarks on the meaning of the nonlinearity for distributions in $X_{s,b}(\phi)$ with $s < 0$. All the arguments in this exposition of the framework of the Fourier restriction norm method are independent of the phase function.

1.1 The $X_{s,b}(\phi)$-spaces: Definitions and elementary properties

Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a measurable function. By the Fourier transform $\mathcal{F}_x : H^s_x(\mathbf{R}^n) \to L^2(\mathbf{R}^n, \langle\xi\rangle^s)$[1] one defines for $D := -i\nabla = -i(\partial_{x_1}, ..., \partial_{x_n})$ the operator

$$\phi(D) := \mathcal{F}_x^{-1}\phi(\xi)\mathcal{F}_x$$

with domain $\Delta := \{f \in H^s_x(\mathbf{R}^n) : \phi\mathcal{F}_x f \in L^2(\mathbf{R}^n, \langle\xi\rangle^s)\}$. Then $\phi(D) : \Delta \to H^s_x(\mathbf{R}^n)$ is selfadjoint and generates a unitary group denoted by

$$(U_\phi(t))_{t\in\mathbf{R}} := (\exp(it\phi(D)))_{t\in\mathbf{R}}.$$

Let $f \in \Delta$. Then $u(t) := U_\phi(t)f$ is the solution of the Cauchy-problem (CP)

$$\partial_t u - i\phi(D)u = 0, \qquad u(0) = f. \tag{1}$$

The solution of the inhomogeneous linear equation

$$\partial_t v - i\phi(D)v = F \in C^0_t(\mathbf{R}, H^s_x(\mathbf{R}^n)), \qquad v(0) = 0 \tag{2}$$

is given by

$$v(t) = \int_0^t U_\phi(t-t')F(t')dt' =: U_{\phi *R}F(t), \tag{3}$$

[1] We use the notation $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$.

see e. g. [CH], chapters 4 and 5. The function ϕ arising in this context is called phase function. Important examples are:

Example 1.1 (The Schrödinger equation)

$$\partial_t u - i\Delta u = 0 \qquad \textit{with} \qquad \phi(\xi) = -|\xi|^2, n \in \mathbf{N}.$$

Example 1.2 (The Airy equation)

$$\partial_t u + \partial_x^3 u = 0 \qquad \textit{with} \qquad \phi(\xi) = \xi^3, n = 1.$$

Now let $H_t^b(\mathbf{R})$ (respectively $H_x^s(\mathbf{R}^n)$) be the usual Sobolev space of functions depending on the time variable t (respectively on the space variable x) and $H_x^s(\mathbf{R}^n) \otimes H_t^b(\mathbf{R})$ the complete tensor product of these spaces. Then the Hilbert space $X_{s,b}(\phi)$ is defined as follows:

Definition 1.1 *Let $X_{s,b}(\phi)$ be the completion of $\bigcap_{s,b\in\mathbf{R}} H_x^s(\mathbf{R}^n) \otimes H_t^b(\mathbf{R})$ with respect to the norm*

$$\|f\|_{X_{s,b}(\phi)} := \|U_\phi(-\cdot)f\|_{H_x^s(\mathbf{R}^n)\otimes H_t^b(\mathbf{R})}.$$

Similarly for phase functions $\phi : \mathbf{Z}^n \to \mathbf{R}$ one defines the selfadjoint operators

$$\phi(D) := \mathcal{F}_x^{-1}\phi(\xi)\mathcal{F}_x$$

with domain $\Delta := \{f \in H_x^s(\mathbf{T}^n) : \phi\mathcal{F}_x f \in l^2(\mathbf{Z}^n, \langle\xi\rangle^s)\}$, generating a unitary group $(U_\phi(t))_{t\in\mathbf{R}}$ with $u(t) := U_\phi(t)f$ for $f \in \Delta$ being the solution of (1), which is now called the periodic boundary value problem (pbvp). Here the solution of the inhomogeneous linear equation (2) - with $H_x^s(\mathbf{R}^n)$ replaced by $H_x^s(\mathbf{T}^n)$ - is again given by (3). The definition of the spaces $X_{s,b}(\phi)$ is now completely analogous:

Definition 1.2 *In the periodic case the spaces $X_{s,b}(\phi)$ are defined as the completion of $\bigcap_{s,b\in\mathbf{R}} H_x^s(\mathbf{T}^n) \otimes H_t^b(\mathbf{R})$ with respect to the norm*

$$\|f\|_{X_{s,b}(\phi)} = \|U_\phi(-\cdot)f\|_{H_x^s(\mathbf{T}^n)\otimes H_t^b(\mathbf{R})}.$$

In the sequel we shall write for short H_t^b instead of $H_t^b(\mathbf{R})$ and H_x^s instead of $H_x^s(\mathbf{R}^n)$ respectively $H_x^s(\mathbf{T}^n)$, if a statement is valid in both cases or if it is clear from the context, whether we are dealing with the periodic or with the nonperiodic case. In the same way we use the notation L_x^2. Moreover we shall use the notations $H^{s,b}$ for $H_x^s \otimes H_t^b$ and $H = \bigcap_{s,b\in\mathbf{R}} H^{s,b}$.

Concerning the phase functions we assume from now on that they do not grow faster than a polynomial.

Now for $b, s \in \mathbf{R}$, $f \in H$ we write

$$\begin{aligned} J_x^s f &:= \mathcal{F}_x^{-1}\langle\xi\rangle^s\mathcal{F}_x f \\ J_t^b f &:= \mathcal{F}_t^{-1}\langle\tau\rangle^b\mathcal{F}_t f \\ \Lambda^b f &:= U_\phi J_t^b U_\phi(-\cdot)f \quad . \end{aligned}$$

Then we have $\|J_x^\sigma f\|_{X_{s-\sigma,b}(\phi)} = \|f\|_{X_{s,b}(\phi)}$, and the extension of J_x^σ, which is denoted again by J_x^σ, is an isometric isomorphism

$$J_x^\sigma : X_{s,b}(\phi) \xrightarrow{\sim} X_{s-\sigma,b}(\phi). \tag{4}$$

In the sequel it will be shown that a corresponding statement holds true for the mapping Λ^β. We start with the following

Lemma 1.1 *For functions $f \in H$ the identities*

$$\|\Lambda^\beta f\|_{X_{s,b-\beta}(\phi)} = \|f\|_{X_{s,b}(\phi)} \tag{5}$$

$$\mathcal{F}\Lambda^\beta f(\xi,\tau) = \langle \tau - \phi(\xi)\rangle^\beta \mathcal{F}f(\xi,\tau) \tag{6}$$

$$\|f\|^2_{X_{s,b}(\phi)} = \int\int \langle \tau - \phi(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\mathcal{F}f(\xi,\tau)|^2 d\tau\mu(d\xi) \tag{7}$$

are valid. Here μ in (7) denotes the Lebesgue measure on $\mathbf{R}^n$ respectively the counting measure on $\mathbf{Z}^n$.

Proof: Concerning (5) we have

$$\begin{aligned}\|\Lambda^\beta f\|_{X_{s,b-\beta}(\phi)} &= \|U_\phi J_t^\beta U_\phi(-\cdot)f\|_{X_{s,b-\beta}(\phi)} \\ &= \|J_t^\beta U_\phi(-\cdot)f\|_{H^{s,b-\beta}} \\ &= \|U_\phi(-\cdot)f\|_{H^{s,b}} \qquad = \|f\|_{X_{s,b}(\phi)}.\end{aligned}$$

To see (6), we use $(\mathcal{F}_t(\exp{(ia\cdot)}g))(\tau) = \mathcal{F}_t g(\tau - a)$ to obtain

$$\begin{aligned}\mathcal{F}\Lambda^\beta f(\xi,\tau) &= \mathcal{F}U_\phi J_t^\beta U_\phi(-\cdot)f(\xi,\tau) \\ &= \mathcal{F}_t \exp{(i\phi(\xi)\cdot)} J_t^\beta \mathcal{F}_x U_\phi(-\cdot)f(\xi,\tau) \\ &= \langle\tau-\phi(\xi)\rangle^\beta \mathcal{F}U_\phi(-\cdot)f(\xi,\tau-\phi(\xi)) \\ &= \langle\tau-\phi(\xi)\rangle^\beta \mathcal{F}_t \exp{(-i\phi(\xi)\cdot)}\mathcal{F}_x f(\xi,\tau-\phi(\xi)) \\ &= \langle\tau-\phi(\xi)\rangle^\beta \mathcal{F}f(\xi,\tau).\end{aligned}$$

Considering (7), we observe that $X_{0,0}(\phi) = L_t^2(\mathbf{R}, L_x^2)$ and use (4), (5), Plancherel resp. Parseval and (6) to see that

$$\begin{aligned}\|f\|^2_{X_{s,b}(\phi)} &= \|\Lambda^b J_x^s f\|^2_{L_t^2(\mathbf{R},L_x^2)} \\ &= \|\mathcal{F}\Lambda^b J_x^s f\|^2_{L_\tau^2(\mathbf{R},L_\xi^2(\mu))} \\ &= \int\int \langle\tau-\phi(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\mathcal{F}f(\xi,\tau)|^2 d\tau\mu(d\xi)\ .\end{aligned} \qquad \Box$$

Corollary 1.1 *If the difference of two phase functions ϕ_i, $i = 1,2$, is bounded, the corresponding $X_{s,b}(\phi_i)$-norms are equivalent.*

Proof: Taking into account that

$$\begin{aligned}\langle\tau-\phi_1(\xi)\rangle &\leq c(1+|\tau-\phi_1(\xi)|)\\ &\leq c(1+|\phi_1(\xi)-\phi_2(\xi)|+|\tau-\phi_2(\xi)|)\leq c\langle\tau-\phi_2(\xi)\rangle,\end{aligned}$$

this follows from (7). □

For functions $f\in H$ it is clear by (6) and the growth condition on ϕ that $\Lambda^\beta f$ still belongs to H. Moreover, for given $\tilde{s},\tilde{b}\in\mathbf{R}$ there exist $s,b\in\mathbf{R}$ so that $H^{s,b}\subset X_{\tilde{s},\tilde{b}}(\phi)$. This gives $\Lambda^\beta f\in\bigcap_{s,b\in\mathbf{R}}X_{s,b}(\phi)$ for $f\in H$.

Thus the linear mapping

$$\Lambda^\beta : X_{s,b}(\phi)\supset H\rightarrow X_{s,b-\beta}(\phi)$$

is well defined for all $s,b,\beta\in\mathbf{R}$ and, by (5), isometric, especially injective. Moreover, for $f\in H$ we have $\Lambda^\beta\Lambda^{-\beta}f=f$, which gives that the range of Λ^β is dense in $X_{s,b-\beta}(\phi)$. Thus for the extension of Λ^β (again denoted by Λ^β) we have shown:

Lemma 1.2 *The mapping*

$$\Lambda^\beta : X_{s,b}(\phi)\xrightarrow{\sim}X_{s,b-\beta}(\phi)$$

is an isometric isomorphism.

By the aid of the previous lemma we are now able to determine the dual spaces of the $X_{s,b}(\phi)$-spaces with respect to the inner product on L^2_{xt} and to study their interpolation properties:

Lemma 1.3 *Let* $<\cdot,\cdot>$ *denote the inner product on* L^2_{xt} *and let* $\Phi : X_{-s,-b}(\phi)\rightarrow(X_{s,b}(\phi))'$ *be defined by* $\Phi(g)[f]:=<J^s_x\Lambda^bf,J^{-s}_x\Lambda^{-b}g>$. *Then* Φ *is an isometric isomorphism and we have* $\Phi(g)[f]=<f,g>$, *whenever* $f\in X_{s,b}(\phi)\cap L^2_{xt}$ *and* $g\in X_{-s,-b}(\phi)\cap L^2_{xt}$.

Proof: For $f\in X_{s,b}(\phi)$, $g\in X_{-s,-b}(\phi)$ Cauchy Schwarz gives

$$\begin{aligned}|\Phi(g)[f]| &= |<J^s_x\Lambda^bf,J^{-s}_x\Lambda^{-b}g>|\\ &\leq \|J^s_x\Lambda^bf\|_{L^2_{xt}}\|J^{-s}_x\Lambda^{-b}g\|_{L^2_{xt}}=\|f\|_{X_{s,b}(\phi)}\|g\|_{X_{-s,-b}(\phi)}.\end{aligned}$$

Hence $\Phi(g)\in(X_{s,b}(\phi))'$ with $\|\Phi(g)\|\leq\|g\|_{X_{-s,-b}(\phi)}$. Moreover, by Lemma 1.2

$$\begin{aligned}\|\Phi(g)\| &= \sup_{\|f\|_{X_{s,b}(\phi)}\leq 1}|<J^s_x\Lambda^bf,J^{-s}_x\Lambda^{-b}g>|=\sup_{\|h\|_{L^2_{xt}}\leq 1}|<h,J^{-s}_x\Lambda^{-b}g>|\\ &= \|J^{-s}_x\Lambda^{-b}g\|_{L^2_{xt}}=\|g\|_{X_{-s,-b}(\phi)}.\end{aligned}$$

It remains to show that Φ is onto. Therefore let y be a bounded linear functional on $X_{s,b}(\phi)$. Then $z=y\circ J^{-s}_x\Lambda^{-b}$ is a bounded linear functional on L^2_{xt}, and by the Riesz' representation theorem there exists $\tilde{g}\in L^2_{xt}$ with $z[\tilde{f}]=<\tilde{f},\tilde{g}>$ for all $\tilde{f}\in L^2_{xt}$. Now $g:=J^s_x\Lambda^b\tilde{g}$ belongs to $X_{-s,-b}(\phi)$ and a straightforward computation gives $y[f] = \Phi(g)[f]$ for all $f \in X_{s,b}(\phi)$. Finally let $f \in X_{s,b}(\phi) \cap L^2_{xt}$ and $g \in X_{-s,-b}(\phi) \cap L^2_{xt}$. Then

$$\begin{aligned} < f, g > &= < U_\phi(-\cdot)f, U_\phi(-\cdot)g > \\ &= < J_t^b U_\phi(-\cdot)f, J_t^{-b} U_\phi(-\cdot)g > \\ &= < \Lambda^b f, \Lambda^{-b} g > \qquad\qquad = < J_x^s \Lambda^b f, J_x^{-s} \Lambda^{-b} g > . \end{aligned}$$

□

Lemma 1.4 *For $s_0, s_1, b_0, b_1 \in \mathbf{R}$, $\theta \in (0,1)$ and $b = (1-\theta)b_0 + \theta b_1$, $s = (1-\theta)s_0 + \theta s_1$ we have*

$$(X_{s_0,b_0}(\phi), X_{s_1,b_1}(\phi))_{[\theta]} = X_{s,b}(\phi)$$

with equality of norms. Here $[\theta]$ denotes the complex interpolation method.

Proof: For $\sigma, \beta \in \mathbf{R}$ define the measure $\rho = \rho(\sigma,\beta)$ on $\mathbf{R} \times \mathbf{Z}^n$ respectively on $\mathbf{R}^{n+1}$ by

$$\int f d\rho = \int f(\xi,\tau) \langle \xi \rangle^\sigma \langle \tau - \phi(\xi) \rangle^\beta d\tau \mu(d\xi).$$

Denote the space of all ρ-measurable and square integrable (with respect to ρ) functions by $L^2(\rho(\sigma,\beta))$. Then the multiplier

$$M_{-\sigma,-\beta} : L^2_{\xi\tau} = L^2(\rho(0,0)) \to L^2(\rho(\sigma,\beta)), \quad f \mapsto \langle \xi \rangle^{-\sigma} \langle \tau - \phi(\xi) \rangle^{-\beta} f$$

is an isometric isomorphism. Combined with Plancherel and Lemma 1.2 this gives that the Fourier transform

$$\mathcal{F} : X_{\sigma,\beta}(\phi) \xrightarrow{\sim} L^2(\rho(\sigma,\beta))$$

is an isometric isomorphism. By theorem 5.5.3 in [BL] we have

$$(L^2(\rho(s_0,b_0)), L^2(\rho(s_1,b_1)))_{[\theta]} = L^2(\rho(s,b))$$

with equal norms. Now, by the properties of an interpolation functor, we obtain that

$$Id = \mathcal{F}^{-1}\mathcal{F} : X_{s,b}(\phi) \to (X_{s_0,b_0}(\phi), X_{s_1,b_1}(\phi))_{[\theta]}$$

is isomorphic and, since $[\theta]$ is exact, also isometric. □

Combining Sobolev's embedding theorem (in the time variable) with the duality lemma we obtain:

Lemma 1.5 *For all $s \in \mathbf{R}$ and independently of the phase function the following holds true*

$$X_{s,b}(\phi) \subset C_t(\mathbf{R}, H^s_x) \qquad \forall\, b > \frac{1}{2}, \tag{8}$$

$$X_{s,b}(\phi) \subset L^p_t(\mathbf{R}, H^s_x) \qquad \forall\, 2 \le p < \infty,\, b \ge \frac{1}{2} - \frac{1}{p}, \tag{9}$$

$$\|f\|_{X_{s,b}(\phi)} \le c\|f\|_{L^1_t(\mathbf{R},H^s_x)} \qquad \forall\, b < -\frac{1}{2}, \tag{10}$$

$$\|f\|_{X_{s,b}(\phi)} \le c\|f\|_{L^p_t(\mathbf{R},H^s_x)} \qquad \forall\, 2 \ge p > 1,\, b \le \frac{1}{2} - \frac{1}{p}. \tag{11}$$

Proof: We may assume $s = 0$ without loss of generality. To see (8) we use Plancherel and Sobolev's embedding theorem to obtain

$$\begin{aligned}\|f\|^2_{L_t^\infty(L_x^2)} &= \sup_t \int \mu(d\xi)|\mathcal{F}_x f(\xi,t)|^2 \\ &\leq \int \mu(d\xi) \sup_t |\mathcal{F}_x f(\xi,t)|^2 \\ &\leq c\int \mu(d\xi)d\tau\langle\tau\rangle^{2b}|\mathcal{F}f(\xi,\tau)|^2 = c\|f\|^2_{H^{0,b}}\end{aligned}$$

for $b > \frac{1}{2}$. From this we get

$$\begin{aligned}\|f\|_{L_t^\infty(\mathbf{R},L_x^2)} &= \|U_\phi(-\cdot)f\|_{L_t^\infty(\mathbf{R},L_x^2)} \\ &\leq c\|U_\phi(-\cdot)f\|_{H^{0,b}} = c\|f\|_{X_{0,b}(\phi)}.\end{aligned}$$

This is the norm estimate in (8). To see the continuity statement in (8) one now uses the density of H in $X_{0,b}(\phi)$. To see (9) we use Minkowsky's inequality and again Sobolev's embedding theorem to see that

$$\begin{aligned}\|f\|_{L_t^p(\mathbf{R},L_x^2)} &\leq \|f\|_{L_x^2(L_t^p)} \\ &\leq c\|f\|_{L_x^2(H_t^b)} = c\|f\|_{H^{0,b}}\end{aligned}$$

and argue then as above. Finally we obtain (10) from (8) and (11) from (9) by duality. □

Compared with more customary function spaces such as $L_t^p(\mathbf{R},H_x^s)$ or $C_t^0(\mathbf{R},H_x^s)$ the spaces $X_{s,b}(\phi)$ have an exceptional property: They are in general not invariant under time reversion and complex conjugation. We shall conclude this from the following

Remark 1.1 *Let $\phi_i : \mathbf{R}^n \to \mathbf{R},\ i = 1,2,$ be continuous phase functions with* $\sup_\xi|\phi_1(\xi)-\phi_2(\xi)| = \infty$. *Then for all $c \in \mathbf{R},\ b \neq 0$ the estimate*

$$\frac{1}{c}\|f\|_{X_{s,b}(\phi_2)} \leq \|f\|_{X_{s,b}(\phi_1)} \leq c\|f\|_{X_{s,b}(\phi_2)} \tag{12}$$

fails. The same statement holds for phase functions $\phi_i : \mathbf{Z}^n \to \mathbf{R},\ i = 1,2$.

Proof: By (4) we may assume $s = 0$. Next we observe that then (12) is equivalent to

$$\frac{1}{c}\|f\|_{H^{0,b}} \leq \|f\|_{X_{0,b}(\phi_1-\phi_2)} \leq c\|f\|_{H^{0,b}}.$$

So it is sufficient to show that for unbounded ϕ and $b \neq 0$ the estimate

$$\frac{1}{c}\|f\|_{H^{0,b}} \leq \|f\|_{X_{0,b}(\phi)} \leq c\|f\|_{H^{0,b}}$$

fails. Consider the nonperiodic case first: We choose sequences ξ_k in $\mathbf{R}^n$ with $\lim_{k\in\mathbf{N}}|\phi(\xi_k)| = \infty$ and ε_k with $|\phi(\xi+\xi_k)-\phi(\xi_k)| \leq 1$ for all $|\xi| < \varepsilon_k$. Now let $0 < \chi_n \in C_0^\infty(\mathbf{R}^n)$ with $\mathrm{Supp}(\chi_n) \subset B_1(0)$. We define the functions f_k by

$$\mathcal{F}f_k(\xi,\tau) = \psi_{\varepsilon_k}(\xi-\xi_k)\chi_1(\tau) \qquad \text{with} \qquad \psi_\varepsilon(\xi) = \varepsilon^{-\frac{n}{2}}\chi_n(\frac{\xi}{\varepsilon}).$$

Then $\|f_k\|_{H^{0,b}}$ is constant and

$$\|f_k\|^2_{X_{0,b}(\phi)} = \int\int \langle\tau - \phi(\xi+\xi_k)\rangle^{2b}\psi^2_{\varepsilon_k}(\xi)\chi_1^2(\tau)d\xi d\tau.$$

For $k \to \infty$ this tends to ∞, if $b > 0$, and to zero, if $b < 0$.

In the periodic case the proof is almost the same, except that in this case one chooses $\mathcal{F}f_k(\xi,\tau) = \delta_{\xi,\xi_k}\chi_1(\tau)$. □

Corollary 1.2 *Assume ϕ to be unbounded and continuous. Then we have*

i) $X_{s,b}(\phi)$ is not invariant under time reversion.

ii) If $\sup_\xi |\phi(\xi)+\phi(-\xi)| = \infty$, then $X_{s,b}(\phi)$ is not closed under complex conjugation.

Proof: For $f_-(x,t) = f(x,-t)$ we have $\mathcal{F}f_-(\xi,\tau) = \mathcal{F}f(\xi,-\tau)$, which implies

$$\|f_-\|_{X_{s,b}(\phi)} = \|f\|_{X_{s,b}(-\phi)}.$$

This gives i). To see ii), observe that $\mathcal{F}\overline{f}(\xi,\tau) = \overline{\mathcal{F}f}(-\xi,-\tau)$, which gives

$$\|\overline{f}\|_{X_{s,b}(\phi)} = \|f\|_{X_{s,b}(\tilde{\phi})}$$

with $\tilde{\phi}(\xi) = -\phi(-\xi)$. □

In the applications one is sometimes forced to choose the parameters $b = b'+1 = \frac{1}{2}$. This leads to several problems, among others we cannot rely on the embedding $X_{s,b}(\phi) \subset C_t(\mathbf{R}, H^s_x)$ in this case. Here the auxiliary spaces $Y_s(\phi)$ turn out to be useful, which are defined as completion of H with respect to the norm

$$\begin{aligned}\|f\|_{Y_s(\phi)} &:= \|\langle\xi\rangle^s\langle\tau\rangle^{-1}\mathcal{F}(U_\phi(-.)f)\|_{L^2_\xi(L^1_\tau)}\\ &= \|\langle\xi\rangle^s\langle\tau-\phi(\xi)\rangle^{-1}\mathcal{F}f\|_{L^2_\xi(L^1_\tau)}.\end{aligned}$$

Observe that by Cauchy-Schwarz' inequality we have $X_{s,b'}(\phi) \subset Y_s(\phi)$ with a continuous embedding, whenever $b' > -\frac{1}{2}$.

Next we introduce the restriction norm spaces $X^\Omega_{s,b}(\phi)$, where Ω is a domain in $\mathbf{R}^{n+1}$ respectively in $\mathbf{R}\times\mathbf{T}^n$:

Definition 1.3 *The restriction norm spaces $X^\Omega_{s,b}(\phi)$ are defined by*

$$X^\Omega_{s,b}(\phi) := \{f|_\Omega : f \in X_{s,b}(\phi)\},$$

endowed with the norm

$$\|f\|_{X^\Omega_{s,b}(\phi)} := \inf\{\|\tilde{f}\|_{X_{s,b}(\phi)} : \tilde{f} \in X_{s,b}(\phi), \tilde{f}|_\Omega = f\}.$$

Notation: In most cases we will have $\Omega = I$, where $I = (-\delta,\delta)\times\mathbf{R}^n$ respectively $I = (-\delta,\delta)\times\mathbf{T}^n$, and then we will write $X^{\delta}_{s,b}(\phi)$ instead of $X^{\Omega}_{s,b}(\phi)$.

The spaces $X_{s,b}(\phi)$ are Hilbert spaces. From this it follows that the infimum in the above definition is in fact a minimum. Moreover, for the $X^{\delta}_{s,b}(\phi)$-spaces we have the following

Lemma 1.6 *For $u \in X^{\delta}_{\sigma,b}(\phi)$ there exists $\tilde{u} \in X_{\sigma,b}(\phi)$ with $\tilde{u}|_I = u$, such that for all $s \le \sigma$*

$$\|u\|_{X^{\delta}_{s,b}(\phi)} = \|\tilde{u}\|_{X_{s,b}(\phi)}.$$

Proof: Let $R_{\sigma,b} : X_{\sigma,b}(\phi) \to X^{\delta}_{\sigma,b}(\phi)$, $u \mapsto u|_I$, denote the restriction operator and $N(R_{\sigma,b})$ its null space. Then

$$R_{\sigma,b}|_{N(R_{\sigma,b})^{\perp}} : N(R_{\sigma,b})^{\perp} \to X^{\delta}_{\sigma,b}(\phi)$$

is one to one, that is, for $u \in X^{\delta}_{\sigma,b}(\phi)$ there exists exactly one extension $\tilde{u} \in N(R_{\sigma,b})^{\perp}$. For this extension $\tilde{u}$ we have

$$\begin{aligned}\|u\|_{X^{\delta}_{\sigma,b}(\phi)} &= \inf\{\|\tilde{v}\|_{X_{\sigma,b}(\phi)} : \tilde{v} \in X_{\sigma,b}(\phi),\ \tilde{v}|_I = u\} \\ &= \inf\{\|\tilde{u}+\tilde{w}\|_{X_{\sigma,b}(\phi)} : \tilde{w} \in N(R_{\sigma,b})\} = \|\tilde{u}\|_{X_{\sigma,b}(\phi)},\end{aligned}$$

since $\|\tilde{u}\|^2_{X_{\sigma,b}(\phi)} \le \|\tilde{u}\|^2_{X_{\sigma,b}(\phi)} + \|\tilde{w}\|^2_{X_{\sigma,b}(\phi)} = \|\tilde{u}+\tilde{w}\|^2_{X_{\sigma,b}(\phi)}$. Now $u \in X^{\delta}_{\sigma,b}(\phi)$ implies that $u \in X^{\delta}_{s,b}(\phi)$, $s \le \sigma$. The same argument gives that there is exactly one extension $\tilde{v} \in N(R_{s,b})^{\perp} \subset X_{s,b}(\phi)$ of u and that $\|u\|_{X^{\delta}_{s,b}(\phi)} = \|\tilde{v}\|_{X_{s,b}(\phi)}$.

To see that $\tilde{u} = \tilde{v}$, we have only to show that $\tilde{u} \in N(R_{s,b})^{\perp}$. Therefore let $w \in X_{s,b}(\phi)$ with $w|_I = 0$. Then $J_x^{2(s-\sigma)}w \in X_{2\sigma-s,b}(\phi) \subset X_{\sigma,b}(\phi)$ and $J_x^{2(s-\sigma)}w|_I = 0$. This gives

$$\begin{aligned}0 &= \int \mu(d\xi)d\tau\langle\xi\rangle^{2\sigma}\langle\tau-\phi(\xi)\rangle^{2b}\mathcal{F}\tilde{u}\overline{\mathcal{F}(J_x^{2(s-\sigma)}w)} \\ &= \int \mu(d\xi)d\tau\langle\xi\rangle^{2s}\langle\tau-\phi(\xi)\rangle^{2b}\mathcal{F}\tilde{u}\overline{\mathcal{F}w},\end{aligned}$$

that is $\tilde{u} \in N(R_{s,b})^{\perp}$. □

Remark : The proof shows that for all $u \in X^{\Omega}_{s,b}(\phi)$ there exists an extension $\tilde{u} \in X_{s,b}(\phi)$ with $\|u\|_{X^{\Omega}_{s,b}(\phi)} = \|\tilde{u}\|_{X_{s,b}(\phi)}$.

1.2 Cut off functions and linear estimates

To localize in time one uses cut off functions $\psi \in C_0^{\infty}(\mathbf{R})$ having the properties

i) $supp(\psi) \subset (-2,2)$

ii) $\psi|_{[-1,1]} = 1$

iii) $\psi(t) = \psi(-t)$, $\psi(t) \ge 0$.

For $0 < \delta \le 1$ one defines $\psi_\delta(t) := \psi(\frac{t}{\delta})$. Then the following estimate is an immediate consequence of the definition of the $X_{s,b}(\phi)$-spaces:

Lemma 1.7 (Estimate for the homogeneous linear equation) *Let $b \geq 0$. Then for the solution u of the Cauchy (respectively periodic boundary value) problem (1) the estimate*

$$\|\psi_\delta u\|_{X_{s,b}(\phi)} \leq c\delta^{\frac{1}{2}-b}\|f\|_{H^s_x}$$

holds true.

Proof: Using $u = U_\phi f$ we obtain

$$\begin{aligned} \|\psi_\delta u\|_{X_{s,b}(\phi)} &= \|U_\phi(-.)\psi_\delta U_\phi f\|_{H^{s,b}} \\ &= \|\psi_\delta f\|_{H^{s,b}} = \|\psi_\delta\|_{H^b_t}\|f\|_{H^s_x}. \end{aligned}$$

Now the claimed estimate follows from $\|\psi_\delta\|_{H^b_t} \leq c\delta^{\frac{1}{2}-b}\|\psi\|_{H^b_t}$. □

Lemma 1.8 *If $F \in Y_s(\phi) \cap C_t(\mathbf{R}, H^s_x)$, then $U_{\phi *R}F$ belongs to $C_t([-T,T], H^s_x)$ for all $0 < T < \infty$ and the estimate*

$$\sup_{|t|\leq T} \|U_{\phi *R}F(t)\|_{H^s_x} \leq c\langle T\rangle\|F\|_{Y_s(\phi)} \tag{13}$$

holds true.

Proof: It follows from the group properties of U_ϕ that $U_{\phi *R}F$ is continuous. To see (13), we write $g(t) = J^s_x U_\phi(-t)F(t)$. Then we have to show for $|t| \leq T$ that

$$\| \textstyle\int_0^t g(t')dt'\|_{L^2_x} \leq c\langle T\rangle\|\langle\tau\rangle^{-1}\mathcal{F}g\|_{L^2_\xi(L^1_\tau)}. \tag{14}$$

To see this, we write $\int_0^t g(t')dt' = g * \chi_{[0,t]}(t)$ and calculate

$$\mathcal{F}_t g * \chi_{[0,t]}(\tau) = c\mathcal{F}_t g(\tau)\mathcal{F}_t\chi_{[0,t]}(\tau) = c\frac{1-e^{-it\tau}}{i\tau}\mathcal{F}_t g(\tau).$$

Now $|\frac{1-e^{-it\tau}}{\tau}| \leq c\langle t\rangle\langle\tau\rangle^{-1}$ and by assumption $\mathcal{F}_t g * \chi_{[0,t]} \in L^1_\tau$. Thus the Fourier inversion formula can be applied to obtain

$$\int_0^t g(t')dt' = c\int_{-\infty}^{\infty}\frac{e^{it\tau}-1}{i\tau}\mathcal{F}_t g(\tau)d\tau.$$

Using Plancherel's theorem we see that

$$\begin{aligned} \| \textstyle\int_0^t g(t')dt'\|^2_{L^2_x} &= \int \mu(d\xi)d\tau d\tau'\frac{e^{it\tau}-1}{\tau}\mathcal{F}g(\tau)\frac{e^{-it\tau'}-1}{\tau'}\overline{\mathcal{F}g(\tau')} \\ &\leq c(1+t^2)\int \mu(d\xi)d\tau d\tau'\langle\tau\rangle^{-1}|\mathcal{F}g(\tau)|\langle\tau'\rangle^{-1}|\mathcal{F}g(\tau')| \\ &\leq c(1+t^2)\|\langle\tau\rangle^{-1}\mathcal{F}g\|^2_{L^2_\xi(L^1_\tau)}, \end{aligned}$$

which gives (14). □

Remark/Definition: (13) expresses the boundedness of

$$U_{\phi *R} : Y_s(\phi) \supset Y_s(\phi) \cap C_t(\mathbf{R}, H^s_x) \to C_t([-T,T], H^s_x).$$

Thus $U_{\phi *R}$ can be extended uniquely to a bounded linear operator (denoted by $U_{\phi *R}$ again) from $Y_s(\phi)$ into $C_t([-T,T], H^s_x)$. Here it is important that $U_{\phi *R}F$ is continuous for $F \in Y_s(\phi)$. For the extended operator we have $U_{\phi *R}F(0) = 0$ and $U_{\phi *R}F$ solves $u_t - i\phi(D)u = F$ in the sense of distributions. Moreover the identity

$$U_{\phi *R}F(t+t_1) = U_\phi(t) U_{\phi *R}F(t_1) + U_{\phi *R}(\tau_{-t_1}F)(t) \tag{15}$$

holds true, where $\tau_{-t_1}F(t) = F(t+t_1)$. This is easily checked for $F \in C_t(\mathbf{R}, H^s_x)$ and follows in the general case by approximation.

Lemma 1.9 (Estimate for the inhomogeneous linear equation) *Let $b'+1 \geq b \geq 0 \geq b'$. Then the following estimate is valid:*

$$\|\psi_\delta U_{\phi *R}F\|_{X_{s,b}(\phi)} \leq c\delta^{1+b'-b}\|F\|_{X_{s,b'}(\phi)} + c_1\delta^{\frac{1}{2}-b}\|F\|_{Y_s(\phi)}. \tag{16}$$

If in addition $b' > -1/2$, (16) holds with $c_1 = 0$.

Proof: Without loss of generality we may assume $F \in H$, since the general case then follows by an approximation argument again.

First we show for $Kg(t) := \psi_\delta(t) \int_0^t g(t')dt'$ that

$$\|Kg\|_{H^b_t} \leq c\delta^{1+b'-b}\|g\|_{H^{b'}_t} + c_0\delta^{\frac{1}{2}-b}\|\langle\tau\rangle^{-1}\mathcal{F}_t g\|_{L^1_\tau}, \tag{17}$$

where we may choose $c_0 = 0$, if $b' > -\frac{1}{2}$. We have (cf. the previous proof)

$$\int_0^t g(t')dt' = c\int_{-\infty}^{\infty} \frac{\exp(it\tau) - 1}{i\tau} \mathcal{F}_t g(\tau) d\tau$$

and thus $Kg(t) = I + II + III$ with

$$\begin{aligned} I &= \psi_\delta \sum_{k\geq 1} \frac{t^k}{k!} \int_{|\tau|\delta \leq 1} (i\tau)^{k-1} \mathcal{F}_t g(\tau) d\tau \\ II &= -\psi_\delta \int_{|\tau|\delta\geq 1} (i\tau)^{-1}\mathcal{F}_t g(\tau) d\tau \\ III &= \psi_\delta \int_{|\tau|\delta\geq 1} (i\tau)^{-1} \exp(it\tau) \mathcal{F}_t g(\tau) d\tau. \end{aligned}$$

The first contribution can be estimated for $1 \geq b \geq 0 \geq b'$ as follows:

$$\|I\|_{H^b_t} \leq \sum_{k\geq 1} \frac{1}{k!} \|t^k \psi_\delta\|_{H^b_t} \int_{|\tau|\delta\leq 1} |\tau|^{k-1} |\mathcal{F}_t g(\tau)| d\tau,$$

where

$$\begin{aligned} \int_{|\tau|\delta\leq 1} |\tau|^{k-1}|\mathcal{F}_t g(\tau)|d\tau &\leq \delta^{1-k} \int_{|\tau|\delta\leq 1} \langle\tau\rangle^{-b'}\langle\tau\rangle^{b'}|\mathcal{F}_t g(\tau)|d\tau \\ &\leq \delta^{1-k}\Big(\int_{|\tau|\delta\leq 1}\langle\tau\rangle^{-2b'}d\tau\Big)^{\frac{1}{2}}\|g\|_{H^{b'}_t} \\ &\leq c\delta^{\frac{1}{2}+b'-k}\|g\|_{H^{b'}_t} \end{aligned}$$

and

$$\begin{aligned}\|t^k\psi_\delta\|_{H_t^b}^2 &= \int \langle\tau\rangle^{2b}|(\partial_\tau^k\mathcal{F}_t\psi_\delta)(\tau)|^2 d\tau\\ &= \delta^{2k+2}\int \langle\tau\rangle^{2b}|(\mathcal{F}_t\psi)^{(k)}(\delta\tau)|^2 d\tau\\ &\le c\delta^{2k-2b+1}\int \langle\tau\rangle^{2b}|(\mathcal{F}_t\psi)^{(k)}(\tau)|^2 d\tau = c\delta^{2k-2b+1}\|t^k\psi\|_{H_t^b}^2.\end{aligned}$$

By the support condition on ψ we have

$$\|t^k\psi\|_{H_t^b} \le \|t^k\psi\|_{H_t^1} \le c(k+1)2^k\|\psi\|_{H_t^1},$$

hence

$$\|I\|_{H_t^b} \le \sum_{k\ge 1}\frac{\|t^k\psi\|_{H_t^b}}{k!}\delta^{1+b'-b}\|g\|_{H_t^{b'}} \le c\delta^{1+b'-b}\|g\|_{H_t^{b'}}.$$

Next we consider the second contribution: For $b \ge 0$ we have

$$\begin{aligned}\|II\|_{H_t^b} &\le c\|\psi_\delta\|_{H_t^b}\int_{|\tau|\delta\ge 1}|\tau|^{-1}|\mathcal{F}_t g(\tau)|d\tau\\ &\le c_0\delta^{1/2-b}\|\langle\tau\rangle^{-1}\mathcal{F}_t g\|_{L_\tau^1}.\end{aligned}$$

For $b' > -\frac{1}{2}$ we use Cauchy Schwarz to obtain

$$\begin{aligned}\|II\|_{H_t^b} &\le c\|\psi_\delta\|_{H_t^b}\int_{|\tau|\delta\ge 1}|\tau|^{-1}|\mathcal{F}_t g(\tau)|d\tau\\ &\le c\delta^{1/2-b}\|g\|_{H_t^{b'}}\Big(\int_{|\tau|\delta\ge 1}|\tau|^{-2}|\langle\tau\rangle^{-2b'}d\tau\Big)^{\frac{1}{2}}\\ &\le c\delta^{1+b'-b}\|g\|_{H_t^{b'}}.\end{aligned}$$

Finally, for the integral J arising in III we have

$$J = c\mathcal{F}_t^{-1}(i\tau)^{-1}\chi_{|\tau|\delta\ge 1}\mathcal{F}_t g$$

and thus

$$\begin{aligned}\|J\|_{H_t^b}^2 &\le c\int_{|\tau|\delta\ge 1}\langle\tau\rangle^{2b-2-2b'}\langle\tau\rangle^{2b'}|\mathcal{F}_t g(\tau)|^2 d\tau\\ &\le c\sup_{|\tau|\ge\frac{1}{\delta}}|\tau|^{2b-2-2b'}\|g\|_{H_t^{b'}}^2.\end{aligned}$$

For all $b, b' \in \mathbf{R}$ satisfying $b - b' \le 1$ this gives

$$\|J\|_{H_t^b} \le c\delta^{1+b'-b}\|g\|_{H_t^{b'}}.$$

For the Fourier transform of the product $\psi_\delta J$ we have

$$\begin{aligned}\langle\tau\rangle^b\mathcal{F}_t(\psi_\delta J)(\tau) &= \langle\tau\rangle^b\int d\tau_1\mathcal{F}_t\psi_\delta(\tau_1)\mathcal{F}_t J(\tau-\tau_1)\\ &\le c\int d\tau_1|\tau_1|^b|\mathcal{F}_t\psi_\delta(\tau_1)\mathcal{F}_t J(\tau-\tau_1)|\\ &+ \int d\tau_1|\mathcal{F}_t\psi_\delta(\tau_1)|\langle\tau-\tau_1\rangle^b|\mathcal{F}_t J(\tau-\tau_1)|.\end{aligned}$$

This gives

$$\begin{aligned}\|\psi_\delta J\|_{H_t^b} &\le \|(|\tau|^b|\mathcal{F}_t\psi_\delta|)*|\mathcal{F}_tJ|\|_{L^2_\tau} + \||\mathcal{F}_t\psi_\delta|*(\langle\tau\rangle^b|\mathcal{F}_tJ|)\|_{L^2_\tau}\\ &\le \||\tau|^b|\mathcal{F}_t\psi_\delta|\|_{L^1_\tau}\|J\|_{L^2_t} + \|\mathcal{F}_t\psi_\delta\|_{L^1_\tau}\|J\|_{H^b_t}\\ &\le c(\delta^{-b}\|J\|_{L^2_t} + \|J\|_{H^b_t}) \le \delta^{1+b'-b}\|g\|_{H^{b'}_t}.\end{aligned}$$

Now (17) is shown. It follows that for fixed ξ:

$$\int\langle\tau\rangle^{2b}|\mathcal{F}Kg(\xi,\tau)|^2d\tau$$

$$\le 2c\delta^{2(1+b'-b)}\int\langle\tau\rangle^{2b'}|\mathcal{F}g(\xi,\tau)|^2d\tau + 2c_0\delta^{1-2b}(\int\langle\tau\rangle^{-1}|\mathcal{F}g(\xi,\tau)|d\tau)^2$$

Multiplying with $\langle\xi\rangle^{2s}$ and integrating with respect to $\mu(d\xi)$ we obtain

$$\|Kg\|^2_{H^{s,b}} \le c\delta^{2(1+b'-b)}\|g\|^2_{H^{s,b'}} + 2c_0\delta^{1-2b}\|\langle\xi\rangle^s\langle\tau\rangle^{-1}\mathcal{F}g\|^2_{L^2_\xi(L^1_\tau)},$$

respectively with $c_1 = \sqrt{2}c_0$:

$$\|Kg\|_{H^{s,b}} \le c\delta^{1+b'-b}\|g\|_{H^{s,b'}} + c_1\delta^{\frac{1}{2}-b}\|\langle\xi\rangle^s\langle\tau\rangle^{-1}\mathcal{F}g\|_{L^2_\xi(L^1_\tau)}.$$

Applied to $g(t) = U_\phi(-t)F(t)$ this gives (16). □

Lemma 1.10 *Let $f \in X_{s,b}(\phi)$, ψ_δ as above and $s \in \mathbf{R}$. Then we have the following estimates:*

i) $\|\psi_\delta f\|_{X_{s,b'}(\phi)} \le c\delta^{b-b'}\|f\|_{X_{s,b}(\phi)}$ *for* $\frac{1}{2} > b > b' \ge 0$ *or* $0 \ge b > b' > -\frac{1}{2}$,

ii) $\|\psi_\delta f\|_{X_{s,\frac{1}{2}}(\phi)} \le c_\varepsilon\delta^{-\varepsilon}\|f\|_{X_{s,\frac{1}{2}}(\phi)}$, $\varepsilon > 0$.

Proof: Consider i) and assume $b > b' \ge 0$ first. For $g \in H^b_t$, $f \in H^\beta_t$ we use that

$$\|fg\|_{H^{b'}_t} \le c\|f\|_{H^\beta_t}\|g\|_{H^b_t} \tag{18}$$

with $\beta = \frac{1}{2} - (b - b')$ (see Lemma 2.10 in section 2.2) to obtain

$$\|\psi_\delta g\|_{H^{b'}_t} \le c\|\psi_\delta\|_{H^\beta_t}\|g\|_{H^b_t} \le c\delta^{b-b'}\|g\|_{H^b_t},$$

since $\|\psi_\delta\|_{H^\beta_t} \le c\delta^{\frac{1}{2}-\beta}\|\psi\|_{H^\beta_t}$. From this we get for $f \in X_{s,b}(\phi)$:

$$\begin{aligned}\|\psi_\delta f\|_{X_{s,b'}(\phi)} &= \|U_\phi(-\cdot)\psi_\delta f\|_{H^{s,b'}}\\ &= \|\psi_\delta U_\phi(-\cdot)f\|_{H^{s,b'}}\\ &\le c\delta^{b-b'}\|U_\phi(-\cdot)f\|_{H^{s,b}} = c\delta^{b-b'}\|f\|_{X_{s,b}(\phi)}.\end{aligned}$$

By duality the same inequality holds for $0 \ge b > b' > -1/2$. The proof of ii) follows the same lines, using

$$\|fg\|_{H_t^{\frac{1}{2}}} \le c\|f\|_{H_t^{\frac{1}{2}+\varepsilon}}\|g\|_{H_t^{\frac{1}{2}}}$$

(see again Lemma 2.10 in section 2.2) instead of (18). □

1.3 The general local existence theorem

The spaces $X_{s,b}(\phi)$ have turned out to be very useful to prove existence and uniqueness results for initial value problems

$$u(0) = u_0 \in H^s_x \tag{19}$$

for nonlinear evolution equations

$$\partial_t u - i\phi(D)u = N(u), \tag{20}$$

where N is a nonlinear function of u and ∇u. Important examples, which were first treated with this method, are

Example 1.3 (The nonlinear Schrödinger equation)

$$\partial_t u - i\Delta u = u^k \overline{u}^l \quad , k, l \in \mathbf{N}_0 \tag{21}$$

as well as

Example 1.4 (The KdV equation)

$$\partial_t u + \partial_x^3 u = \partial_x(u^2), \tag{22}$$

see [B93], [KPV93b], [KPV96a],[KPV96b] and [St97]. In several cases we will consider data and solutions in Sobolev spaces H^s_x with $s < 0$, so we have to be careful with the meaning of $N(u)$: For smooth functions $u \in H$ we assume $N(u)$ to be given by

$$N(u)(x,t) = N_0(u(x,t), \nabla u(x,t)), \tag{23}$$

where $N_0 : \mathbf{C}^{n+1} \to \mathbf{C}$ is continuous and satisfies $N(0) = 0$ as well as

$$\begin{aligned} |N_0(u_1,v_1) - N_0(u_2,v_2)| \;\; &\leq \;\; c_1(|u_1|^{\alpha-1}|v_1|^{\beta} + |u_2|^{\alpha-1}|v_2|^{\beta})|u_1 - u_2| \\ &+ \;\; c_2(|u_1|^{\alpha}|v_1|^{\beta-1} + |u_2|^{\alpha}|v_2|^{\beta-1})|v_1 - v_2| \end{aligned} \tag{24}$$

for some α, $\beta \geq 1$. (If N_0 does not depend on ∇u, we assume (24) only with $c_2 = \beta = 0$, and if N_0 depends only on ∇u, we assume (24) with $c_1 = \alpha = 0$.) We shall always rely on a Lipschitz-estimate

$$\|N(u) - N(v)\|_{X_{s,b'}(\phi) \cap Y_s(\phi)} \leq C(\|u\|_{X_{s,b}(\phi)} + \|v\|_{X_{s,b}(\phi)})\|u - v\|_{X_{s,b}(\phi)} \tag{25}$$

for smooth u and v. Here $C : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is a continuous and nondecreasing function, s is the Sobolev exponent given with the data, and for the parameters b and b' we will approximately have $b \approx b' + 1 \approx \frac{1}{2}$. By the estimate (25) we may extend the nonlinear mapping N uniquely to the whole $X_{s,b}(\phi)$ by

$$N(u) := \lim_{n \in \mathbf{N}} N(u_n),$$

where $u_n \in H$, $u_n \to u$ in $X_{s,b}(\phi)$ and the limit is taken in $X_{s,b'}(\phi) \cap Y_s(\phi)$. It is straight forward to check, that this limit does not depend on the approximating sequence and that the estimate (25) is still valid for the extended operator N. Obviously the question comes up, for which functions $u \in X_{s,b}(\phi)$ our definition of $N(u)$ coincides with the natural one in (23). Our (partial) answer is the following

Lemma 1.11 *Let $u \in X_{s,b}(\phi)$ such that for an open subset $\Omega \subset \mathbf{R}^{n+1}$ (respectively $\Omega \subset \mathbf{R} \times \mathbf{T}^n$) $u|_\Omega \in L^{\alpha p}_{loc}(\Omega)$ and $\nabla u|_\Omega \in L^{\beta p'}_{loc}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then $N(u)|_\Omega \in L^1_{loc}(\Omega)$ and (23) holds for almost all $(x,t) \in \Omega$.*

Remark: If N_0 does not depend on ∇u we only assume $u|_\Omega \in L^\alpha_{loc}(\Omega)$. If N_0 depends only on ∇u we assume $\nabla u|_\Omega \in L^\beta_{loc}(\Omega)$.

Proof: We choose a smooth approximate identity $(J_\varepsilon)_{\varepsilon>0}$ on $\mathbf{R}^{n+1}$ (respectively on $\mathbf{R} \times \mathbf{T}^n$), so that for $u \in X_{s,b}(\phi)$ we have $u_\varepsilon := J_\varepsilon * u \in H$. Then $u_\varepsilon|_\Omega \to u|_\Omega$ in $L^{\alpha p}_{loc}(\Omega)$ and $\nabla u_\varepsilon|_\Omega = (\nabla u)_\varepsilon|_\Omega \to \nabla u|_\Omega$ in $L^{\beta p'}_{loc}(\Omega)$. The dominated convergence theorem gives that $u_\varepsilon \to u$ in $X_{s,b}(\phi)$. Hence for $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ (respectively $\phi \in C_0^\infty(\mathbf{R} \times \mathbf{T}^n)$) supported in $K \subset\subset \Omega$ and $N_1(u)(x,t) := N_0(u(x,t), \nabla u(x,t))$ we obtain

$$|N(u)(\phi) - N_1(u)(\phi)| \leq |N(u)(\phi) - N(u_\varepsilon)(\phi)|$$
$$+\|\phi\|_{L^\infty_{x,t}} \int_K dxdt |N_0(u_\varepsilon(x,t), \nabla u_\varepsilon(x,t)) - N_0(u(x,t), \nabla u(x,t))| =: I + II.$$

Since $N(u_\varepsilon) \to N(u)$ in $X_{s,b'}(\phi)$, we have $I \to 0$ ($\varepsilon \to 0$). Using (24) the integral in II can be estimated by

$$\begin{aligned}
& c_1 \int_K dxdt(|u_\varepsilon|^{\alpha-1}|\nabla u_\varepsilon|^\beta + |u|^{\alpha-1}|\nabla u|^\beta)|u_\varepsilon - u| \\
+ \quad & c_2 \int_K dxdt(|u_\varepsilon|^{\alpha}|\nabla u_\varepsilon|^{\beta-1} + |u|^{\alpha}|\nabla u|^{\beta-1})|\nabla u_\varepsilon - \nabla u| \\
\leq \quad & c_1(\|u_\varepsilon\|^{\alpha-1}_{L^{\alpha p}(K)}\|\nabla u_\varepsilon\|^\beta_{L^{\beta p'}(K)} + \|u\|^{\alpha-1}_{L^{\alpha p}(K)}\|\nabla u\|^\beta_{L^{\beta p'}(K)})\|u_\varepsilon - u\|_{L^{\alpha p}(K)} \\
+ \quad & c_2(\|u_\varepsilon\|^{\alpha}_{L^{\alpha p}(K)}\|\nabla u_\varepsilon\|^{\beta-1}_{L^{\beta p'}(K)} + \|u\|^{\alpha}_{L^{\alpha p}(K)}\|\nabla u\|^{\beta-1}_{L^{\beta p'}(K)})\|\nabla u_\varepsilon - \nabla u\|_{L^{\beta p'}(K)}.
\end{aligned}$$

This tends to zero with $\varepsilon \to 0$. □

Corollary 1.3

i) Let L^q_{loc} denote $L^q_{loc}(\mathbf{R}^{n+1})$ respectively $L^q_{loc}(\mathbf{R} \times \mathbf{T}^n)$. Then, for $u \in X_{s,b}(\phi) \cap L^{\alpha p}_{loc}$ with $\nabla u \in L^{\beta p'}_{loc}$ it follows that $N(u)(x,t) = N_0(u(x,t), \nabla u(x,t))$ a. e..

ii) For $u \in H$, $v \in X_{s,b}(\phi)$ with $u|_\Omega = v|_\Omega$ we have $N(u)|_\Omega = N(v)|_\Omega$.

For $u \in H$ the nonlinear operator N is local in spacetime and commutes with time translations. This is still true for the extended operator:

Lemma 1.12

i) Let $\Omega \subset \mathbf{R}^{n+1}$ (respectively $\Omega \subset \mathbf{R} \times \mathbf{T}^n$) be a domain and $u, v \in X_{s,b}(\phi)$ with $u|_\Omega = v|_\Omega$. Then $N(u)|_\Omega = N(v)|_\Omega$.

ii) Let τ_t denote the time translation $\tau_t u(t_0) = u(t_0 - t)$. Then for $u \in X_{s,b}(\phi)$ we have $N(\tau_t u) = \tau_t N(u)$.

Proof: Choose sequences $(u_n)_{n\in\mathbf{N}}$, $(v_n)_{n\in\mathbf{N}}$ of smooth functions with $u_n \to u$, $v_n \to v$ in $X_{s,b}(\phi)$.

To see i) we write

$$\begin{aligned} &\|N(u)|_\Omega - N(v)|_\Omega\|_{X^\Omega_{s,b'}(\phi)} \leq \|N(u)|_\Omega - N(u_n)|_\Omega\|_{X^\Omega_{s,b'}(\phi)} \\ + \quad &\|N(u_n)|_\Omega - N(v_n)|_\Omega\|_{X^\Omega_{s,b'}(\phi)} + \|N(v_n)|_\Omega - N(v)|_\Omega\|_{X^\Omega_{s,b'}(\phi)}. \end{aligned} \tag{26}$$

Clearly, $\|N(u)|_\Omega - N(u_n)|_\Omega\|_{X^\Omega_{s,b'}(\phi)} \leq \|N(u) - N(u_n)\|_{X_{s,b'}(\phi)}$, which tends to zero with $n \to \infty$. By the same argument the third term in (26) vanishes. Now for all u_n', $v_n' \in X_{s,b}(\phi)$ with $u_n'|_\Omega = u_n|_\Omega$ and $v_n'|_\Omega = v_n|_\Omega$ we have $N(u_n')|_\Omega = N(u_n)|_\Omega$ and $N(v_n')|_\Omega = N(v_n)|_\Omega$ by part ii) of Corollary 1.3. Hence by (25)

$$\|N(u_n)|_\Omega - N(v_n)|_\Omega\|_{X^\Omega_{s,b'}(\phi)} \leq C(\|u_n'\|_{X_{s,b}(\phi)} + \|v_n'\|_{X_{s,b}(\phi)})\|u_n' - v_n'\|_{X_{s,b}(\phi)}.$$

A proper choice of u_n', v_n' (cf. the remark below Lemma 1.6) yields the upper bound

$$C(\|u_n|_\Omega\|_{X^\Omega_{s,b}(\phi)} + \|v_n|_\Omega\|_{X^\Omega_{s,b}(\phi)})\|u_n|_\Omega - v_n|_\Omega\|_{X^\Omega_{s,b}(\phi)},$$

which tends to zero, since $\|u_n|_\Omega - v_n|_\Omega\|_{X^\Omega_{s,b}(\phi)} \leq \|u_n - u\|_{X_{s,b}(\phi)} + \|v_n - v\|_{X_{s,b}(\phi)}$. Now part i) is shown.

To see part ii) we first observe that τ_t is an isometric isomorphism on all the spaces $X_{s,b}(\phi)$, $Y_s(\phi)$ and $H^{s,b}$, since their norms depend only on the size of the Fourier transform. Especially we have $\tau_t H = H$. Hence

$$\begin{aligned} N(\tau_t u) &= N(\tau_t \lim_{n\in\mathbf{N}} u_n) = N(\lim_{n\in\mathbf{N}} \tau_t u_n) \\ &= \lim_{n\in\mathbf{N}} N(\tau_t u_n) = \lim_{n\in\mathbf{N}} \tau_t N(u_n) = \tau_t N(u), \end{aligned}$$

where the first two limits are in $X_{s,b}(\phi)$ and the last two are in $X_{s,b'}(\phi)$. □

Remark/Definition: By part i) of the above Lemma we can now define the mapping

$$N : X^\Omega_{s,b}(\phi) \to X^\Omega_{s,b'}(\phi) \qquad \text{by} \qquad N(u) = N(\tilde{u})|_\Omega,$$

where $\tilde{u}$ is an arbitrary extension of u.

We now turn to prove a general local existence theorem, which reduces local wellposedness of (19), (20) to nonlinear estimates. Here by a local solution of (19), (20) we understand a solution $u \in C_t((-\delta,\delta), H^s_x)$ of the corresponding integral equation

$$u(t) = \Lambda u(t) := U_\phi(t)u_0 + U_\phi *_R N(u)(t), \quad t \in (-\delta,\delta). \tag{27}$$

Theorem 1.1 (General local wellposedness)

i) Let $s \in \mathbf{R}$. Assume that there exist $b \geq \frac{1}{2}$ and $\theta > 0$ such that for all $0 < \delta \ll 1$ the estimate

$$\|U_\phi *_R (N(u) - N(v))\|_{X^\delta_{s,b}(\phi)} \leq \delta^\theta C(\|u\|_{X^\delta_{s,b}(\phi)} + \|v\|_{X^\delta_{s,b}(\phi)})\|u - v\|_{X^\delta_{s,b}(\phi)} \tag{28}$$

holds with a nondecreasing function $C : \mathbf{R}_0^+ \to \mathbf{R}_0^+$*, and that, for* $b = \frac{1}{2}$*,* $N(u) \in Y_s(\phi)$ *for all* $u \in X_{s,b}(\phi)$*.*

Then there exist $\delta = \delta(\|u_0\|_{H_x^s}) > 0$ *and a unique solution* $u \in X_{s,b}^{\delta}(\phi)$ *of (27). This solution belongs to* $C_t((-\delta,\delta), H_x^s)$ *and the mapping* $f : H_x^s \to X_{s,b}^{\delta_0}(\phi)$*,* $u_0 \mapsto u$ *(data upon solution) is locally Lipschitz continuous for any* $0 < \delta_0 < \delta$*.*

ii) Assume in addition that $u_0 \in H_x^{\sigma}$ *for some* $\sigma > s$ *and that also the estimates*

$$\|U_{\phi} *_R N(u)\|_{X_{\sigma,b}^{\delta}(\phi)} \leq \delta^{\theta} C(\|u\|_{X_{s,b}^{\delta}(\phi)}) \|u\|_{X_{\sigma,b}^{\delta}(\phi)} \tag{29}$$

and

$$\|U_{\phi} *_R (N(u) - N(v))\|_{X_{\sigma,b}^{\delta}(\phi)} \leq \delta^{\theta} \{ C(\|u\|_{X_{s,b}^{\delta}(\phi)} + \|v\|_{X_{s,b}^{\delta}(\phi)}) \|u-v\|_{X_{\sigma,b}^{\delta}(\phi)} + C(\|u\|_{X_{\sigma,b}^{\delta}(\phi)} + \|v\|_{X_{\sigma,b}^{\delta}(\phi)}) \|u-v\|_{X_{s,b}^{\delta}(\phi)} \} \tag{30}$$

are valid. In the case where $b = \frac{1}{2}$ *assume in addition that* $N(u) \in Y_{\sigma}(\phi)$ *for all* $u \in X_{\sigma,b}(\phi)$*. Then the solution* u *of (27) belongs to* $X_{\sigma,b}^{\delta}(\phi) \cap C_t((-\delta,\delta), H_x^{\sigma})$ *and the mapping data upon solution is locally Lipschitz continuous from* H_x^{σ} *to* $X_{\sigma,b}^{\delta_0}(\phi)$*.*

Proof: i) Existence: We assume (29) and (30), since by (28) these estimates hold at least in the case $\sigma = s$. Defining

$$B_{s,\sigma} = \{ u \in X_{\sigma,b}^{\delta}(\phi) : \|u\|_{X_{\sigma,b}^{\delta}(\phi)} \leq R_{\sigma}, \ \|u\|_{X_{s,b}^{\delta}(\phi)} \leq R_s \},$$

we shall show that for a proper choice of R_{σ}, R_s and δ the mapping Λ introduced above has a fixed point in $B_{s,\sigma}$. In fact, by Lemma 1.7, applied to $\psi(t)U_{\phi}(t)u_0$, and (29) we see that for $u \in B_{s,\sigma}$

$$\begin{aligned} \|\Lambda u\|_{X_{\sigma,b}^{\delta}(\phi)} &\leq c\|u_0\|_{H_x^{\sigma}} + \delta^{\theta} C(\|u\|_{X_{s,b}^{\delta}(\phi)}) \|u\|_{X_{\sigma,b}^{\delta}(\phi)} \\ &\leq c\|u_0\|_{H_x^{\sigma}} + \delta^{\theta} C(R_s) R_{\sigma}. \end{aligned}$$

Especially for $\sigma = s$ we have

$$\|\Lambda u\|_{X_{s,b}^{\delta}(\phi)} \leq c\|u_0\|_{H_x^s} + \delta^{\theta} C(R_s) R_s.$$

Now choosing $R_s = 2c\|u_0\|_{H_x^s}$, $R_{\sigma} = 2c\|u_0\|_{H_x^{\sigma}}$ and δ small enough to ensure that $\delta^{\theta}(C(2R_s)+1) \leq \frac{1}{2}$, we see that Λ maps $B_{s,\sigma}$ into itself. For the difference $\Lambda u - \Lambda v$ we use (28) to obtain

$$\begin{aligned} \|\Lambda u - \Lambda v\|_{X_{s,b}^{\delta}(\phi)} &\leq \delta^{\theta} C(\|u\|_{X_{s,b}^{\delta}(\phi)} + \|v\|_{X_{s,b}^{\delta}(\phi)}) \|u-v\|_{X_{s,b}^{\delta}(\phi)} \\ &\leq \delta^{\theta} C(2R_s) \|u-v\|_{X_{s,b}^{\delta}(\phi)} \leq \frac{1}{2} \|u-v\|_{X_{s,b}^{\delta}(\phi)} \end{aligned}$$

for $u, v \in B_{s,\sigma}$ by our choice of R_s and δ. Iteration yields

$$\|\Lambda^n u - \Lambda^n v\|_{X^\delta_{s,b}(\phi)} \leq \frac{1}{2^n}\|u - v\|_{X^\delta_{s,b}(\phi)}. \tag{31}$$

Next we use (30), (31) and induction to deduce

$$\|\Lambda^n u - \Lambda^n v\|_{X^\delta_{\sigma,b}(\phi)} \leq \frac{n+1}{2^{n-1}}(1 + C(2R_\sigma))\|u - v\|_{X^\delta_{\sigma,b}(\phi)}.$$

Now Weissinger's fixed point theorem[2] gives a solution $u \in B_{s,\sigma}$ of $\Lambda u = u$.

ii) Persistence property: For $b > \frac{1}{2}$ it follows from Lemma 1.5 that $X^\delta_{\sigma,b}(\phi) \subset C_t((-\delta,\delta), H^\sigma_x)$, while for $b = \frac{1}{2}$ we use Lemma 1.8 and the additional assumption $N(u) \in Y_\sigma(\phi)$ for $u \in X_{\sigma,b}(\phi)$ to see that any solution $u \in X^\delta_{\sigma,b}(\phi)$ of (27) belongs to $C_t((-\delta,\delta), H^\sigma_x)$.

iii) Uniqueness: Assume that $u, v \in X^\delta_{s,b}(\phi)$ are solutions of (27), which do not coincide on $[0,\delta)$. Define

$$t_0 := \inf\{t \in [0,\delta) : \ u(t) \neq v(t)\}.$$

Since u and v belong to $C_t((-\delta,\delta), H^s_x)$ this makes sense and we have $u(t_0) = v(t_0)$. Now for $\delta_0 \in (0, \delta - t_0)$ and $t \in (-\delta_0, \delta_0)$ we write

$$u_1(t) = u(t + t_0) \qquad \text{and} \qquad v_1(t) = v(t + t_0).$$

Then $u_1,\ v_1 \in X^{\delta_0}_{\sigma,b}(\phi)$, and using (15) and part ii) of Lemma 1.12 we see that

$$u_1(t) - v_1(t) = U_\phi *_R N(u_1)(t) - U_\phi *_R N(v_1)(t) = \Lambda u_1(t) - \Lambda v_1(t).$$

Applying (28) we obtain

$$\|u_1 - v_1\|_{X^{\delta_0}_{s,b}(\phi)} \leq \delta_0^\theta C(\|u_1\|_{X^{\delta_0}_{s,b}(\phi)} + \|v_1\|_{X^{\delta_0}_{s,b}(\phi)})\|u_1 - v_1\|_{X^{\delta_0}_{s,b}(\phi)}.$$

Now for $\delta_0 > 0$ sufficiently small we have

$$\delta_0^\theta C(\|u_1\|_{X^{\delta_0}_{s,b}(\phi)} + \|v_1\|_{X^{\delta_0}_{s,b}(\phi)}) < 1,$$

which implies $\|u_1 - v_1\|_{X^{\delta_0}_{s,b}(\phi)} = 0$. But then $u(t + t_0) = v(t + t_0)$ for all $t \in (-\delta_0, \delta_0)$. This contradicts the choice of t_0. For $t \in (-\delta, 0]$ the same argument applies.

iv) Continuous dependence: Let $0 < \delta_0 < \delta$ and $\varepsilon > 0$ so small that $\delta_0^\theta(C(2(R_s + \varepsilon)) + 1) \leq \frac{1}{2}$. Then for $v_0,\ v_0' \in H^s_x$ with $\|u_0 - v_0\|_{H^s_x} \leq \frac{\varepsilon}{2c}$ and $\|u_0 - v_0'\|_{H^s_x} \leq \frac{\varepsilon}{2c}$ there exist unique solutions $v,\ v' \in X^{\delta_0}_{s,b}(\phi)$ of (19) with $v(0) = v_0$

[2] This is essentially the contraction mapping principle, the only difference is that the assumption $\|\Lambda u - \Lambda v\| \leq q\|u - v\|$, $q < 1$, is replaced by $\|\Lambda^n u - \Lambda^n v\| \leq a_n\|u - v\|$, $\sum_{n\geq 1} a_n < \infty$.

respectively $v'(0) = v_0'$ and $\|v\|_{X^{\delta_0}_{s,b}(\phi)}, \|v'\|_{X^{\delta_0}_{s,b}(\phi)} \le R_s + \varepsilon$. Using (28) for the difference $v - v'$ we obtain

$$\begin{aligned}\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)} &\le c\|v_0 - v_0'\|_{H^s_x} + \delta_0^\theta C(\|v\|_{X^{\delta_0}_{s,b}(\phi)} + \|v'\|_{X^{\delta_0}_{s,b}(\phi)})\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)} \\ &\le c\|v_0 - v_0'\|_{H^s_x} + \delta_0^\theta C(2(R_s + \varepsilon))\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)} \\ &\le c\|v_0 - v_0'\|_{H^s_x} + \frac{1}{2}\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)}.\end{aligned}$$

Hence

$$\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)} \le 2c\|v_0 - v_0'\|_{H^s_x}.$$

Next we assume in addition that $v_0, v_0' \in H^\sigma_x$ and $\|v_0\|_{H^\sigma_x}, \|v_0'\|_{H^\sigma_x} \le R$, where R is a given radius. Then by (30)

$$\begin{aligned}&\|v - v'\|_{X^{\delta_0}_{\sigma,b}(\phi)} \le c\|v_0 - v_0'\|_{H^\sigma_x} \\ +\ & \delta_0^\theta\{C(2(R_s + \varepsilon))\|v - v'\|_{X^{\delta_0}_{\sigma,b}(\phi)} + C(\|v\|_{X^{\delta_0}_{\sigma,b}(\phi)} + \|v'\|_{X^{\delta_0}_{\sigma,b}(\phi)})\|v - v'\|_{X^{\delta_0}_{s,b}(\phi)}\} \\ \le\ & c\|v_0 - v_0'\|_{H^\sigma_x} + \frac{1}{2}\|v - v'\|_{X^{\delta_0}_{\sigma,b}(\phi)} + \delta_0^\theta C(4cR)2c\|v_0 - v_0'\|_{H^s_x}.\end{aligned}$$

This gives $\|v - v'\|_{X^{\delta_0}_{\sigma,b}(\phi)} \le L\|v_0 - v_0'\|_{H^\sigma_x}$ with $L = 2c(1 + 2\delta_0^\theta C(4cR))$. □

Remark: The proof shows that the lifespan δ guaranteed by Theorem 1.1 can be chosen as a continuous nonincreasing function of $\|u_0\|_{H^s_x}$.

We may go a step further and reduce the estimates (28) to (30) in Theorem 1.1 by the aid of Lemma 1.9 to nonlinear estimates of type (25). Here two cases occur: In the first case for the parameters b and b' we have $b - b' < 1$ and we can obtain a positive power of δ already from the linear estimate (Lemma 1.9). In the second case we have $b = b' + 1 = \frac{1}{2}$, and here the contracting factor has to come from the nonlinear estimate.

Lemma 1.13 *Let $s \in \mathbf{R}$. Assume that there exist $b > \frac{1}{2}$ and $b' > b - 1$, so that the estimate*

$$\|N(u) - N(v)\|_{X_{s,b'}(\phi)} \le C_0(\|u\|_{X_{s,b}(\phi)} + \|v\|_{X_{s,b}(\phi)})\|u - v\|_{X_{s,b}(\phi)} \tag{32}$$

holds, where $C_0 : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is continuous and nondecreasing. Then hypothesis (28) of Theorem 1.1 is valid. If, in addiition, for some $\sigma > s$ also the estimates

$$\|N(u)\|_{X_{\sigma,b'}(\phi)} \le C_0(\|u\|_{X_{s,b}(\phi)})\|u\|_{X_{\sigma,b}(\phi)} \tag{33}$$

and

$$\begin{aligned}\|N(u) - N(v)\|_{X_{\sigma,b'}(\phi)} &\le C_0(\|u\|_{X_{s,b}(\phi)} + \|v\|_{X_{s,b}(\phi)})\|u - v\|_{X_{\sigma,b}(\phi)} \\ &+ C_0(\|u\|_{X_{\sigma,b}(\phi)} + \|v\|_{X_{\sigma,b}(\phi)})\|u - v\|_{X_{s,b}(\phi)}\end{aligned} \tag{34}$$

hold, then the assumptions (29) and (30) of Theorem 1.1 are valid, too.

Proof: Let $u,\ v \in X^{\delta}_{s,b}(\phi)$ be given with extensions $\tilde{u},\ \tilde{v} \in X_{s,b}(\phi)$. Then $\psi_\delta U_\phi *_R (N(\tilde{u}) - N(\tilde{v}))$ is an extension of $U_\phi *_R (N(u) - N(v))$. Combining Lemma 1.9 with (32) we obtain

$$\begin{aligned}\|U_\phi *_R (N(u) - N(v))\|_{X^{\delta}_{s,b}(\phi)} &\leq \|\psi_\delta U_\phi *_R (N(\tilde{u}) - (N(\tilde{v}))\|_{X_{s,b}(\phi)} \\ &\leq c\delta^{1-b+b'}\|N(\tilde{u}) - (N(\tilde{v})\|_{X_{s,b'}(\phi)} \\ &\leq c\delta^{1-b+b'}C_0(\|\tilde{u}\|_{X_{s,b}(\phi)}+\|\tilde{v}\|_{X_{s,b}(\phi)})\|\tilde{u}-\tilde{v}\|_{X_{s,b}(\phi)}.\end{aligned}$$

Now Lemma 1.6 gives (28) in Theorem 1.1 with $\theta = 1-b+b' > 0$ and $C(t) = cC_0(t)$. The same argument shows that (33) implies (29) and that (34) implies (30). Here the use of Lemma 1.6 becomes essential. □

Lemma 1.14 *Let $s \in \mathbf{R}$ and $b = b' + 1 = \frac{1}{2}$. Assume that the estimate*

$$\|N(u) - N(v)\|_{X_{s,b'}(\phi)\cap Y_s(\phi)} \leq C_0(\|u\|_{X_{s,b}(\phi)}+\|v\|_{X_{s,b}(\phi)})\|u-v\|_{X_{s,b}(\phi)} \tag{35}$$

holds, where $C_0 : \mathbf{R}^+_0 \to \mathbf{R}^+_0$ is a continuous and nondecreasing function satisfying $C_0(\lambda t) \leq \lambda^\gamma C_0(t)$ for some $\gamma \geq 0$. Assume further that there exists $\varepsilon > 0$ such that for all $0 < \delta \ll 1$ and for all $u, v \in X_{s,b}(\phi)$ supported in $\{(x,t) : |t| \leq \delta\}$ we have

$$\|N(u) - N(v)\|_{X_{s,b'}(\phi)\cap Y_s(\phi)} \leq \delta^\varepsilon C_0(\|u\|_{X_{s,b}(\phi)}+\|v\|_{X_{s,b}(\phi)})\|u-v\|_{X_{s,b}(\phi)}. \tag{36}$$

Then $N(u)$ is well defined for $u \in X_{s,b}(\phi)$ and belongs to $Y_s(\phi)$. Moreover, assumption (28) in Theorem 1.1 is fulfilled.

If additionally for some $\sigma > s$ the estimates

$$\|N(u)\|_{X_{\sigma,b'}(\phi)\cap Y_\sigma(\phi)} \leq C_0(\|u\|_{X_{s,b}(\phi)})\|u\|_{X_{\sigma,b}(\phi)} \tag{37}$$

and

$$\begin{aligned}\|N(u) - N(v)\|_{X_{\sigma,b'}(\phi)\cap Y_\sigma(\phi)} &\leq C_0(\|u\|_{X_{s,b}(\phi)}+\|v\|_{X_{s,b}(\phi)})\|u-v\|_{X_{\sigma,b}(\phi)} \\ &+ C_0(\|u\|_{X_{\sigma,b}(\phi)}+\|v\|_{X_{\sigma,b}(\phi)})\|u-v\|_{X_{s,b}(\phi)}\end{aligned} \tag{38}$$

hold true and if they are still valid with an additional factor δ^ε, whenever u, v are supported in $\{(x,t) : |t| \leq \delta\}$, then $N(u) \in Y_\sigma(\phi)$ for $u \in X_{\sigma,b}(\phi)$ and conditions (29) and (30) of Theorem 1.1 are satisfied, too.

Proof: By (35) respectively (38) $N(u)$ is well defined for $u \in X_{s,b}(\phi)$ (resp. $u \in X_{\sigma,b}(\phi)$) and belongs to $Y_s(\phi)$ (resp. $Y_\sigma(\phi)$). Now let $u,\ v \in X^{\delta}_{s,b}(\phi)$ be given with extensions $\tilde{u},\ \tilde{v}$. Then $\psi_\delta U_\phi *_R (N(\psi_{2\delta}\tilde{u}) - N(\psi_{2\delta}\tilde{v}))$ is an extension of $U_\phi *_R (N(u) - N(v))$, for which we obtain

$$\begin{aligned}&\|\psi_\delta U_\phi *_R (N(\psi_{2\delta}\tilde{u}) - N(\psi_{2\delta}\tilde{v}))\|_{X_{s,b}(\phi)} \\ \leq\ & c\|N(\psi_{2\delta}\tilde{u}) - N(\psi_{2\delta}\tilde{v})\|_{X_{s,b'}(\phi)\cap Y_s(\phi)} \\ \leq\ & c\delta^\varepsilon C_0(\|\psi_{2\delta}\tilde{u}\|_{X_{s,b}(\phi)}+\|\psi_{2\delta}\tilde{v}\|_{X_{s,b}(\phi)})\|\psi_{2\delta}(\tilde{u}-\tilde{v})\|_{X_{s,b}(\phi)} \\ \leq\ & c\delta^\varepsilon C_0(c_{\varepsilon'}\delta^{-\varepsilon'}(\|\tilde{u}\|_{X_{s,b}(\phi)}+\|\tilde{v}\|_{X_{s,b}(\phi)}))c_{\varepsilon'}\delta^{-\varepsilon'}\|\tilde{u}-\tilde{v}\|_{X_{s,b}(\phi)} \\ \leq\ & \delta^\theta C(\|\tilde{u}\|_{X_{s,b}(\phi)}+\|\tilde{v}\|_{X_{s,b}(\phi)})\|\tilde{u}-\tilde{v}\|_{X_{s,b}(\phi)},\end{aligned}$$

where $\theta = \varepsilon - (\gamma+1)\varepsilon'$. Here Lemma 1.9, (36) and Lemma 1.10, part ii), were applied. Together with Lemma 1.6 this gives (28) in Theorem 1.1. Similarly (29) respectively (30) can be derived from (37) respectively (38), here again the use of Lemma 1.6 becomes essential. □

In the situation where Lemma 1.13 applies, it is clear by the Sobolev embedding in the time variable (Lemma 1.5) that the mapping data upon solution from H^s_x to $C_t((-\delta_0,\delta_0),H^s_x)$ (respectively from H^σ_x to $C_t((-\delta_0,\delta_0),H^\sigma_x)$) is locally Lipschitz continuous. This is still true, but no longer trivial in the situation of Lemma 1.14:

Remark 1.2 *Under the assumptions of Lemma 1.14 the mapping $f: u_0 \mapsto u$ (data upon solution) is locally Lipschitz continuous from H^s_x to $C_t((-\delta_0,\delta_0),H^s_x)$ respectively from H^σ_x to $C_t((-\delta_0,\delta_0),H^\sigma_x)$.*

Proof: Let $v, v' \in X^{\delta_0}_{s,b}(\phi)$ as in step iv) of the proof of Theorem 1.1 with extensions $\tilde{v}, \tilde{v}' \in X_{s,b}(\phi)$. Then

$$\|v(t)-v'(t)\|_{H^s_x} \le \|v_0 - v'_0\|_{H^s_x} + \|U_\phi *_R (N(v)(t) - N(v')(t))\|_{H^s_x}.$$

In order to estimate the second contribution we use Lemma 1.8, assumption (35) in Lemma 1.14 and Lemma 1.6 to obtain

$$\begin{aligned} & \|U_\phi *_R (N(v)(t) - N(v')(t))\|_{H^s_x} \\ \le\ & c\|N(\tilde{v}) - N(\tilde{v}')\|_{Y_s(\phi)} \\ \le\ & cC_0(\|\tilde{v}\|_{X_{s,b}(\phi)} + \|\tilde{v}'\|_{X_{s,b}(\phi)})\|\tilde{v}-\tilde{v}'\|_{X_{s,b}(\phi)} \\ \le\ & cC_0(\|v\|_{X^{\delta_0}_{s,b}(\phi)} + \|v'\|_{X^{\delta_0}_{s,b}(\phi)})\|v-v'\|_{X^{\delta_0}_{s,b}(\phi)} \\ \le\ & cC_0(2(R_s+\varepsilon))2c\|v_0 - v'_0\|_{H^s_x} \end{aligned}$$

(for the last step cf. the proof of Theorem 1.1). If in addition $v_0, v'_0 \in H^\sigma_x$ with $\|v_0\|_{H^\sigma_x}, \|v'_0\|_{H^\sigma_x} \le R$, where R is a given radius, we can estimate similarly

$$\|v(t)-v'(t)\|_{H^\sigma_x} \le \|v_0 - v'_0\|_{H^\sigma_x} + \|N(\tilde{v}) - N(\tilde{v}')\|_{Y_\sigma(\phi)} = I + II.$$

Arguing as above but using (38) instead of (35) we see that

$$\begin{aligned} II \ \le\ & cC_0(\|v\|_{X^{\delta_0}_{s,b}(\phi)} + \|v'\|_{X^{\delta_0}_{s,b}(\phi)})\|v-v'\|_{X^{\delta_0}_{\sigma,b}(\phi)} \\ +\ & cC_0(\|v\|_{X^{\delta_0}_{\sigma,b}(\phi)} + \|v'\|_{X^{\delta_0}_{\sigma,b}(\phi)})\|v-v'\|_{X^{\delta_0}_{s,b}(\phi)} \\ \le\ & cC_0(2(R_s+\varepsilon))L_\sigma\|v_0 - v'_0\|_{H^\sigma_x} + cC_0(4cR)2c\|v_0-v'_0\|_{H^s_x} \end{aligned}$$

(cf. again step iv) of the proof of Theorem 1.1). □

Corollary 1.4 (Global wellposedness) *If the assumptions of Lemma 1.13 or Lemma 1.14 are fulfilled and if for a solution u of (27) $\|u(t)\|_{H^s_x}$ is a conserved quantity, then the existence and uniqueness statements in Theorem 1.1 are valid for all $\delta > 0$. Moreover, the mapping data upon solution $H^s_x \to C_t((-\delta,\delta),H^s_x)$ (respectively $H^\sigma_x \to C_t((-\delta,\delta),H^\sigma_x)$) is locally Lipschitz continuous.*

Proof: For given $u_0 \in H^\sigma_x$ let Δ denote the set of all $\delta > 0$, for which the following holds true:

i) There exists a solution $u \in X^\delta_{\sigma,b}(\phi) \cap C_t((-\delta,\delta), H^\sigma_x)$ of (27),

ii) this solution is unique in $X^\delta_{s,b}(\phi)$,

iii) there exists a neighbourhood $U(u_0) \subset H^\sigma_x$ and a Lipschitz constant $L = L(u_0,\delta)$ such that for all $v_0,\ v_0' \in U(u_0)$ there exist unique solutions $v,\ v' \in X^\delta_{\sigma,b}(\phi) \cap C_t((-\delta,\delta), H^\sigma_x)$ of (19) with $v(0) = v_0$, $v'(0) = v_0'$ satisfying the estimate
$$\|v - v'\|_{L^\infty_t((-\delta,\delta),H^\sigma_x)} \le L\|v_0 - v_0'\|_{H^\sigma_x}.$$

By the local existence theorem (and Remark 1.2) $\Delta \neq \emptyset$. Define $T_0 = \sup\{\delta \in \Delta\}$ and assume $T_0 < \infty$. Fix $0 < \varepsilon \ll \delta(\|u_0\|_{H^s_x})$, $\delta = \delta(\|u_0\|_{H^s_x}) - \varepsilon$, $T_1 = T_0 - \varepsilon$ and $T = T_0 - 2\varepsilon$. Then for the solution $u_1 \in X^{T_1}_{\sigma,b}(\phi)$ of (27) guaranteed by the choice of T_1 we consider the initial value problems

$$\partial_t u - i\phi(D)u = N(u), \qquad\qquad u(0) = u_1(\pm T). \tag{39}$$

By Theorem 1.1 (and Remark 1.2) we obtain solutions $u_\pm \in X^\delta_{\sigma,b}(\phi) \cap C_t((-\delta,\delta), H^\sigma_x)$ of (39), uniquely determined in $X^\delta_{s,b}(\phi)$, such that in a whole neighbourhood $U_+(u_1(T))$ (respectively $U_-(u_1(-T))$) the mapping data upon solution into $C_t((-\delta,\delta), H^\sigma_x)$ is Lipschitz. Define

$$U(t) := \begin{cases} u_1(t) & : \quad |t| \le T \\ u_+(t-T) & : \quad T \le t < T + \delta \\ u_-(t+T) & : \quad -T-\delta < t \le T. \end{cases}$$

Then, using (15) and part ii) of Lemma 1.12, we see that U solves (27) on $(-T-\delta, T+\delta)$. Moreover, $\tau_{\mp T}u_1$ solves (39) on $(-\varepsilon,\varepsilon)$ and so $U(t) = u_1(t)$ for $T \le t < T + \varepsilon$ by local uniqueness, especially we have $U \in C_t((-\delta - T, \delta + T), H^\sigma_x)$.

Now let $\tilde{u}$ and $\tilde{u}_\pm \in X_{s,b}(\phi)$ be extensions of u_1 and $\tau_{\pm T}u_\pm$. Then, for suitable smooth characteristic functions χ_T of $[-T,T]$ and χ_δ of $[T-\delta, T+\delta]$ with $\chi_T(t) = 0$ for $|t| \ge T + \varepsilon$ respectively $\chi_\delta(t) = 0$ for $|t - T| \ge \delta + \varepsilon$, we see that

$$\tilde{U}(t) = \chi_T(t)\tilde{u}(t) + (1 - \chi_T(t))\chi_\delta(t)\tilde{u}_+(t) + (1 - \chi_T(t))\chi_\delta(-t)\tilde{u}_-(t)$$

is an extension of U in $X_{\sigma,b}(\phi)$, which gives $U \in X^{T+\delta}_{\sigma,b}(\phi)$.

Now let $v \in X^{T+\delta}_{s,b}(\phi)$ be another solution of (27). Then, by the choice of T_0, $U(t) = u_1(t) = v(t)$ for $|t| \le T$. Moreover, $\tau_{\mp T}v$ solves (39) on $(-\delta,\delta)$ (use (15) and Lemma 1.12, part ii) again) and thus $\tau_{\mp T}v(t) = u_\pm(t)$ for $|t| < \delta$. This gives $U(t) = v(t)$ for all $|t| < T + \delta$.

Concerning continuous dependence we already know that there are neighbourhoods $U(u_0)$ and $U_\pm(u_1(\pm T))$ in H^σ_x such that

i) for all $v_0,\ v_0' \in U(u_0)$ with corresponding solutions $v,\ v'$ we have
$$\sup_{|t|<T_1} \|v(t) - v'(t)\|_{H^\sigma_x} \le L(u_0, T_1)\|v_0 - v_0'\|_{H^\sigma_x},$$

ii) for all $w_{0,\pm},\ w'_{0,\pm} \in U_\pm(u_1(\pm T))$ with corresponding solutions $w_\pm,\ w'_\pm$ the estimate

$$\sup_{|t|<\delta} \|w_\pm(t) - w'_\pm(t)\|_{H^\sigma_x} \leq L(u_1(\pm T),\delta)\|w_{0,\pm} - w'_{0,\pm}\|_{H^\sigma_x}$$

holds true.

Choosing a smaller neighbourhood $U'(u_0) \subset U(u_0)$ we can achieve by i) that for all $v_0,\ v'_0 \in U'(u_0)$ with solutions $v,\ v'$ we have $v(\pm T) \in U_\pm(u_1(\pm T))$ and $v'(\pm T) \in U_\pm(u_1(\pm T))$. These solutions $v,\ v'$ can be extended in the same way as above on the time interval $(-T-\delta, T+\delta)$. For the extended solutions $V,\ V' \in X^{T+\delta}_{\sigma,b}(\phi)$ we have the estimate

$$\begin{aligned} &\sup_{|t|<T+\delta} \|V(t) - V'(t)\|_{H^\sigma_x} \leq \sup_{|t|<T} \|v(t) - v'(t)\|_{H^\sigma_x} + \sup_{T\leq|t|<T+\delta} \|V(t) - V'(t)\|_{H^\sigma_x} \\ \leq\ & L(u_0,T_1)\|v_0 - v'_0\|_{H^\sigma_x} + \max\left(L(u_1(\pm T),\delta)\|v(\pm T) - v'(\pm T)\|_{H^\sigma_x}\right) \\ \leq\ & L(u_0, T+\delta)\|v_0 - v'_0\|_{H^\sigma_x}, \end{aligned}$$

where $L(u_0, T+\delta) = L(u_0,T_1)(1 + \max(L(u_1(\pm T),\delta)))$. Now we have shown that the properties i) to iii) hold true for $T+\delta > T_0$, which contradicts the choice of T_0. □

1.4 Notes and references

The use of the spaces $X_{s,b}(\phi)$ respectively $X^\delta_{s,b}(\phi)$ (and similar ones, built up from more complicated basic spaces) in order to treat wellposedness problems for nonlinear evolution equations by the contraction mapping principle was introduced by Bourgain in his work on periodic nonlinear Schrödinger and KdV equations, see [B93], and further applied in a series of subsequent articles, see e. g. [B93a], [B93b] and [BC96]. All the basic properties of these spaces, the linear estimates and the proof of the wellposedness theorem are contained - more or less explicitly - in these papers. The idea was picked up, further developed but also simplyfied soon by other authors, let us mention here the works of Kenig, Ponce and Vega on the KdV equation with data in Sobolev spaces with negative index ([KPV93b]) and of Klainerman and Machedon on the nonlinear wave equation with a certain null form as nonlinearity ([KM95]). In 1996 the survey article [G96] appeared, and the present exposition of the method is in fact based on Ginibre's article and the second section of the work of Ginibre, Tsutsumi and Velo on the Zakharov system, see [GTV97].

In detail: In the definition of the spaces $X_{s,b}(\phi)$ as completion we follow Kenig, Ponce and Vega ([KPV93b], for the periodic case see [KPV96a]). In order to achieve uniformity in the treatment of the periodic and nonperiodic case, we use the intersection H of all mixed Sobolev spaces as test functions. The connection between the $X_{s,b}(\phi)$-norms and the unitary group U_ϕ, giving insight especially in the trivial character of the first linear estimate, was made clear in [G96], section 3 (see also the discussion at the beginning of section 3 in [KPV93b]). The duality lemma can be found in a more general context in [T96], Theorem 3.6, in that paper the interpolation property is explicitly mentioned and used to define a more general class of function spaces in the range $0 < |b| < 1$ of the parameter b. The behaviour of the $X_{s,b}(\phi)$-norms under complex conjugation respectively time reversion is not discussed in the literature, allthough its consequences (e. g. for the treatment of equations of second order in time, see below) are well known. Lemma 1.5 can be found - up to $\varepsilon' s$ - in [OTT99], see Lemma 2.1 in that paper. The auxiliary spaces $Y_s(\phi)$ were introduced in [GTV97] in order to treat the case $b' \leq -\frac{1}{2}$. The extension lemma (Lemma 1.6), useful for the persistence of higher regularity (part ii) of the general local existence theorem), seems to be new.

The linear estimates (section 1.2) are more or less taken over from [G96] respectively [GTV97]. Lemma 1.7 is Lemme 3.1 in [G96], Lemma 1.8 is Lemma 2.2 in [GTV97], we only remark here that the definition of the solution operator for $F \in Y_s(\phi)$ contains an extension - otherwise we should have at least $F \in L^1_t(I, H^s_x)$ for some time interval I around zero. For Lemma 1.9 see Lemma 2.1 in [GTV97], the proof is taken from [G96] and goes back to Bourgain [B93]. For Lemma 1.10, ii), cf. Lemma 2.5 in [GTV97].

In section 1.3 we start with the discussion of the meaning of the nonlinearity for irregular distributions, which we define as the extension of the nonlinear operator being Lipschitz continuous on a dense subset. This problem - in general not discussed in the literature - can sometimes be circumvented in the nonperiodic case, if smoothing effects of the unitary group are available, cf. the remarks thereon in [KPV93b]. In the periodic case such smoothing effects are not known, nevertheless there are wellposedness results for data in H^s_x, $s < 0$, as well in the present literature (see e. g. [KPV96a], [KPV96b]) as in our subsequent applications. The proof of the general local existence theorem collects some of the arguments found in the above cited literature and is more or less standard. A major point in this context is that the proof given here does not depend on the phase function or any other special property of a nonlinear equation (such as scaling invariance, cf. [KPV96a], [KPV96b]). This is somewhat in the spirit of Reed's lecture notes [R]. A similar attempt was made by Selberg for the nonlinear wave equation with general nonlinearity, see Theorems 2 and 3 in [Se01]. Some hints, especially on persistence of higher regularity, were taken from that paper. Finally we show a corollary on global wellposedness in the presence of a conserved quantity. The proof adapts a standard argument given in [R] (there Theorem 2 in chapter 1.1) to the $X_{s,b}(\phi)$-framework.

With regard to our applications in part II this exposition is restricted to a single equation of first order in time. It should be mentioned that the method can be generalized to systems of diagonal type in a straightforward way and to equations of second order in the time variable, either by rewriting them as a system of first order equations (see e. g. [GTV97] or [OTT99]) or by replacing τ by $|\tau|$ in expression (7) in order to achieve the invariance of the norm under time reversion, which is necessary in this case (see e. g. [KM95] or [FG96]).

2 Nonlinear estimates: Generalities

In the nonlinear estimates the specific properties of the phase function as well as of the nonlinearity play an important role. Nevertheless, some general arguments and techniques can be formulated, sometimes at hand of examples. This shall be done in this section, where we already focus on the Schrödinger equation.

2.1 Insertion of space-time estimates for free solutions into the framework of the method

In the nonperiodic case there is a rich theory on linear space-time estimates - such as Strichartz estimates, smoothing effect of Kato type and maximal function estimates - for solutions of the Cauchy problem (1) for the homogeneous linear equation. Recently also bilinear refinements of such estimates have appeared. Any multilinear estimate of this type implies a corresponding $X_{s,b}(\phi)$-estimate. This is made precise in the following Lemma, which is the straightforward generalization of Lemma 2.3 in [GTV97] (see also Proposition 3.5 in [KS01]):

Lemma 2.1 *Let - for some σ, σ_1, ..., $\sigma_k \in \mathbf{R}$ -*

$$m : H_x^{\sigma_1} \times ... \times H_x^{\sigma_k} \to H_x^{\sigma}$$

be a continuous k-linear operator and, for $b > \frac{1}{2}$,

$$M : X_{\sigma_1,b}(\phi_1) \times ... \times X_{\sigma_k,b}(\phi_k) \to C_t(\mathbf{R}, H_x^{\sigma})$$

be defined by

$$M(u_1, ..., u_k)(t) = m(u_1(t), ..., u_k(t)).$$

Moreover, assume $Y \subset \mathcal{S}'(\mathbf{R}^{n+1})$ to be a B-space being stable under multiplication with L_t^{∞}, that is

$$\|\psi u\|_Y \leq c\|\psi\|_{L_t^{\infty}} \|\psi u\|_Y \;\; \forall \psi \in L_t^{\infty}, \;\; u \in Y,$$

such that for $f_i \in H_x^{\sigma_i}$, $U_{\phi_i} f_i(x,t) = U_{\phi_i}(t) f_i(x)$ and $s_i \leq \sigma_i$, $1 \leq i \leq k$, the estimate

$$\|M(U_{\phi_1} f_1, ..., U_{\phi_k} f_k)\|_Y \leq c \prod_{i=1}^{k} \|f_i\|_{H_x^{s_i}} \tag{40}$$

holds true. Then for all $(u_1, ..., u_k) \in X_{\sigma_1,b}(\phi_1) \times ... \times X_{\sigma_k,b}(\phi_k)$ we have

$$\|M(u_1, ..., u_k)\|_Y \leq c \prod_{i=1}^{k} \|u_i\|_{X_{s_i,b}(\phi_i)},$$

where the constant depends on b.

Proof: Since $b > \frac{1}{2}$, we have $g_i := \mathcal{F}_t U_{\phi_i}(-\cdot) u_i \in L^1_{\tau}(\mathbf{R}, H_x^{\sigma_i})$ and hence

$$\begin{aligned} u_i(t) &= U_{\phi_i}(t) U_{\phi_i}(-t) u_i(t) \\ &= c U_{\phi_i}(t) \int e^{it\tau} (\mathcal{F}_t U_{\phi_i}(-\cdot) u_i)(\tau) d\tau \\ &= c \int e^{it\tau} U_{\phi_i}(t) g_i(\tau) d\tau. \end{aligned}$$

This gives

$$\begin{aligned} M(u_1,...,u_k)(t) &= m(c\int e^{it\tau}U_{\phi_1}(t)g_1(\tau)d\tau,...,c\int e^{it\tau}U_{\phi_k}(t)g_k(\tau)d\tau) \\ &= c\int d\tau_1..d\tau_k e^{it(\tau_1+..+\tau_k)}m(U_{\phi_1}(t)g_1(\tau_1),..,U_{\phi_k}(t)g_k(\tau_k)), \end{aligned}$$

where we have used the continuity and k-linearity of m as well as $g_i \in L^1_\tau(\mathbf{R}, H^{\sigma_i}_x)$. Now using Minkowski's inequality and the stability assumption on Y we arrive at

$$\begin{aligned} \|M(u_1,...,u_k)\|_Y &\leq c\int d\tau_1..d\tau_k \|M(U_{\phi_1}g_1(\tau_1),...,U_{\phi_k}g_k(\tau_k))\|_Y \\ &\leq c\int d\tau_1..d\tau_k \prod_{i=1}^k \|g_i(\tau_i)\|_{H^{s_i}_x} \end{aligned}$$

by(40). Finally writing $\|g_i(\tau_i)\|_{H^{s_i}_x} = \langle\tau_i\rangle^{-b}(\langle\tau_i\rangle^b\|g_i(\tau_i)\|_{H^{s_i}_x})$ and using Cauchy-Schwarz' inequality completes the proof. □

Remark : Most frequently we will use Lemma 2.1 in the simple case where $k = 1$, $\sigma = \sigma_1 = s_1$ and m is the identity. Then we have

$$\|u_1\|_Y \leq c\|u_1\|_{X_{s_1,b}(\phi_1)}, \tag{41}$$

expressing the boundedness of the embedding $X_{s_1,b}(\phi_1) \subset Y$ (assuming Y to be defined only by the size of its norm, which is always the case in the applications - in fact we will usually have $Y = L^p_t(L^q_x)$ or $Y = L^p_x(L^q_t)$ for some $1 \leq p, q \leq \infty$). If $Y_\theta = (L^2_{xt}, Y)_{[\theta]}$, $\theta \in [0,1]$, we can interpolate between (41) and the trivial case $L^2_{xt} = X_{0,0}(\phi_1)$ to obtain

$$\|u\|_{Y_\theta} \leq c\|u\|_{X_{s,b}(\phi_1)}$$

for $s \geq \theta s_1$, $b > \frac{\theta}{2}$. From this we get by duality

$$\|u\|_{X_{s',b'}(\phi_1)} \leq c\|u\|_{(Y_\theta)'},$$

whenever $s' \leq -\theta s_1$, $b' < -\frac{\theta}{2}$. The latter is of special interest in view on Lemma 1.13 respectively Lemma 1.14, since there $b' \geq -\frac{1}{2}$ is required.

In the sequel we shall give a series of examples concerning the Schrödinger and Airy equation.

2.1.1 Schrödinger estimates

In this section we always have $\phi : \mathbf{R}^n \to \mathbf{R}$, $\xi \mapsto -|\xi|^2$. We start with the linear Strichartz estimates for the free Schrödinger equation:

Lemma 2.2 *Assume that* $0 < \frac{1}{q} < \frac{1}{2}$, $b > \frac{1}{2}(\frac{n}{2} - \frac{n}{q} + 1 - \frac{2}{p})$ *and*

$$\frac{n}{4}(\frac{q-2}{q}) \leq \frac{1}{p} < \begin{cases} \frac{1}{q} + \frac{n}{4}(\frac{q-2}{q}) & : \ n = 1,2 \\ \frac{1}{2} & : \ n \geq 3 \end{cases}$$

Then the estimate

$$\|u\|_{L_t^p(L_x^q)} \le c\|u\|_{X_{0,b}(\phi)}$$

holds true for all $u \in X_{0,b}(\phi)$.

Quotation/Proof: Let p and q be given according to the above assumptions. Define

$$q_0 := 2 + \frac{2}{\frac{n}{2} + \frac{2}{q-2}(1 - \frac{q}{p})}$$

and p_0 by

$$\frac{1}{p_0} := \frac{n}{2}(\frac{1}{2} - \frac{1}{q_0}).$$

An elementary computation shows that $q_0 \in (2, \infty)$ and for $n \ge 3$ that $q_0 < \frac{2n}{n-2}$. In this case the Strichartz estimates

$$\|U_\phi u_0\|_{L_t^{p_0}(L_x^{q_0})} \le c\|u_0\|_{L_x^2} \tag{42}$$

hold true (see [CH], Prop. 7.3.6). Next we define

$$\theta := \frac{q_0}{q}\frac{q-2}{q_0-2} = \frac{n}{2} - \frac{n}{q} + 1 - \frac{2}{p} \in (0, 1]$$

and $b_0 := \frac{b}{\theta} > \frac{1}{2}$. Now Lemma 2.1 gives

$$\|u\|_{L_t^{p_0}(L_x^{q_0})} \le c\|u\|_{X_{0,b_0}(\phi)}.$$

Using $(L_{xt}^2, L_t^{p_0}(L_x^{q_0}))_{[\theta]} = L_t^p(L_x^q)$ (see [BL], Thm. 5.1.2, the interpolation condition is easily checked for θ as above) and Lemma 1.4 we obtain the desired result. □

Remarks : i) By duality we obtain the estimate

$$\|u\|_{X_{0,b'}(\phi)} \le c\|u\|_{L_t^{p'}(L_x^{q'})},$$

whenever $\frac{1}{2} < \frac{1}{q'} < 1$, $b' < \frac{1}{2}(\frac{n}{2} - \frac{n}{q'} + 1 - \frac{2}{p'})$ and

$$1 - \frac{n}{4}(\frac{2-q'}{q'}) \ge \frac{1}{p'} > \begin{cases} \frac{1}{q'} - \frac{n}{4}(\frac{2-q'}{q'}) & : \quad n = 1, 2 \\ \frac{1}{2} & : \quad n \ge 3 \end{cases}.$$

ii) For many applications the special case $p = q$ is sufficient. In this case the estimate (42) goes back to Strichartz ([S77]). Here the assumptions in Lemma 2.2 reduce to

$$\frac{1}{2} > \frac{1}{p} \ge \frac{1}{2} - \frac{1}{n+2}, \qquad b > (\frac{n}{2} + 1)(\frac{1}{2} - \frac{1}{p})$$

respectively to

$$\frac{1}{2} < \frac{1}{p'} \le \frac{1}{2} + \frac{1}{n+2}, \qquad b' < (\frac{n}{2} + 1)(\frac{1}{2} - \frac{1}{p'})$$

for the dualized version.

As a simple application we give the following

Example 2.1 (Nonlinear Schrödinger equation with data in $L^2(\mathbf{R}^n)$)
Consider the Cauchy problem (19), (20) with $s = 0$, $\phi(\xi) = -|\xi|^2$ and the nonlinearity

$$N(u) = |u|^{k_0} u^{k_1} \overline{u}^{k_2},$$

where $0 \le k_0 \in \mathbf{R}$, $k_{1,2} \in \mathbf{N}_0$, $k_0 + k_1 + k_2 = k \in (1, 1 + \frac{4}{n})$. Then for

$$b' \in (-\frac{1}{2}, \min(0, \frac{1}{2} - \frac{n}{4}(k-1))) \qquad b \in (\frac{1}{2}, b'+1)$$

the estimate

$$\|N(u) - N(v)\|_{X_{0,b'}(\phi)} \le c\|u - v\|_{X_{0,b}(\phi)}(\|u\|^{k-1}_{X_{0,b}(\phi)} + \|v\|^{k-1}_{X_{0,b}(\phi)})$$

holds true. Thus Lemma 1.13 and Theorem 1.1 apply, we obtain local wellposedness for $k \in (1, 1 + \frac{4}{n})$.

Proof: The assumption $b' < \frac{1}{2} - \frac{n}{4}(k-1)$ implies $\frac{1}{2} - \frac{2b'}{n+2} > \frac{k}{2} - \frac{k}{n+2}$. Thus

$$I := (\frac{1}{2}, \frac{k}{2}) \cap (\frac{k}{2} - \frac{k}{n+2}, \frac{1}{2} - \frac{2b'}{n+2})$$

is not empty (observe that $k > 1$ and $b' < 0$). Choosing $p' \in \mathbf{R}$ with $\frac{1}{p'} \in I$ we have

$$\frac{1}{2} < \frac{1}{p'} < \frac{1}{2} - \frac{2b'}{n+2} \le \frac{1}{2} + \frac{1}{n+2},$$

the latter, since $b' > -\frac{1}{2}$. Thus $b' < (\frac{n}{2}+1)(\frac{1}{2} - \frac{1}{p'})$, that is, the parameters b' and p' fulfil the assumptions of remark i) below Lemma 2.2 (with $p' = q'$, cf. remark ii)).

From $\frac{k}{2} > \frac{1}{p'} \ge \frac{k}{2} - \frac{k}{n+2}$ we deduce for $p = kp'$ that

$$\frac{1}{2} > \frac{1}{p} \ge \frac{1}{2} - \frac{1}{n+2} > \frac{1}{2} - \frac{2b}{n+2},$$

especially that $b > (\frac{n}{2}+1)(\frac{1}{2} - \frac{1}{p})$. Thus Lemma 2.2 (with $p = q$) applies for our choice of b and p. Now using remark i), the mean value theorem, Hölder's inequality and Lemma 2.2 we obtain the following chain of inequalities:

$$\begin{aligned}
\|N(u) - N(v)\|_{X_{0,b'}(\phi)} &\le c\|N(u) - N(v)\|_{L^{p'}_{xt}} \\
&\le c\|(u-v)(|u|^{k-1} + |v|^{k-1})\|_{L^{p'}_{xt}} \\
&\le c\|u - v\|_{L^p_{xt}}(\|u\|^{k-1}_{L^p_{xt}} + \|v\|^{k-1}_{L^p_{xt}}) \\
&\le c\|u - v\|_{X_{0,b}(\phi)}(\|u\|^{k-1}_{X_{0,b}(\phi)} + \|v\|^{k-1}_{X_{0,b}(\phi)})
\end{aligned}$$

□

Remark: The wellposedness result in this example is well known, see for instance Theorem 1.2 in [CW90], where the wellposedness problem for NLS is also studied for

$s > 0$. Nevertheless it has three interesting aspects: In the first place it covers the whole subcritical region in the L^2_x-case, thus coinciding with the known theory in this case. Secondly, it contains Lemma 3.1 in [BOP98] as well as part ii) of Theorem 2.1 in [St97]. Finally, it gives a hint, for which values of $k = k_1 + k_2$ local wellposedness might hold for the Schrödinger equation with nonlinearity $N(u) = u^{k_1}\overline{u}^{k_2}$ and data in H^s_x even for $s < 0$: These values are $k \in \{2, 3, 4\}$ in one space dimension and $k = 2$ in dimension two or three.

The next Lemma contains - in terms of $X_{s,b}(\phi)$-estimates - the sharp version of Kato's smoothing effect in $n \geq 1$ space dimensions and the onedimensional maximal function estimate due to Kenig, Ponce and Vega:

Lemma 2.3 *Let $b > \frac{1}{2}$. Then for $n = 1$ the estimates*

i) $\|u\|_{L^\infty_x(L^2_t)} \leq c\|u\|_{X_{-\frac{1}{2},b}(\phi)}$ *(Kato smoothing effect),*

ii) $\|u\|_{L^4_x(L^\infty_t)} \leq c\|u\|_{X_{\frac{1}{4},b}(\phi)}$ *(maximal function estimate)*

hold true. For $n \geq 2$ we have

iii) $\sup_{R>0} R^{-\frac{1}{2}}\|u\|_{L^2_t(L^2_x(B_R(0)))} \leq c\|u\|_{X_{-\frac{1}{2},b}(\phi)}$ *(Kato smoothing effect).*

Quotation/Proof: Combining Theorem 4.1 in [KPV91] with Lemma 2.1 we obtain

$$\|I^{\frac{1}{2}}v\|_{L^\infty_x(L^2_t)} \leq c\|v\|_{X_{0,b}(\phi)},$$

where I^s (J^s) is the Riesz (Bessel) potential operator of order $-s$. Using the projections $p = \mathcal{F}^{-1}\chi_{\{|\xi|\leq 1\}}\mathcal{F}$ and $P = Id - p$ we get

$$\|J^{\frac{1}{2}}v\|_{L^\infty_x(L^2_t)} \leq c\|PJ^{\frac{1}{2}}v\|_{L^\infty_x(L^2_t)} + \|pJ^{\frac{1}{2}}v\|_{L^\infty_x(L^2_t)} = I + II$$

with

$$I \leq c\|I^{-\frac{1}{2}}PJ^{\frac{1}{2}}v\|_{X_{0,b}(\phi)} \leq c\|v\|_{X_{0,b}(\phi)}$$

by the preceeding and

$$II \leq c\|pJ^{\frac{1}{2}}v\|_{L^2_t(L^\infty_x)} \leq c\|pJ^{1+\varepsilon}v\|_{L^2_{xt}} \leq c\|v\|_{X_{0,b}(\phi)}$$

by Sobolev embedding in x. For $u = J^{\frac{1}{2}}v$ this gives i). Part ii) follows from Theorem 2.5 in [KPV91] and Lemma 2.1. To see iii), we write for short $\|u\| = \sup_{R>0} R^{-\frac{1}{2}}\|u\|_{L^2_t(L^2_x(B_R(0)))}$. Then Theorem 4.1 in [KPV91] and Lemma 2.1 give

$$\|I^{\frac{1}{2}}v\| \leq c\|v\|_{X_{0,b}(\phi)}$$

respectively

$$\|J^{\frac{1}{2}}v\| \leq \|PJ^{\frac{1}{2}}v\| + \|pJ^{\frac{1}{2}}v\| = I + II$$

with

$$I \leq c\|I^{-\frac{1}{2}}PJ^{\frac{1}{2}}v\|_{X_{0,b}(\phi)} \leq c\|v\|_{X_{0,b}(\phi)}$$

and

$$II \leq c\|pJ^{\frac{1}{2}}v\|_{L^2_t(L^\infty_x)} + \|pJ^{\frac{1}{2}}v\|_{L^2_{xt}} \leq c\|pJ^{1+\varepsilon}v\|_{L^2_{xt}} \leq c\|v\|_{X_{0,b}(\phi)}.$$

Writing $u = J^{\frac{1}{2}}v$ again we obtain iii). □

Remark : Let $u \in X_{s,b}(\phi)$ for some $s \geq -\frac{1}{2}$, $b > \frac{1}{2}$. Then, by i) and iii), we have $u \in L^2_{loc}(\mathbf{R}^{n+1})$ in arbitrary space dimensions. So for quadratic nonlinearities such as u^2, $|u|^2$ or $\overline{u}^2$ the definition of the nonlinearity given at the beginning of section 1.3 coincides with the natural one by Lemma 1.11. This cannot be guaranteed anymore, if $s < -\frac{1}{2}$. The Lipschitz estimate (25) has been shown for the nonlinearities u^2 and $\overline{u}^2$ in one and two space dimensions not only for $s \geq -\frac{1}{2}$, but also for $s > -\frac{3}{4}$, see [KPV96b] and [CDKS01]. This shows that in these cases it is not redundant to define the nonlinearity by the extension process in section 1.3.

Now we turn to the bilinear refinements of Strichartz' inequalities exhibiting stronger smoothing properties than the standard Strichartz' estimates. We start with the case of one space dimension, where we have a gain of half a derivative on the product of two solutions:

Lemma 2.4

$$\|I^{\frac{1}{2}}(e^{it\partial^2}u_1e^{-it\partial^2}u_2)\|_{L^2_{xt}} = \frac{1}{\sqrt{2}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}$$

Proof: We will write for short $\hat{u}$ instead of $\mathcal{F}_xu$ and $\int_* d\xi_1$ for $\int_{\xi_1+\xi_2=\xi} d\xi_1$. By density we may assume $\hat{u}_i \in C^\infty_0(\mathbf{R})$. Then, using Fourier-Plancherel in the space variable we obtain:

$$\begin{aligned} & \|I^{\frac{1}{2}}(e^{it\partial^2}u_1e^{-it\partial^2}u_2)\|^2_{L^2_{xt}} \\ = \quad & \frac{1}{2\pi}\int d\xi dt|\xi|\left|\int_* d\xi_1 e^{-it(\xi_1^2-\xi_2^2)}\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\right|^2 \\ = \quad & \frac{1}{2\pi}\int d\xi dt|\xi|\int_* d\xi_1 d\eta_1 e^{-it(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)}\prod_{i=1}^2\hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\ = \quad & \int d\xi|\xi|\int_* d\xi_1 d\eta_1 \delta(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)\prod_{i=1}^2\hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)}. \end{aligned}$$

For the argument of the δ-function we have

$$\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2 = 2\xi(\xi_1-\eta_1).$$

Using $\delta(a(x-b)) = \frac{1}{|a|}\delta(x-b)$ we obtain

$$\begin{aligned} .. \quad = \quad & \frac{1}{2}\int d\xi d\xi_1 d\eta_1 \delta(\xi_1-\eta_1)\hat{u}_1(\xi_1)\hat{u}_2(\xi-\xi_1)\overline{\hat{u}_1(\eta_1)\hat{u}_2(\xi-\eta_1)} \\ = \quad & \frac{1}{2}\int d\xi d\xi_1|\hat{u}_1(\xi_1)\hat{u}_2(\xi-\xi_1)|^2 = \frac{1}{2}\|u_1\|^2_{L^2_x}\|u_2\|^2_{L^2_x}. \end{aligned}$$

□

Remarks : i) For the use of $\delta(P)$ cf. appendix A 2.
ii) In view on the Sobolev embedding $H_x^{\frac{1}{2}+\varepsilon} \subset L_x^\infty$ this can be seen (almost) as a refinement of the $L_t^4(L_x^\infty)$-Strichartz estimate, which is the admissible endpoint case in one space dimension.

Using Lemma 2.1 we obtain the following estimate, which was shown by Bekiranov, Ogawa and Ponce using the Schwarz method described in section 2.2.2 (see Lemma 3.2 in [BOP98]):

Corollary 2.1 (Bekiranov, Ogawa, Ponce) *Let $n = 1$ and $b > \frac{1}{2}$. Then the estimate*

$$\|u\overline{v}\|_{L_t^2(\dot{H}_x^{\frac{1}{2}})} \leq c\|u\|_{X_{0,b}(\phi)}\|v\|_{X_{0,b}(\phi)}$$

holds for all $u, v \in X_{0,b}(\phi)$.

Next we have Bourgain's bilinear refinements of Strichartz' estimate in two (respectively three) space dimensions, cf. Lemma 111 and Corollary 113 in [B98a] (respectively Lemma 5 and Corollary 6 in [B98b]), for which we give a detailed proof. For that purpose we introduce the following notation: First, for a subset $M \subset \mathbf{R}^n$, we define $P_M := {\mathcal{F}_x}^{-1}\chi_M\mathcal{F}_x$, where χ_M denotes a smooth characteristic function of the set M. Especially we require for $l \in \mathbf{N}_0$:

- $P_l := P_{B_{2^l}}$ for the (closed) ball B_{2^l} of radius 2^l centered at zero ($P_{-1} = 0$),
- $P_{\Delta l} := P_l - P_{l-1}$, $\tilde{P}_{\Delta l} := \sum_{k=-1}^{1} P_{\Delta(l+k)}$, such that $P_{\Delta l} = P_{\Delta l}\tilde{P}_{\Delta l}$, as well as
- $P_{Q_\alpha^l}$, where $\alpha \in \mathbf{Z}^n$ and Q_α^l is a cube of sidelength 2^l centered at $2^l\alpha$, so that

$$\sum_{\alpha \in \mathbf{Z}^n} \chi_{Q_\alpha^l} = 1\ .$$

Lemma 2.5 (Bourgain) *Let $n = 2$. Then for $l \geq m$ the estimate*

$$\|e^{it\Delta}P_{\Delta m}u_1 e^{it\Delta}P_{\Delta l}u_2\|_{L_{xt}^2} \leq c2^{\frac{m-l}{2}}\|u_1\|_{L_x^2}\|u_2\|_{L_x^2}$$

holds.

Proof: By the standard Strichartz' estimate we may assume $m \ll l$. Arguing as in the previous proof we obtain

$$\begin{aligned}
&\|e^{it\Delta}P_{\Delta m}u_1 e^{it\Delta}P_{\Delta l}u_2\|_{L_{xt}^2}^2 \\
= \ & c\int d\xi \int_* d\xi_1 d\eta_1 \delta(\sum_{i=1}^{2}|\xi_i|^2 - |\eta_i|^2)\prod_{i=1}^{2}\hat{u_i}(\xi_i)\overline{\hat{u_i}(\eta_i)}\chi_{\Delta l}(\xi_1)\chi_{\Delta m}(\xi_2)\chi_{\Delta l}(\eta_1)\chi_{\Delta m}(\eta_2) \\
\leq \ & \frac{c}{2}(I_1 + I_2) = cI_1,
\end{aligned}$$

with

$$I_1 = \int d\xi \int_* d\xi_1 |\hat{u_1}(\xi_1)\hat{u_2}(\xi_2)|^2 \int_* d\eta_1 \delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2)\chi_{\Delta l}(\eta_1)\chi_{\Delta m}(\eta_2).$$

(I_2 is obtained from I_1 by exchanging the variables ξ_i and η_i, thus we have $I_1 = I_2$.) Now for the inner integral we get by Lemma A.2

$$\begin{aligned} I(\xi,\xi_1) &:= \int_* d\eta_1 \delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2)\chi_{\Delta l}(\eta_1)\chi_{\Delta m}(\eta_2) \\ &= \int_{P(\eta_1)=0} \frac{dS_{\eta_1}}{|\nabla_{\eta_1} P(\eta_1)|}\chi_{\Delta l}(\eta_1)\chi_{\Delta m}(\xi - \eta_1), \end{aligned}$$

where $P(\eta_1) = |\eta_1|^2 + |\xi - \eta_1|^2 - |\xi_1|^2 - |\xi_2|^2$, hence $|\nabla_{\eta_1} P(\eta_1)| = |4\eta_1 - 2\xi| = 2|\eta_1 - \eta_2| \geq c2^l$. This gives

$$I(\xi,\xi_1) \leq c2^{-l} \int_{P(\eta_1)=0} dS_{\eta_1}\chi_{\Delta m}(\xi - \eta_1) \leq c2^{m-l},$$

since $\int_{P(\eta_1)=0} dS_{\eta_1}\chi_{\Delta m}(\xi-\eta_1)$ is the length of the intersection of $\{P(\eta_1) = 0\}$ with $B_{2^m}(\xi) - B_{2^{m-1}}(\xi)$. Finally we conclude that

$$I_1 \leq c2^{m-l}\|u_1\|^2_{L^2_x}\|u_2\|^2_{L^2_x}.$$

□

Remark: The corresponding estimate in three space dimensions is

$$\|e^{it\Delta}P_{\Delta m}u_1 e^{it\Delta}P_{\Delta l}u_2\|_{L^2_{xt}} \leq c2^{m-\frac{l}{2}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}.$$

This follows from the geometric argument at the end of the above proof. Observe that standard Strichartz in connection with Sobolev's embedding Theorem gives

$$\|e^{it\Delta}u_1 e^{it\Delta}u_2\|_{L^2_{xt}} \leq c\|u_1\|_{H^{\frac{1}{4}}_x}\|u_2\|_{H^{\frac{1}{4}}_x} \leq c2^{\frac{m+l}{4}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x},$$

which coincides for $m \sim l$.

Corollary 2.2 (Bourgain) *Let $n = 2$, $\varepsilon > 0$ and $0 < s < \frac{1}{2} < b$. Then*

i) $\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t(H^s_x)} \leq c\|u_0\|_{H^{s+\varepsilon}_x}\|v_0\|_{L^2_x}$,

ii) $\|uv\|_{L^2_t(H^s_x)} \leq c\|u\|_{X_{s+\varepsilon,b}(\phi)}\|v\|_{X_{0,b}(\phi)}$.

Remarks : i) Using multilinear interpolation (Thm. 4.4.1 in [BL]) we obtain from part ii):

$$\|uv\|_{L^2_t(H^s_x)} \leq c\|u\|_{X_{s_1,b}(\phi)}\|v\|_{X_{s_2,b}(\phi)},$$

provided $\frac{1}{2} > s \geq 0$, $b > \frac{1}{2}$, $s_{1,2} \geq 0$ and $s_1 + s_2 > s$.

ii) For fixed v part ii) of the above Corollary expresses the boundedness of the multiplier

$$M_v : X_{s+\varepsilon,b}(\phi) \to L^2_t(H^s_x) \qquad u \mapsto uv$$

with norm $\leq c\|v\|_{X_{0,b}(\phi)}$. But then the adjoint mapping

$$M_v^* = M_{\overline{v}} : L^2_t(H^{-s}_x) \to X_{-s-\varepsilon,-b}(\phi) \qquad u \mapsto u\overline{v}$$

is also bounded with the same norm, which gives the estimate

$$\|u\overline{v}\|_{X_{s-\varepsilon,-b}(\phi)} \leq c\|v\|_{X_{0,b}(\phi)}\|u\|_{L^2_t(H^s_x)},$$

provided $-\frac{1}{2} < s \leq 0 < \varepsilon$ and $b > \frac{1}{2}$. Here we may replace u by $\overline{u}$ on the left hand side, since $\|u\|_{L^2_t(H^s_x)} = \|\overline{u}\|_{L^2_t(H^s_x)}$.

Proof: Clearly, ii) follows from i) by Lemma 2.1. To see i) we write $u_0 = \sum_{m\geq 0} P_{\Delta m}u_0$ and $v_0 = \sum_{l\geq 0} P_{\Delta l}v_0$. Then

$$\begin{aligned} & \|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t(H^s_x)} \\ \leq & \left(\sum_{m\geq l\geq 0} + \sum_{l\geq m\geq 0}\right)\|e^{it\Delta}P_{\Delta m}u_0 e^{it\Delta}P_{\Delta l}v_0\|_{L^2_t(H^s_x)} =: \sum_1 + \sum_2, \end{aligned}$$

with

$$\begin{aligned} \sum_1 \leq & \sum_{m\geq l\geq 0} 2^{ms}\|e^{it\Delta}P_{\Delta m}u_0 e^{it\Delta}P_{\Delta l}v_0\|_{L^2_{xt}} \\ \leq & c\sum_{m\geq l\geq 0} 2^{ms}\|P_{\Delta m}u_0\|_{L^2_x}\|v_0\|_{L^2_x} \\ \leq & c\sum_{m\geq 0} m2^{-m\varepsilon}\|u_0\|_{H^{s+\varepsilon}_x}\|v_0\|_{L^2_x} \leq c\|u_0\|_{H^{s+\varepsilon}_x}\|v_0\|_{L^2_x}, \end{aligned}$$

where we have used Hölder and (standard) Strichartz. Now using Lemma 2.5 we obtain for the second contribution

$$\begin{aligned} \sum_2 \leq & \sum_{l\geq m\geq 0} 2^{ls}\|e^{it\Delta}P_{\Delta m}u_0 e^{it\Delta}P_{\Delta l}v_0\|_{L^2_{xt}} \\ \leq & c\sum_{l\geq m\geq 0} 2^{ls+\frac{m-l}{2}}\|\tilde{P}_{\Delta m}u_0\|_{L^2_x}\|v_0\|_{L^2_x} \\ \leq & c\sum_{l\geq m\geq 0} 2^{l(s-\frac{1}{2})}2^{m(\frac{1}{2}-s-\varepsilon)}\|u_0\|_{H^{s+\varepsilon}_x}\|v_0\|_{L^2_x} \leq c\sum_{l\geq 0} 2^{-l\varepsilon}\|u_0\|_{H^{s+\varepsilon}_x}\|v_0\|_{L^2_x}. \end{aligned}$$

□

Remark: The corresponding estimates in three space dimensions are

i) $\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t(H^s_x)} \leq c\|u_0\|_{H^{s+\frac{1}{2}+\varepsilon}_x}\|v_0\|_{L^2_x}$,

ii) $\|uv\|_{L^2_t(H^s_x)} \leq c\|u\|_{X_{s+\frac{1}{2}+\varepsilon,b}(\phi)}\|v\|_{X_{0,b}(\phi)}$,

provided $\varepsilon > 0$ and $0 < s < \frac{1}{2} < b$, cf. Corollary 6 in [B98b].

Finally we show how to extend the twodimensional estimate to negative values of s:

Lemma 2.6 *Let $n = 2$. Then for $l \geq m$, the estimate*

$$\|P_{\Delta m}(e^{it\Delta}P_{\Delta l}u_1 e^{it\Delta}u_2)\|_{L^2_{xt}} \leq c2^{\frac{m-l}{2}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}$$

holds.

Proof: Without loss of generality we may assume $\|u_1\|_{L^2_x} = \|u_2\|_{L^2_x} = 1$ and, by standard Strichartz, $m \ll l$. Then

$$\begin{aligned} & \|P_{\Delta m}(e^{it\Delta}P_{\Delta l}u_1 e^{it\Delta}u_2)\|_{L^2_{xt}} \\ \leq & \sum_{\alpha\in\mathbf{Z}^2} \|P_{\Delta m}(e^{it\Delta}P_{Q^m_\alpha}P_{\Delta l}u_1 e^{it\Delta}u_2)\|_{L^2_{xt}} \\ \leq & \sum_{\alpha\in\mathbf{Z}^2}\sum_{|\alpha+\beta|\leq 2} \|P_{\Delta m}(e^{it\Delta}P_{Q^m_\alpha}P_{\Delta l}u_1 e^{it\Delta}P_{Q^m_\beta}u_2)\|_{L^2_{xt}}, \end{aligned}$$

since $|\xi_1 - 2^m\alpha| \leq 2^m$ and $|\xi| \leq 2^m$ imply that $|\xi_2 + 2^m\alpha| \leq |\xi_1 - 2^m\alpha| + |\xi| \leq 2^{m+1}$. Now, for fixed α, β, we estimate the square of the L^2_{xt}-norm:

$$\begin{aligned} & \|P_{\Delta m}(e^{it\Delta}P_{Q^m_\alpha}P_{\Delta l}u_1 e^{it\Delta}P_{Q^m_\beta}u_2)\|^2_{L^2_{xt}} \\ = & \; c\int d\xi\chi_{\Delta_m}(\xi)\int_* d\xi_1 d\eta_1\delta(|\xi_1|^2+|\xi_2|^2-|\eta_1|^2-|\eta_2|^2)\chi_{\Delta_l}(\xi_1)\chi_{\Delta_l}(\eta_1).. \\ \times & \; ..\chi_{Q^m_\alpha}(\xi_1)\chi_{Q^m_\alpha}(\eta_1)\chi_{Q^m_\beta}(\xi_2)\chi_{Q^m_\beta}(\eta_2)\prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \leq \frac{c}{2}(I_1+I_2) = cI_1, \end{aligned}$$

where

$$I_1 = \int d\xi\chi_{\Delta_m}(\xi)\int_* d\xi_1\chi_{Q^m_\alpha}(\xi_1)\chi_{Q^m_\beta}(\xi_2)|\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)|^2 I(\xi,\xi_1)$$

and

$$I(\xi,\xi_1) = \int_* d\eta_1\delta(|\xi_1|^2+|\xi_2|^2-|\eta_1|^2-|\eta_2|^2)\chi_{\Delta_l}(\eta_1)\chi_{Q^m_\alpha}(\eta_1).$$

(As in the previous proof I_2 is obtained from I_1 by exchanging the variables ξ_i and η_i, thus we have $I_1 = I_2$.) For the inner integral $I(\xi,\xi_1)$ we use $\int dx\delta(P(x))f(x) = \int_{P(x)=0}\frac{dS_x}{|\nabla P(x)|}f(x)$ with

$$P(\eta_1) = |\eta_1|^2 + |\xi-\eta_1|^2 - |\xi_1|^2 - |\xi_2|^2, \;\; |\nabla_{\eta_1}P(\eta_1)| = |4\eta_1 - 2\xi| \geq c2^l$$

(because of the factors $\chi_{\Delta_m}(\xi)$, $\chi_{\Delta_l}(\eta_1)$ and $m \ll l$) to get

$$I(\xi,\xi_1) \leq c2^{-l}\int_{P(\eta_1)=0} dS_{\eta_1}\chi_{Q^m_\alpha}(\eta_1) \leq c2^{m-l}.$$

We arrive at

$$I_1 \leq c2^{m-l}\|P_{Q^m_\alpha}u_1\|^2_{L^2_x}\|P_{Q^m_\beta}u_2\|^2_{L^2_x},$$

which gives, inserted into $\sum_{\alpha\in\mathbf{Z}^2}\sum_{|\alpha+\beta|\leq 2}$:

$$\begin{aligned} & \|P_{\Delta m}(e^{it\Delta}P_{\Delta l}u_1 e^{it\Delta}u_2)\|_{L^2_{xt}} \\ \leq & \; c2^{\frac{m-l}{2}}\sum_{\alpha\in\mathbf{Z}^2}\sum_{|\alpha+\beta|\leq 2}\|P_{Q^m_\alpha}u_1\|_{L^2_x}\|P_{Q^m_\beta}u_2\|_{L^2_x} \\ \leq & \; c2^{\frac{m-l}{2}}\sum_{\alpha\in\mathbf{Z}^2}\sum_{|\alpha+\beta|\leq 2}\|P_{Q^m_\alpha}u_1\|^2_{L^2_x} + \|P_{Q^m_\beta}u_2\|^2_{L^2_x} \leq c2^{\frac{m-l}{2}} \end{aligned}$$

□

Corollary 2.3 *Let $n = 2$, $\varepsilon > 0 > s > -\frac{1}{2}$ and $b > \frac{1}{2}$. Then*

i) $\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t(H^{s-\varepsilon}_x)} \leq c\|u_0\|_{H^s_x}\|v_0\|_{L^2_x}$,

ii) $\|uv\|_{L^2_t(H^{s-\varepsilon}_x)} \leq c\|u\|_{X_{s,b}(\phi)}\|v\|_{X_{0,b}(\phi)}$.

Remark : Again we can use multilinear interpolation to obtain

$$\|u_1u_2\|_{L^2_t(H^s_x)} \leq c\|u_1\|_{X_{s_1,b}(\phi)}\|u_2\|_{X_{s_2,b}(\phi)},$$

provided $-\frac{1}{2} < s \leq 0$, $b > \frac{1}{2}$, $s_{1,2} \leq 0$ and $s_1 + s_2 > s$.

Proof: To see i) we write

$$\begin{aligned} & \|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t(H^{s-\varepsilon}_x)} \\ \leq & \sum_{m,l\in\mathbf{N}} 2^{m(s-\varepsilon)}\|P_{\Delta m}(e^{it\Delta}P_{\Delta l}u_0 e^{it\Delta}v_0)\|_{L^2_{xt}} \leq \sum_1 + \sum_2 \end{aligned}$$

with

$$\begin{aligned} \sum_1 &= \sum_{l\in\mathbf{N}_0}\sum_{m\geq l} 2^{m(s-\varepsilon)}\|e^{it\Delta}P_{\Delta l}u_0 e^{it\Delta}v_0\|_{L^2_{xt}} \\ &\leq c\sum_{l\in\mathbf{N}_0} 2^{l(s-\frac{\varepsilon}{2})}\sum_{m\in\mathbf{N}_0} 2^{-\frac{m\varepsilon}{2}}\|P_{\Delta l}u_0\|_{L^2_x}\|v_0\|_{L^2_x} \leq c\|u_0\|_{H^s_x}\|v_0\|_{L^2_x} \end{aligned}$$

where we have used Hölder and (standard) Strichartz. Now Lemma 2.6 is applied to estimate

$$\begin{aligned} \sum_2 &= \sum_{l\in\mathbf{N}_0}\sum_{m\leq l} 2^{m(s-\varepsilon)}\|P_{\Delta m}(e^{it\Delta}P_{\Delta l}u_0 e^{it\Delta}v_0)\|_{L^2_{xt}} \\ &\leq c\sum_{l\in\mathbf{N}_0} 2^{-\frac{l}{2}}\sum_{m\leq l} 2^{m(s+\frac{1}{2}-\varepsilon)}\|\tilde{P}_{\Delta l}u_0\|_{L^2_x}\|v_0\|_{L^2_x} \\ &\leq c\sum_{l\in\mathbf{N}} 2^{l(s-\varepsilon)}\|\tilde{P}_{\Delta l}u_0\|_{L^2_x}\|v_0\|_{L^2_x} \leq c\|u_0\|_{H^s_x}\|v_0\|_{L^2_x}. \end{aligned}$$

This gives i). For $u \in X_{0,b}(\phi)$ part ii) follows from this by Lemma 2.1, for the general case we use an approximation argument as in the proof of Lemma 1.11 (observe that $u \in L^2_{loc}(\mathbf{R}^{n+1})$ by Lemma 2.3). □

2.1.2 Airy estimates

Here we have $\phi : \mathbf{R} \to \mathbf{R}$, $\xi \mapsto \xi^3$. Again we start with the Strichartz type estimates for the Airy equation:

Lemma 2.7 *For $b > \frac{1}{2}$ the following estimates are valid:*

i) $\|u\|_{L^p_t(H^{s,q}_x)} \leq c\|u\|_{X_{0,b}(\phi)}$, *whenever* $0 \leq s = \frac{1}{p} \leq \frac{1}{4}$ *and* $\frac{1}{q} = \frac{1}{2} - \frac{2}{p}$,

ii) $\|u\|_{L^p_t(L^q_x)} \leq c\|u\|_{X_{0,b}(\phi)}$, *whenever* $0 < \frac{1}{q} = \frac{1}{2} - \frac{3}{p} \leq \frac{1}{2}$.

Quotation/Proof: Theorem 2.1 in [KPV91] gives in the case of the Airy-equation

$$\|e^{-t\partial^3}u_0\|_{L^p_t(\dot{H}^{s,q}_x)} \le c\|u_0\|_{L^2_x},$$

provided $0 \le s = \frac{1}{p} \le \frac{1}{4}$ and $\frac{1}{q} = \frac{1}{2} - \frac{2}{p}$. Now Lemma 2.1 is applied to obtain

$$\|u\|_{L^p_t(\dot{H}^{s,q}_x)} \le c\|u\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{2} \tag{43}$$

for the same values of s, p and q. From this ii) follows by Sobolev's embedding theorem (in the space variable). Especially we have

$$\|u\|_{L^8_{xt}} \le c\|u\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{2},$$

which, interpolated with the trivial case, gives

$$\|u\|_{L^4_{xt}} \le c\|u\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{3}.$$

Now let us see how to replace $\dot{H}^{s,q}_x$ by $H^{s,q}_x$ in (43) in the endpoint case, i. e. $s = \frac{1}{p} = \frac{1}{4}$, $q = \infty$: Using the projections $p = \mathcal{F}_x^{-1}\chi_{\{|\xi|\le 1\}}\mathcal{F}_x$ and $P = Id - p$ we have

$$\|u\|_{L^4_t(H^{\frac{1}{4},\infty}_x)} \le \|Pu\|_{L^4_t(H^{\frac{1}{4},\infty}_x)} + \|pu\|_{L^4_t(H^{\frac{1}{4},\infty}_x)} =: I + II.$$

For I we use (43) to obtain

$$I \le c\|I^{-\frac{1}{4}}J^{\frac{1}{4}}Pu\|_{X_{0,b}(\phi)} \le c\|u\|_{X_{0,b}(\phi)},$$

while for II by Sobolev's embedding theorem we get

$$II \le c\|pu\|_{L^4_t(H^{\frac{1}{2}+\varepsilon,4}_x)} \le c\|pu\|_{X_{\frac{1}{2}+\varepsilon,b}(\phi)} \le c\|u\|_{X_{0,b}(\phi)}.$$

This gives i) in the endpoint case, from which the general case follows by interpolation with Sobolev's embedding theorem (in the time variable). □

Remark: The endpoint case in ii) is also valid - see e. g. Lemma 3.29 in [KPV93a] - but we shall not make use of this here.

The $X_{s,b}(\phi)$-versions of Kato's smoothing effect and the maximal function estimate for the Airy-equation are the following:

Lemma 2.8 *Let $b > \frac{1}{2}$. Then the estimates*

i) $\|u\|_{L^\infty_x(L^2_t)} \le c\|u\|_{X_{-1,b}(\phi)}$ *(Kato smoothing effect),*

ii) $\|u\|_{L^4_x(L^\infty_t)} \le c\|u\|_{X_{\frac{1}{4},b}(\phi)}$ *(maximal function estimate).*

hold true.

Quotation/Proof: Combining Theorem 4.1 in [KPV91] with Lemma 2.1 we obtain i) as in the proof of Lemma 2.3. Part ii) follows from Theorem 2.5 in [KPV91] and Lemma 2.1. □

2.2 Multilinear estimates leading to wellposedness results

Here we consider nonlinearities of the type $N(u) = D^\beta(\prod_{i=1}^m D^{\beta_i}u)$. In this case the nonlinear estimates (32) and (35) reduce to

$$\|D^\beta(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{s,b'}(\phi)} \leq c\displaystyle\prod_{i=1}^m \|u_i\|_{X_{s,b}(\phi)} \tag{44}$$

respectively to

$$\|D^\beta(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{s,b'}(\phi)\cap Y_s(\phi)} \leq c\displaystyle\prod_{i=1}^m \|u_i\|_{X_{s,b}(\phi)}, \tag{45}$$

and also (36) reduces to (45) with an additional factor δ^ε on the right hand side. In view on systems and nonlinearities depending on u and $\overline{u}$ the proof of the following more general estimates is of interest:

$$\|D^\beta(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{s,b'}(\phi)} \leq c\displaystyle\prod_{i=1}^m \|u_i\|_{X_{s_i,b_i}(\phi_i)} \tag{46}$$

and

$$\|D^\beta(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{Y_s(\phi)} \leq c\displaystyle\prod_{i=1}^m \|u_i\|_{X_{s_i,b_i}(\phi_i)}. \tag{47}$$

Lemma 2.9 *For $1 \leq i \leq m$ let $u_i \in H \subset X_{s_i,b_i}(\phi_i)$ and*

$$f_i(\xi,\tau) := \langle\tau - \phi_i(\xi)\rangle^{b_i}\langle\xi\rangle^{s_i}\mathcal{F}u_i(\xi,\tau).$$

Then with $d\nu := \mu(d\xi_1..d\xi_{m-1})d\tau_1..d\tau_{m-1}$ und $\xi = \sum_{i=1}^m \xi_i$, $\tau = \sum_{i=1}^m \tau_i$ the following identities are valid:

$$\mathcal{F}D^\beta(\prod_{i=1}^m D^{\beta_i}u_i)(\xi,\tau) = c\xi^\beta\int d\nu\prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i - \phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i)$$

as well as

a) $\|D^\beta(\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{s,b'}(\phi)} =$

$c\|\langle\tau - \phi(\xi)\rangle^{b'}\langle\xi\rangle^s\xi^\beta\int d\nu\prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i - \phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}$

b) $\|D^\beta(\prod_{i=1}^m D^{\beta_i}u_i)\|_{Y_s(\phi)} =$

$c\|\langle\tau - \phi(\xi)\rangle^{-1}\langle\xi\rangle^s\xi^\beta\int d\nu\prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i - \phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)}$

Proof: For the convolution of m functions g_i, $1 \leq i \leq m$, we have with $x = \sum_{i=1}^m x_i$

$$\mathop{\ast}_{i=1}^m g_i(x) = \int\mu(dx_1..dx_{m-1})\prod_{i=1}^m g_i(x_i).$$

Hence by the properties of the Fourier transform the following holds true with $\xi = \sum_{i=1}^m \xi_i,\ \tau = \sum_{i=1}^m \tau_i$:

$$\begin{aligned} & \mathcal{F}D^{\beta}(\prod_{i=1}^m D^{\beta_i}u_i)(\xi,\tau) \\ = \quad & c\xi^{\beta}(\mathop{\ast}_{i=1}^{m} \xi^{\beta_i}\mathcal{F}u_i)(\xi,\tau) \\ = \quad & c\xi^{\beta}(\mathop{\ast}_{i=1}^{m} \xi^{\beta_i}\langle\tau-\phi_i(\xi)\rangle^{-b_i}\langle\xi\rangle^{-s_i}f_i)(\xi,\tau) \\ = \quad & c\xi^{\beta}\int d\nu \prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i-\phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i). \end{aligned}$$

From this we obtain a) because of

$$\|D^{\beta}(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{s,b'}(\phi)} = \|\langle\tau-\phi(\xi)\rangle^{b'}\langle\xi\rangle^{s}\mathcal{F}D^{\beta}(\prod_{i=1}^m D^{\beta_i}u_i)\|_{L^2_{\xi,\tau}}$$

and b) because of

$$\|D^{\beta}(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{Y_s(\phi)} = \|\langle\tau-\phi(\xi)\rangle^{-1}\langle\xi\rangle^{s}\mathcal{F}D^{\beta}(\prod_{i=1}^m D^{\beta_i}u_i)\|_{L^2_{\xi}(L^1_{\tau})}.$$

□

Remark : The previous Lemma has some simple but important consequences: First of all it shows that the estimate (46) holds true, iff

$$\begin{aligned} & \|\langle\tau-\phi(\xi)\rangle^{b'}\langle\xi\rangle^{s}\xi^{\beta}\textstyle\int d\nu\prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i-\phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ & \qquad \leq c\prod_{i=1}^m \|f_i\|_{L^2_{\xi,\tau}}. \end{aligned} \tag{48}$$

In order to prove the latter one may assume without loss of generality that $\xi^{\beta}\prod_{i=1}^m \xi_i^{\beta_i}f_i(\xi_i,\tau_i) \geq 0$. Because of

$$\langle\xi\rangle = \langle\sum_{i=1}^m \xi_i\rangle \leq \sum_{i=1}^m \langle\xi_i\rangle$$

it follows that, if the estimate (44) holds true for some $s \in \mathbf{R}$, then for any $\sigma \geq s$ the estimate

$$\|D^{\beta}(\textstyle\prod_{i=1}^m D^{\beta_i}u_i)\|_{X_{\sigma,b'}(\phi)} \leq c\displaystyle\sum_{j=1}^m \|u_j\|_{X_{\sigma,b_j}(\phi_j)}\prod_{i=1,\,i\neq j}^m \|u_i\|_{X_{s,b_i}(\phi_i)}$$

is also valid, which implies (33) and (34) in this case. Correspondingly, if (45) holds true for some $s \in \mathbf{R}$, then for all $\sigma \geq s$ the above estimate with $X_{\sigma,b'}(\phi)$ replaced by $Y_{\sigma}(\phi)$ is valid, too, implying (37) and (38).

As a simple application of the above arguments we give a short proof of Sobolev's multiplication law (cf. Corollary 3.16 in [T00]), which we have used in section 1:

Lemma 2.10 *Let $s \geq 0$. Assume in addition that*

i) $s \leq s_{1,2}$ and $s < s_1 + s_2 - \frac{n}{2}$ or

ii) $s < s_{1,2}$ and $s \leq s_1 + s_2 - \frac{n}{2}$.

Then $\|fg\|_{H^s_x} \leq c\|f\|_{H^{s_1}_x}\|g\|_{H^{s_2}_x}$ with c depending on s, s_1, s_2 and n.

Proof: Without loss of generality we may assume $\mathcal{F}f$, $\mathcal{F}g \geq 0$. Then, using $\langle\xi\rangle \leq \langle\xi_1\rangle + \langle\xi_2\rangle$, we have

$$\begin{aligned}\|fg\|_{H^s_x} &\leq \|(J^sf)g\|_{L^2_x} + \|fJ^sg\|_{L^2_x} \\ &\leq \|J^sf\|_{L^p_x}\|g\|_{L^{p'}_x} + \|f\|_{L^{q'}_x}\|J^sg\|_{L^q_x},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = \frac{1}{2}$. Now we choose

$$\frac{1}{p'} = \begin{cases} 0 & : \quad s_2 > \frac{n}{2} \\ \frac{s_1 - s}{n} & : \quad s_2 = \frac{n}{2} \\ \frac{1}{2} - \frac{s_2}{n} & : \quad s_2 < \frac{n}{2} \end{cases} ; \qquad \frac{1}{q'} = \begin{cases} 0 & : \quad s_1 > \frac{n}{2} \\ \frac{s_2 - s}{n} & : \quad s_1 = \frac{n}{2} \\ \frac{1}{2} - \frac{s_1}{n} & : \quad s_1 < \frac{n}{2} \end{cases} .$$

Then $H^{s_2}_x \subset L^{p'}_x$ and $H^{s_1}_x \subset L^{q'}_x$ (observe that $s_{1,2} - s > 0$ if $s_{2,1} = \frac{n}{2}$) as well as $H^{s_1}_x \subset H^{s,p}_x$ and $H^{s_2}_x \subset H^{s,q}_x$. □

2.2.1 Bourgain's approach

In order to prove (48) one uses linear (or multilinear) space-time estimates - similar as in example 2.1 - after exploiting the algebraic inequality

$$\langle\tau - \phi(\xi)\rangle + \sum_{i=1}^{m}\langle\tau_i - \phi_i(\xi_i)\rangle \geq |\sum_{i=1}^{m}\phi_i(\xi_i) - \phi(\xi)| =: c.q. \tag{49}$$

coming from the identity

$$\tau - \phi(\xi) - \sum_{i=1}^{m}(\tau_i - \phi_i(\xi_i)) = \sum_{i=1}^{m}\phi_i(\xi_i) - \phi(\xi)$$

(observe the convolution constraint $\sum_{i=1}^{m}\tau_i = \tau$, $\sum_{i=1}^{m}\xi_i = \xi$ in (48)).

Here it comes in that the results, which can be achieved by the method, do not only depend on the degree of the nonlinearity but also on its structure. To illustrate this we consider the Schrödinger equation with the nonlinearities

$$N_1(u,\overline{u}) = u^2, \qquad N_2(u,\overline{u}) = u\overline{u}, \qquad N_3(u,\overline{u}) = \overline{u}^2$$

in one space dimension: For N_1 (respectively N_3) we have $c.q. = 2|\xi_1\xi_2|$ (respectively $c.q. = \xi^2 + \xi_1^2 + \xi_2^2$), giving control over half a derivative on each factor, while for N_2 one only has $c.q. = 2|\xi\xi_1|$, which gives nothing, if ξ_1 is very close to $-\xi_2$. The corresponding results are local wellposedness for data in H^s_x with $s > -\frac{3}{4}$ for $N_{1,3}$ respectively with $s > -\frac{1}{4}$ for N_2 in the nonperiodic case and with $s > -\frac{1}{2}$ for $N_{1,3}$ respectively with $s \geq 0$ for N_2 in the periodic case, see [KPV96b].

As an application of this approach we consider the Schrödinger equation with the nonlinearity $N(u) = \overline{u}^2$ in the continuous case first in three and then in two space dimensions. In this case we have to show that

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{s,b'}(\phi)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}$$

where $\phi(\xi) = -|\xi|^2$. With $v_i = \overline{u}_i$ this can be rewritten as

$$\| \prod_{i=1}^{2} v_i \|_{X_{s,b'}(\phi)} \leq c \prod_{i=1}^{2} \|v_i\|_{X_{s,b}(-\phi)},$$

that is, we have $\phi_1(\xi) = \phi_2(\xi) = |\xi|^2 = -\phi(\xi)$, which gives the rather comfortable inequality

$$\langle \tau - \phi(\xi) \rangle + \sum_{i=1}^{2} \langle \tau_i - \phi_i(\xi_i) \rangle \geq \langle \xi \rangle^2 + \sum_{i=1}^{2} \langle \xi_i \rangle^2 .$$

Our first example is an alternative proof of a recent result due to Tao (see the remark below Proposition 11.3 in [T00]):

Example 2.2 (Tao) *Let $n = 3$ and $\phi : \mathbf{R}^3 \to \mathbf{R}$, $\xi \mapsto -|\xi|^2$ (Schrödinger equation in the nonperiodic case in three space dimensions). Assume that $0 \geq s > -\frac{1}{2}$, $-\frac{1}{2} < b' < \frac{s}{2} - \frac{1}{4}$ and $b > \frac{1}{2}$. Then the estimate*

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{s,b'}(\phi)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}$$

holds true. For $b < b'+1$ Lemma 1.13 and the general local existence Theorem apply and give local wellposedness in $X_{s,b}(\phi)$, $s > -\frac{1}{2}$, for (19), (20) with ϕ as above and $N(u) = \overline{u}^2$.

Proof: Defining $f_i(\xi,\tau) = \langle \tau - |\xi|^2 \rangle^b \langle \xi \rangle^s \mathcal{F}\overline{u}_i(\xi,\tau)$, $1 \leq i \leq 2$, we have according to Lemma 2.9

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{s,b'}(\phi)} = c \| \langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^{b'} \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} \langle \xi_i \rangle^{-s} f_i(\xi_i,\tau_i) \|_{L^2_{\xi,\tau}} .$$

By the introductory remark and since $b' < \frac{s}{2} - \frac{1}{4}$ is assumed, it holds that

$$\langle \xi \rangle^{s+\frac{1}{2}} \prod_{i=1}^{2} \langle \xi_i \rangle^{-s} \leq c (\langle \tau + |\xi|^2 \rangle^{-b'} + \sum_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b'} \chi_{A_i}),$$

where in A_i we have $\langle \tau_i - |\xi_i|^2 \rangle \geq \langle \tau + |\xi|^2 \rangle$. Hence

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{0,b'}(\phi)} \leq c \sum_{j=0}^{2} \|I_j\|_{L^2_{\xi,\tau}},$$

with

$$I_0(\xi,\tau) = \langle \xi \rangle^{-\frac{1}{2}} \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} f_i(\xi_i,\tau_i)$$

and, for $1 \leq j \leq 2$,

$$\begin{aligned} I_j(\xi,\tau) &= \langle \xi \rangle^{-\frac{1}{2}} \langle \tau + |\xi|^2 \rangle^{b'} \int d\nu \langle \tau_j - |\xi_j|^2 \rangle^{-b'} \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} f_i(\xi_i,\tau_i) \chi_{A_j} \\ &\leq \langle \xi \rangle^{-\frac{1}{2}} \langle \tau + |\xi|^2 \rangle^{-b} \int d\nu \langle \tau_j - |\xi_j|^2 \rangle^{b} \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} f_i(\xi_i,\tau_i). \end{aligned}$$

To estimate I_0 we use Lemma 2.9, Sobolev's embedding theorem in the x-variable, Hölder's inequality and the $X_{s,b}(\phi)$-version of the $L_t^4(L_x^3)$-Strichartz-estimate (Lemma 2.2):

$$\begin{aligned} \|I_0\|_{L^2_{\xi,\tau}} &\leq c \| \textstyle\prod_{i=1}^2 J^s \overline{u}_i \|_{L_t^2(H_x^{-\frac{1}{2}})} \\ &\leq c \| \textstyle\prod_{i=1}^2 J^s \overline{u}_i \|_{L_t^2(L_x^{\frac{3}{2}})} \\ &\leq c \prod_{i=1}^{2} \|J^s u_i\|_{L_t^4(L_x^3)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}. \end{aligned}$$

To estimate I_j, $1 \leq j \leq 2$, we also use the dual version of Lemma 2.2:

$$\begin{aligned} \|I_j\|_{L^2_{\xi,\tau}} &\leq c \|J^s \overline{u}_i \mathcal{F}^{-1} f_j\|_{X_{-\frac{1}{2},-b}(\phi)} \\ &\leq c \|J^s \overline{u}_i \mathcal{F}^{-1} f_j\|_{L_t^{\frac{4}{3}}(H_x^{-\frac{1}{2},\frac{3}{2}})} \\ &\leq c \|J^s \overline{u}_i \mathcal{F}^{-1} f_j\|_{L_t^{\frac{4}{3}}(L_x^{\frac{6}{5}})} \\ &\leq c \|\mathcal{F}^{-1} f_j\|_{L^2_{xt}} \|J^s u_i\|_{L_t^4(L_x^3)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}. \end{aligned}$$

□

Arguing as in the previous proof and using the L^4_{xt}-Strichartz estimate valid in two space dimensions leads to the estimate

$$\|\overline{u}_1 \overline{u}_2\|_{X_{0,b'}(\phi)} \leq c \|u_1\|_{X_{s,b}(\phi)} \|u_2\|_{X_{s,b}(\phi)},$$

provided $-\frac{1}{2} < b' < s \leq 0$, $\frac{1}{2} < b$. This is essentially the first part of Theorem 2.1 in [St97]. This has been improved in [CDKS01], see the first part of Theorem 1 in that paper. As a second example we show here, how this improvement can be deduced by using Bourgain's refinement of Strichartz' inequality in two space dimensions (Corollary 2.2) and its extension to $s < 0$ (Corollary 2.3):

Example 2.3 (Colliander, Delort, Kenig, Staffilani) *Let* $n = 2$ *and* $\phi : \mathbf{R}^2 \to \mathbf{R}$, $\xi \mapsto -|\xi|^2$ *(Schrödinger equation in the nonperiodic case in two space dimensions). Assume that* $0 \geq s > -\frac{3}{4}$, $-\frac{1}{2} < b' < s + \frac{1}{4}$, $\sigma < 2(s - b')$, $\sigma \leq 0$, $2b' \leq s$ *and* $b > \frac{1}{2}$. *Then the estimate*

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{\sigma,b'}(\phi)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}$$

holds true. For $b < b' + 1$ *Lemma 1.13 and the general local existence Theorem apply and give local wellposedness in* $X_{s,b}(\phi)$, $s > -\frac{3}{4}$, *for (19), (20) with* ϕ *as above and* $N(u) = \overline{u}^2$.

Proof: Without loss of generality we may assume that $\sigma > -\frac{1}{2}$. Writing $f_i(\xi, \tau) = \langle \tau - |\xi|^2 \rangle^b \langle \xi \rangle^s \mathcal{F}\overline{u}_i(\xi, \tau)$, $1 \leq i \leq 2$ as in the previous proof we have

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{\sigma,b'}(\phi)} = c \| \langle \xi \rangle^\sigma \langle \tau + |\xi|^2 \rangle^{b'} \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}}.$$

By the expressions $\langle \tau + |\xi|^2 \rangle$ and $\langle \tau_i - |\xi_i|^2 \rangle$, $i = 1, 2$, the quantity $\langle \xi \rangle^2 + \langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2$ can be controlled. So we split the domain of integration into $A_0 + A_1 + A_2$, where in A_0 we have $\langle \tau + |\xi|^2 \rangle = \max(\langle \tau + |\xi|^2 \rangle, \langle \tau_1 - |\xi_1|^2 \rangle, \langle \tau_2 - |\xi_2|^2 \rangle)$ and in A_j, $j = 1, 2$, it should hold that $\langle \tau_j - |\xi_j|^2 \rangle = \max(\langle \tau + |\xi|^2 \rangle, \langle \tau_1 - |\xi_1|^2 \rangle, \langle \tau_2 - |\xi_2|^2 \rangle)$. First we consider the region A_0: Here we have $\langle \xi_1 \rangle^{-b'} \langle \xi_2 \rangle^{-b'} \leq c \langle \tau + |\xi|^2 \rangle^{-b'}$, so that for this region we get the upper bound

$$\begin{aligned} & c \| \langle \xi \rangle^\sigma \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} \langle \xi_i \rangle^{b'-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \; & c \| (J^{b'} u_1)(J^{b'} u_2) \|_{L^2_t(H^\sigma_x)} \leq c \| J^{b'} u_1 \|_{X_{\frac{\sigma}{2}+\varepsilon,b}(\phi)} \| J^{b'} u_2 \|_{X_{\frac{\sigma}{2}+\varepsilon,b}(\phi)}, \end{aligned}$$

by Corollary 2.3 and the remark below. Since $\sigma < 2(s - b')$ is assumed, this gives the desired bound.

Now, by symmetry, it is sufficient to consider the region A_1, where

$$\langle \tau + |\xi|^2 \rangle^{b+b'} \langle \xi_1 \rangle^{-2b'+s} \langle \xi_2 \rangle^{-s} \leq c \langle \tau_1 - |\xi_1|^2 \rangle^b$$

holds, giving the upper bound

$$\begin{aligned} & c \| \langle \xi \rangle^\sigma \langle \tau + |\xi|^2 \rangle^{-b} \int d\nu \langle \xi_1 \rangle^{2(b'-s)} f_1(\xi_1, \tau_1) \langle \tau_2 - |\xi_2|^2 \rangle^{-b} f_2(\xi_2, \tau_2) \|_{L^2_{\xi,\tau}} \\ = \; & c \| (J^{2(b'-s)} \mathcal{F}^{-1} f_1)(\overline{J^s u_2}) \|_{X_{\sigma,-b}(\phi)}. \end{aligned}$$

Using the dualized version of Corollary 2.2 this can be estimated by

$$c \| J^{2(b'-s)} \mathcal{F}^{-1} f_1 \|_{L^2_t(H^{\sigma+\varepsilon}_x)} \| J^s u_2 \|_{X_{0,b}(\phi)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)},$$

since $2(b' - s) + \sigma < 0$ by assumption. $\square$

2.2.2 The Schwarz method

This method, developed by Kenig, Ponce and Vega in [KPV96a] and [KPV96b], (in general) also uses the inequality (49) but avoids the use of the Strichartz- or similar estimates, which is replaced by a clever use of the Cauchy Schwarz inequality combined with Fubini's Theorem and elementary subsequent estimates.

We still want to prove the estimate (48), which, by duality, is equivalent to

$$\left|\int \mu(d\xi)d\tau d\nu \langle \tau - \phi(\xi)\rangle^{b'}\langle\xi\rangle^s\xi^\beta f_0(\xi,\tau)\prod_{i=1}^m \xi_i^{\beta_i}\langle\tau_i-\phi_i(\xi_i)\rangle^{-b_i}\langle\xi_i\rangle^{-s_i}f_i(\xi_i,\tau_i)\right|$$
$$\leq c\prod_{i=0}^m \|f_i\|_{L^2_{\xi,\tau}},$$

where again $d\nu = \mu(d\xi_1..d\xi_{m-1})d\tau_1..\tau_{m-1}$, $\xi = \sum_{i=1}^m \xi_i$ and $\tau = \sum_{i=1}^m \tau_i$. For short we write

$$\begin{aligned} d\nu_j &:= \mu(d\xi_1..d\xi_{j-1}d\xi_{j+1}..d\xi_m)d\tau_1..\tau_{j-1}\tau_{j+1}..\tau_m, \\ w(\xi,\xi_1,..,\xi_m) &:= \langle\xi\rangle^s\xi^\beta\prod_{i=1}^m\xi_i^{\beta_i}\langle\xi_i\rangle^{-s_i} \qquad \text{and} \\ W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m) &:= w(\xi,\xi_1,..,\xi_m)\langle\tau-\phi(\xi)\rangle^{b'}\prod_{i=1}^m\langle\tau_i-\phi_i(\xi_i)\rangle^{-b_i}. \end{aligned}$$

Now the use of Cauchy Schwarz and Fubini is summarized in the following

Lemma 2.11 *Assume that*

$$c_0^2 := \sup_{\xi,\tau}\int d\nu|W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)|^2 < \infty \tag{50}$$

or, for some $j \in \{1,..,m\}$,

$$c_j^2 := \sup_{\xi_j,\tau_j}\int d\nu_j|W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)|^2 < \infty. \tag{51}$$

Then

$$\left|\int \mu(d\xi)d\tau d\nu W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)f_0(\xi,\tau)\prod_{i=1}^m f_i(\xi_i,\tau_i)\right| \leq c\prod_{i=0}^m\|f_i\|_{L^2_{\xi,\tau}},$$

where $c = \min_{j=0}^m c_j$.

Proof: Assume (50) first. Then Cauchy Schwarz applied to $\int\mu(d\xi)d\tau$ and to $\int d\nu$ gives

$$\begin{aligned} &\left|\int \mu(d\xi)d\tau d\nu W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)f_0(\xi,\tau)\prod_{i=1}^m f_i(\xi_i,\tau_i)\right| \\ \leq\ & \|f_0\|_{L^2_{\xi,\tau}}\|\textstyle\int d\nu W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)\prod_{i=1}^m f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ \leq\ & \|f_0\|_{L^2_{\xi,\tau}}\|(\textstyle\int d\nu|W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m)|^2)^{\frac12}(\int d\nu\prod_{i=1}^m|f_i(\xi_i,\tau_i)|^2)^{\frac12}\|_{L^2_{\xi,\tau}} \\ \leq\ & c_0\|f_0\|_{L^2_{\xi,\tau}}\|(\textstyle\int d\nu\prod_{i=1}^m|f_i(\xi_i,\tau_i)|^2)^{\frac12}\|_{L^2_{\xi,\tau}}. \end{aligned}$$

By the Fubini Theorem we get

$$\begin{aligned}
& \|(\textstyle\int d\nu \prod_{i=1}^{m} |f_i(\xi_i,\tau_i)|^2)^{\frac{1}{2}}\|^2_{L^2_{\xi,\tau}} \\
= & \int \mu(d\xi)d\tau \int d\nu \prod_{i=1}^{m} |f_i(\xi_i,\tau_i)|^2 \\
= & \int d\nu \prod_{i=1}^{m-1} |f_i(\xi_i,\tau_i)|^2 \int \mu(d\xi)d\tau |f_m(\xi_m,\tau_m)|^2 \\
= & \prod_{i=1}^{m} \|f_i\|^2_{L^2_{\xi,\tau}},
\end{aligned}$$

which gives the first part of the claim. Now assume (51) for some $j \in \{1,..,m\}$. Integrating with respect to (ξ_m, τ_m) instead of (ξ, τ) we obtain similarly as above

$$\begin{aligned}
& \left| \int \mu(d\xi)d\tau d\nu W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m) f_0(\xi,\tau) \prod_{i=1}^{m} f_i(\xi_i,\tau_i) \right| \\
= & \left| \int \mu(d\xi_j)d\tau_j d\nu_j W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m) f_0(\xi,\tau) \prod_{i=1}^{m} f_i(\xi_i,\tau_i) \right| \\
\leq & \|f_j\|_{L^2_{\xi,\tau}} \| \textstyle\int d\nu_j W(\xi,\xi_1,..,\xi_m,\tau,\tau_1,..,\tau_m) f_0(\xi,\tau) \prod_{i\neq j} f_i(\xi_i,\tau_i)\|_{L^2_{\xi_j,\tau_j}} \\
\leq & \|f_j\|_{L^2_{\xi,\tau}} \|(\textstyle\int d\nu_j |W(\xi,..,\tau,..)|^2)^{\frac{1}{2}} (\int d\nu_j |f_0(\xi,\tau)|^2 \prod_{i\neq j} |f_i(\xi_i,\tau_i)|^2)^{\frac{1}{2}}\|_{L^2_{\xi_j,\tau_j}} \\
\leq & c_j \|f_j\|_{L^2_{\xi,\tau}} \|(\textstyle\int d\nu_j |f_0(\xi,\tau)|^2 \prod_{i\neq j} |f_i(\xi_i,\tau_i)|^2)^{\frac{1}{2}}\|_{L^2_{\xi_j,\tau_j}}.
\end{aligned}$$

Using Fubini again, we see that

$$\|(\textstyle\int d\nu_j |f_0(\xi,\tau)|^2 \prod_{i\neq j} |f_i(\xi_i,\tau_i)|^2)^{\frac{1}{2}}\|^2_{L^2_{\xi_j,\tau_j}} = \prod_{i\neq j} \|f_i\|^2_{L^2_{\xi,\tau}},$$

which gives the second part of the claim. □

In order to control the τ_i-integrations in the expressions c_j^2 the following elementary lemma is helpful, which we take over together with its proof from [GTV97] (cf. Lemma 4.2 there):

Lemma 2.12 *For* $0 \leq a_- \leq a_+$ *with* $a_+ + a_- > 1$ *and* $a, b \in \mathbf{R}$ *the inequality*

$$J(a,b) := \int_{\mathbf{R}} d\tau \langle \tau - a \rangle^{-a_+} \langle \tau - b \rangle^{-a_-} \leq c \langle a - b \rangle^{-(a_- - [1-a_+]_+)}$$

is valid, where for $x \in \mathbf{R}$ $[x]_+$ *is defined by*

$$[x]_+ := \begin{cases} x & : \quad x > 0 \\ \varepsilon > 0 & : \quad x = 0 \\ 0 & : \quad x < 0 \end{cases}.$$

Proof: Without loss of generality we may assume $b=0$ and $a>0$. Then

$$\begin{aligned}
J(a,0) &\le 2\int_0^\infty d\tau\langle\tau-a\rangle^{-a_+}\langle\tau\rangle^{-a_-}\\
&\le 2\left(\int_0^{\frac{a}{2}}+\int_{\frac{a}{2}}^{\frac{3a}{2}}+\int_{\frac{3a}{2}}^{\infty}\right)d\tau\langle\tau-a\rangle^{-a_+}\langle\tau\rangle^{-a_-}\\
&\le c\left(\langle a\rangle^{-a_+}\int_0^{\frac{a}{2}}d\tau\langle\tau\rangle^{-a_-}+\langle a\rangle^{-a_-}\int_{-\frac{a}{2}}^{\frac{a}{2}}d\tau\langle\tau\rangle^{-a_+}+\int_{\frac{3a}{2}}^{\infty}d\tau\langle\tau\rangle^{-a_++a_-}\right)\\
&\le c(\langle a\rangle^{-(a_+-[1-a_-]_+)}+\langle a\rangle^{-(a_--[1-a_+]_+)}+\langle a\rangle^{-(a_++a_--1)}).
\end{aligned}$$

Since $a_--[1-a_+]_+\le a_+-[1-a_-]_+\le a_++a_--1$, the claimed inequality follows. □

For quadratic nonlinearities we obtain the following sufficient criterion for the estimate (48):

Lemma 2.13 *Let $m=2$. Assume one of the following conditions a) , b) or c) to be fulfilled:*

a) $b_2\ge b_1>\frac{1}{4}$, $\beta=-(2b_1-[1-2b_2]_+)$ and

$$\sup_{\xi,\tau}\langle\tau-\phi(\xi)\rangle^{2b'}\int\mu(d\xi_1)|w(\xi,\xi_1,\xi-\xi_1)|^2\langle\tau-\phi_1(\xi_1)-\phi_2(\xi-\xi_1)\rangle^{\beta}<\infty$$

b) $b_2\ge -b'>\frac{1}{4}$, $\beta=2b'+[1-2b_2]_+$ (or $-b'\ge b_2>\frac{1}{4}$, $\beta=-2b_2+[1+2b']_+$) and

$$\sup_{\xi_1,\tau_1}\langle\tau_1-\phi_1(\xi_1)\rangle^{-2b_1}\int\mu(d\xi_2)|w(\xi_1+\xi_2,\xi_1,\xi_2)|^2\langle\tau_1-\phi(\xi_1+\xi_2)+\phi_2(\xi_2)\rangle^{\beta}<\infty$$

c) $b_1\ge -b'>\frac{1}{4}$, $\beta=2b'+[1-2b_1]_+$ (or $-b'\ge b_1>\frac{1}{4}$, $\beta=-2b_1+[1+2b']_+$) and

$$\sup_{\xi_2,\tau_2}\langle\tau_2-\phi_2(\xi_2)\rangle^{-2b_2}\int\mu(d\xi_1)|w(\xi_1+\xi_2,\xi_1,\xi_2)|^2\langle\tau_2-\phi(\xi_1+\xi_2)+\phi_1(\xi_1)\rangle^{\beta}<\infty$$

Then the estimate (48) holds true.

Proof: By Lemma 2.12 we have

$$\begin{aligned}
&\int d\tau_1\langle\tau_1-\phi_1(\xi_1)\rangle^{-2b_1}\langle\tau-\tau_1-\phi_2(\xi-\xi_1)\rangle^{-2b_2}\\
\le\ & c\langle\tau-\phi_1(\xi_1)-\phi_2(\xi-\xi_1)\rangle^{\beta}
\end{aligned}$$

for $\beta=-(2b_1-[1-2b_2]_+)$. Thus (50) follows from condition a), and Lemma 2.11 gives (48). Further we have, again by Lemma 2.12,

$$\begin{aligned}
&\int d\tau_2\langle\tau_2-\phi_2(\xi_2)\rangle^{-2b_2}\langle\tau_1+\tau_2-\phi(\xi_1+\xi_2)\rangle^{2b'}\\
\le\ & c\langle\tau_1-\phi(\xi_1+\xi_2)+\phi_2(\xi_2)\rangle^{\beta}
\end{aligned}$$

for $\beta = 2b' + [1 - 2b_2]_+$, if $b_2 \geq -b'$, respectively for $\beta = -2b_2 + [1 + 2b']_+$, if $-b' \geq b_2$, that is, condition b) implies (51) for $j = 1$. The same argument gives that condition c) implies (51) for $j = 2$. Now in both cases by Lemma 2.11 we obtain (48). □

Remark: The Schwarz method can be improved by introducing dyadic decompositions with respect not only to the variables ξ and ξ_i but also to other quantities such as $\tau - \phi(\xi)$, $\tau_i - \phi_i(\xi_i)$ or $\phi(\xi) - \sum_{i=1}^m \phi_i(\xi_i)$ before using Cauchy Schwarz. This is done e. g. in [CDKS01], where the estimate in example 2.3 is shown by the Schwarz method combined with "a standard dyadic decomposition in the spatial frequency variable and a parabolic level set decomposition"[3]. Using yet another decomposition with respect to $\cos\alpha$, where α is the angle between ξ_1 and ξ_2, these authors could also prove the estimate in example 2.3 with $\overline{u_1 u_2}$ replaced by $u_1 u_2$ (under slightly stronger restrictions on σ and s). The same technique is applied there successfully to treat the nonlinearity $N(u) = |u|^2$ in two space dimensions. We also refer to Tao's article [T00], where this approach is studied systematically and where the 3-d problem for the quadratic nonlinearities is solved.

2.3 Some Strichartz type estimates for the Schrödinger equation in the periodic case

In this section we are concerned with some of the Strichartz type estimates for the Schrödinger equation in the periodic case, which were shown by Bourgain in [B93]. All the following estimates are essentially contained in sections 2 and 3 of [B93]. Since we want to use them in the form of an embedding of the type $L_t^p(L_x^q) \subset X_{s,b}(\phi)$, where we have spaces of functions being periodic in the space- but not in the time-variable, we shall give modified proofs for these estimates, combining some of the arguments from [B93] with the Schwarz method described in 2.2.2. Throughout this section we have $\phi : \mathbf{Z}^n \to \mathbf{R}$, $\xi \mapsto -|\xi|^2$.

Lemma 2.14 *(cf. [B93], Prop. 2.6) Let $n = 1$. Then for any $b > \frac{3}{8}$ and for any $b' < -\frac{3}{8}$ the following estimates hold:*

i) $\|u\|_{L^4_{xt}} \leq c\|u\|_{X_{0,b}(\phi)}$

ii) $\|u\|_{X_{0,b'}(\phi)} \leq c\|u\|_{L^{\frac{4}{3}}_{xt}}$

Proof (cf. [KPV96b], Lemma 5.3): Clearly, ii) follows from i) by duality. To see i), we shall show first that

$$\sup_{(\xi,\tau)\in\mathbf{Z}\times\mathbf{R}} S(\xi,\tau) < \infty$$

for

$$\begin{aligned} S(\xi,\tau) &= \sum_{\xi_1\in\mathbf{Z}} \langle \tau + \xi_1^2 + (\xi - \xi_1)^2 \rangle^{1-4b} \\ &\leq c \sum_{\xi_1\in\mathbf{Z}} \langle 4\tau + (2\xi_1)^2 + (2(\xi - \xi_1))^2 \rangle^{1-4b}. \end{aligned}$$

[3]quoted from the introduction of [CDKS01]

With $k = 2\xi_1 - \xi \in \mathbf{Z}$ we have

$$k + \xi = 2\xi_1, \; k - \xi = 2(\xi_1 - \xi) \text{ and } (2\xi_1)^2 + (2(\xi - \xi_1))^2 = 2(\xi^2 + k^2),$$

hence

$$\begin{aligned} S(\xi,\tau) &\leq c\sum_{k\in\mathbf{Z}} \langle 4\tau + 2\xi^2 + 2k^2\rangle^{1-4b} \\ &\leq c\sum_{k\in\mathbf{Z}} \langle k^2 - |2\tau + \xi^2|\rangle^{1-4b} \\ &\leq c\sum_{k\in\mathbf{Z}} \langle (k - x_0)(k + x_0)\rangle^{1-4b}, \end{aligned}$$

where $x_0^2 = |2\tau + \xi^2|$. Now there are at most four numbers $k \in \mathbf{Z}$ with $|k - x_0| < 1$ or $|k + x_0| < 1$. For all the others we have

$$\langle k - x_0\rangle\langle k + x_0\rangle \leq c\langle (k - x_0)(k + x_0)\rangle.$$

Cauchy-Schwarz' inequality gives

$$\begin{aligned} S(\xi,\tau) &\leq c + c\sum_{k\in\mathbf{Z}} (\langle k - x_0\rangle\langle k + x_0\rangle)^{1-4b} \\ &\leq c + c\left(\sum_{k\in\mathbf{Z}} \langle k - x_0\rangle^{2(1-4b)}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbf{Z}} \langle k + x_0\rangle^{2(1-4b)}\right)^{\frac{1}{2}} \leq c \ , \end{aligned}$$

provided $2(1 - 4b) < -1$, that is $b > \frac{3}{8}$. Without loss of generality we can assume $b \in (\frac{3}{8}, \frac{1}{2})$. Using part a) of Lemma 2.13 we arrive at

$$\begin{aligned} &\| \textstyle\sum_{\xi_1\in\mathbf{Z}} \int d\tau_1 \langle \tau_1 + \xi_1^2\rangle^{-b} f(\xi_1, \tau_1)\langle \tau - \tau_1 + (\xi - \xi_1)^2\rangle^{-b} g(\xi - \xi_1, \tau - \tau_1)\|_{L^2_{\xi,\tau}} \\ &\qquad \leq c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}}. \end{aligned}$$

Now by Lemma 2.9 it follows that

$$\|u_1 u_2\|_{L^2_{xt}} \leq c\|u_1\|_{X_{0,b}(\phi)} \|u_2\|_{X_{0,b}(\phi)} \ .$$

Taking $u_1 = u_2 = u$, we get

$$\|u\|^2_{L^4_{xt}} = \|u^2\|_{L^2_{xt}} \leq c\|u\|^2_{X_{0,b}(\phi)} \ .$$

□

Remark: Arguing as in Example 2.1, but using the previous lemma instead of Lemma 2.2, one obtains local (and - by the conservation of the L^2_x-norm - global) wellposedness for

$$iu_t + u_{xx} = |u|^{p-1}u \qquad\qquad u(0) = u_0 \in L^2_x(\mathbf{T}),$$

provided $p \leq 3$. This is the onedimensional L^2_x-result in [B93], cf. Theorem 4.45 there (see also Théorème 5.1 in [G96]).

In the sequel we shall make use of the following number theoretic results concerning the number of solutions of certain Diophantine equations:

Proposition 2.1 *i) For all $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ with*

$$a(r,3) := \#\{(k_1,k_2) \in \mathbf{Z}^2 : 3k_1^2 + k_2^2 = r \in \mathbf{N}\} \leq c\langle r\rangle^{\varepsilon}.$$

ii) For all $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ with

$$a(r,1) := \#\{(k_1,k_2) \in \mathbf{Z}^2 : k_1^2 + k_2^2 = r \in \mathbf{N}\} \leq c\langle r\rangle^{\varepsilon}.$$

iii) Let $n \geq 3$. Then for all $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ with

$$\#\{k \in \mathbf{Z}^n : |k|^2 = r \in \mathbf{N}\} \leq c\langle r\rangle^{\frac{n-2}{2}+\varepsilon}.$$

Quotation/Proof: i) $a(r,3)$ is calculated explicitly in [P], Satz 6.2: It is

$$a(r,3) = 2(-1)^r \sum_{d|r} (\frac{d}{3}) \ .$$

Here $(\frac{d}{p})$ denotes the Legendre-symbol taking values only in $\{0, \pm 1\}$. Thus $a(r,3)$ can be estimated by the number of divisors of r, which is bounded by $c\langle r\rangle^{\varepsilon}$, see [HW], Satz 315. For ii), see Satz 338 in [HW]. iii) follows from ii) by induction, writing $\{k \in \mathbf{Z}^n : |k|^2 = r \in \mathbf{N}\} = \bigcup_{k_n^2 \leq r}\{(k', k_n) : |k'|^2 = r - k_n^2\}$.

The following Lemma corresponds to Prop. 2.36 in [B93]:

Lemma 2.15 *Let $n = 1$. Then for all $s > 0$ and $b > \frac{1}{2}$ there exists a constant $c = c(s,b)$, so that the following estimate holds:*

$$\|u\|_{L^6_{xt}} \leq c\|u\|_{X_{s,b}(\phi)} \ .$$

Proof: As in the proof of the previous lemma, we start by showing that

$$\sup_{(\xi,\tau)\in\mathbf{Z}\times\mathbf{R}} S(\xi,\tau) < \infty \ ,$$

where now (with $\xi_3 = \xi - \xi_1 - \xi_2$)

$$\begin{aligned} S(\xi,\tau) &= \sum_{\xi_1,\xi_2\in\mathbf{Z}} \langle\tau + \xi_1^2 + \xi_2^2 + \xi_3^2\rangle^{-2b}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi_3\rangle^{-2s} \\ &\leq c \sum_{\xi_1,\xi_2\in\mathbf{Z}} \langle 9\tau + (3\xi_1)^2 + (3\xi_2)^2 + (3\xi_3)^2\rangle^{-2b}\langle(3\xi_1)^2 + (3\xi_2)^2 + (3\xi_3)^2\rangle^{-s}. \end{aligned}$$

Taking $k_1 = 3(\xi_1 + \xi_2) - 2\xi$ and $k_2 = 3(\xi_1 - \xi_2)$ as new indices, we have

$$3\xi_1 = \frac{1}{2}(k_1 + k_2) + \xi, \ \ 3\xi_2 = \frac{1}{2}(k_1 - k_2) + \xi \ \text{ and } \ 3\xi_3 = \xi - k_1.$$

From this we get

$$(3\xi_1)^2 + (3\xi_2)^2 + (3\xi_3)^2 = \frac{1}{2}(3k_1^2 + k_2^2) + 3\xi^2.$$

It follows

$$\begin{aligned} S(\xi,\tau) &\le c\sum_{k_1,k_2\in\mathbf{Z}}\langle 9\tau+3\xi^2+\frac{1}{2}(3k_1^2+k_2^2)\rangle^{-2b}\langle\frac{1}{2}(3k_1^2+k_2^2)\rangle^{-s} \\ &\le c\sum_{r\in\mathbf{N}_0}\sum_{3k_1^2+k_2^2=r}\langle 9\tau+3\xi^2+\frac{r}{2}\rangle^{-2b}\langle\frac{r}{2}\rangle^{-s} \\ &\le c\sum_{r\in\mathbf{N}_0}\langle 9\tau+3\xi^2+\frac{r}{2}\rangle^{-2b}, \end{aligned}$$

where in the last step we have used part i) of the above proposition. Since we have demanded $b>\frac{1}{2}$, the introducing claim follows. Now we use Lemma 2.12 to obtain

$$\sup_{(\xi,\tau)\in\mathbf{Z}\times\mathbf{R}}\int d\nu\prod_{i=1}^{3}\langle\tau_i+\xi_i^2\rangle^{-2b}\langle\xi_i\rangle^{-2s}<\infty$$

with $\int d\nu=\int d\tau_1 d\tau_2\sum_{\xi_1,\xi_2\in\mathbf{Z}}$ and $(\tau,\xi)=\sum_{i=1}^{3}(\tau_i,\xi_i)$. Lemma 2.11 gives

$$\|\int d\nu\prod_{i=1}^{3}\langle\tau_i+\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}\le c\prod_{i=1}^{3}\|f_i\|_{L^2_{\xi,\tau}}\ ,$$

implying

$$\|\prod_{i=1}^{3}u_i\|_{L^2_{xt}}\le c\prod_{i=1}^{3}\|u_i\|_{X_{s,b}(\phi)}$$

by Lemma 2.9. Because of $\|u\|^3_{L^6_{xt}}=\|u^3\|_{L^2_{xt}}$ the proof is complete. □

Corollary 2.4 *Let* $n=1$*:*

a) *For all Hölder- and Sobolevexponents* $p,\ q,\ s$ *and* b *satisfying*

$$0\le\frac{1}{p}\le\frac{1}{6},\ \ 0<\frac{1}{q}\le\frac{1}{2}-\frac{2}{p},\ \ b>\frac{1}{2},\ \ s>\frac{1}{2}-\frac{2}{p}-\frac{1}{q}$$

the estimate

$$\|u\|_{L_t^p(L_x^q)}\le c\|u\|_{X_{s,b}(\phi)} \tag{52}$$

holds true.

b) *For all* $p,\ q,\ s$ *and* b *satisfying*

$$0\le\frac{1}{p}\le\frac{1}{q}\le\frac{1}{2}\le\frac{2}{p}+\frac{1}{q}\le\frac{3}{2},\ \ s>0\ \ \text{and}\ \ b>\frac{3}{4}-\frac{1}{p}-\frac{1}{2q}$$

the estimate (52) is valid.

c) *For all* $p,\ q,\ s$ *satisfying*

$$0<\frac{1}{p}\le\frac{1}{6},\ \ 0<\frac{1}{q}\le\frac{1}{2}-\frac{2}{p},\ \ s>\frac{1}{2}-\frac{2}{p}-\frac{1}{q}$$

there exists $b<\frac{1}{2}$ *so that (52) holds true.*

Proof: i) By the Sobolev embedding theorem in the time variable we have $X_{0,b}(\phi) \subset L^\infty_t(L^2_x)$ for all $b > \frac{1}{2}$. Interpolating this with the above lemma, we obtain (52) whenever $0 \le \frac{1}{p} \le \frac{1}{6}$, $s > 0$ and $\frac{1}{2} = \frac{2}{p} + \frac{1}{q}$.

ii) Combining this with Sobolev embedding in the space variable, part a) follows. To see part b), one has to interpolate between the result in i) and the trivial case $X_{0,0}(\phi) = L^2_{xt}$.

iii) Now for p, q, and s according to the assumptions of part c), there exists $\theta \in [0, 1)$ satisfying

$$\theta \ge 1 - \frac{2}{p} \;\; \theta > 1 - \frac{2}{q} \text{ and } s > \frac{3}{2} - \theta - \frac{2}{p} - \frac{1}{q} \, .$$

Define $s_1 = \frac{s}{\theta}$, $b_1 = \frac{1}{4} + \frac{1}{4\theta}$ and p_1, q_1 by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{q_1}$. A simple computation shows, that p_1, q_1, s_1 and b_1 are chosen according to the assumptions of part a). Now part c) with $b = \theta b_1 = \frac{\theta+1}{4} < \frac{1}{2}$ follows by interpolation between this and the trivial case. □

Next we prove the higherdimensional L^4-estimates (cf. [B93], Prop. 3.6).

Lemma 2.16 *Let $n \ge 2$. Then for all $s > \frac{n}{2} - \frac{n+2}{4}$ and $b > \frac{1}{2}$ there exists a constant $c = c(s, b)$, so that the following estimate holds:*

$$\|u\|_{L^4_{xt}} \le c\|u\|_{X_{s,b}(\phi)} \, .$$

Proof: We start by showing that

$$\sup_{(\xi,\tau)\in \mathbf{Z}^n\times\mathbf{R}} S(\xi,\tau) \le cN^{4s}$$

for

$$\begin{aligned} S(\xi,\tau) &= \sum_{\xi_1\in\mathbf{Z}^n} \chi_N(\xi_1)\chi_N(\xi-\xi_1)\langle\tau + |\xi_1|^2 + |\xi-\xi_1|^2\rangle^{-2b} \\ &\le c\sum_{\xi_1\in\mathbf{Z}^n} \chi_{2N}(2\xi_1)\chi_{2N}(2(\xi-\xi_1))\langle 4\tau + |2\xi_1|^2 + |2(\xi-\xi_1)|^2\rangle^{-2b}. \end{aligned}$$

Here χ_N denotes the characteristic function of the ball with radius N centered at zero. With $k = 2\xi_1 - \xi \in \mathbf{Z}^n$ we have

$$k + \xi = 2\xi_1, \;\; k - \xi = 2(\xi_1 - \xi) \text{ and } |2\xi_1|^2 + |2(\xi-\xi_1)|^2 = 2(|\xi|^2 + |k|^2).$$

Thus we can estimate

$$\begin{aligned} S(\xi,\tau) &\le c\sum_{k\in\mathbf{Z}^n} \chi_{2N}(k+\xi)\chi_{2N}(k-\xi)\langle 4\tau + 2(|\xi|^2+|k|^2)\rangle^{-2b} \\ &\le c\sum_{k\in\mathbf{Z}^n} \chi_{2N}(k)\langle 2\tau + |\xi|^2 + |k|^2\rangle^{-2b} \\ &= c\sum_{r\in\mathbf{N}_0}\sum_{k\in\mathbf{Z}^n,|k|^2=r} \chi_{4N^2}(r)\langle 2\tau + |\xi|^2 + r\rangle^{-2b} \\ &\le cN^{n-2+2\varepsilon}\sum_{r\in\mathbf{N}_0}\langle 2\tau + |\xi|^2 + r\rangle^{-2b} \le cN^{4s} \, , \end{aligned}$$

where in the last but one inequality we have used Proposition 2.1. Thus the stated bound on $S(\xi,\tau)$ is proved. Now using part a) of Lemma 2.13 again we arrive at

$$\| \int d\nu \prod_{i=1}^2 \langle \tau_i + |\xi_i|^2 \rangle^{-b} f_i(\xi_i,\tau_i) \|_{L^2_{\xi\tau}} \le c N^{2s} \prod_{i=1}^2 \|f_i\|_{L^2_{\xi\tau}}$$

for all $f_i \in L^2_{\xi\tau}$ which are supported in $\{(\xi,\tau) : |\xi| \le N\}$. Now Lemma 2.9 gives for all $u_i \in X_{0,b}(\phi)$, $i = 1, 2$, having a Fourier transform supported in $\{(\xi,\tau) : |\xi| \le N\}$:

$$\|u_1 u_2\|_{L^2_{xt}} \le c N^{2s} \prod_{i=1}^2 \|u_i\|_{X_{0,b}(\phi)} .$$

Taking $u = u_1 = u_2$ we get

$$\|u\|_{L^4_{xt}} \le c N^s \|u\|_{X_{0,b}(\phi)} \tag{53}$$

provided the above support condition ist fulfilled.

Now let $(\phi_j)_{j\in\mathbf{N}_0}$ be a smooth partition of the unity according to the assumptions of the Littlewood-Paley-Theorem[4], such that $\|f\|_{L^4_x(\mathbf{T}^n)} \sim \|(\sum_{j\in\mathbf{N}_0} |\phi_j * f|^2)^{\frac{1}{2}}\|_{L^4_x(\mathbf{T}^n)}$. Combining this with the estimate (53) we get

$$\begin{aligned} \|u\|^2_{L^4_{xt}} &\le c \quad \| \textstyle\sum_{j\in\mathbf{N}_0} |\phi_j * u|^2 \|_{L^2_{xt}} \\ &\le c \sum_{j\in\mathbf{N}_0} \|\phi_j * u\|^2_{L^4_{xt}} \\ &\le c \sum_{j\in\mathbf{N}_0} 2^{2sj} \|\phi_j * u\|^2_{X_{0,b}(\phi)} \le c\|u\|^2_{X_{s,b}(\phi)}. \end{aligned}$$

□

Corollary 2.5 *Let $n \ge 2$:*

a) *For all Hölder- and Sobolevexponents $p,\ q,\ s$ and b satisfying*

$$0 \le \frac{1}{p} \le \frac{1}{4},\ 0 < \frac{1}{q} \le \frac{1}{2} - \frac{1}{p},\ b > \frac{1}{2},\ s > \frac{n}{2} - \frac{2}{p} - \frac{n}{q}$$

the estimate

$$\|u\|_{L^p_t(L^q_x)} \le c\|u\|_{X_{s,b}(\phi)} \tag{54}$$

holds true.

b) *For all $p,\ q,\ s$ and b satisfying*

$$0 \le \frac{1}{p} \le \frac{1}{q} \le \frac{1}{2} \le \frac{1}{p} + \frac{1}{q} \le 1,\ s > (n-2)(\frac{1}{2} - \frac{1}{q}) \text{ and } b > 1 - \frac{1}{p} - \frac{1}{q}$$

the estimate (54) is valid.

c) *For all $p,\ q,\ s$ satisfying*

$$0 < \frac{1}{p} \le \frac{1}{4},\ 0 < \frac{1}{q} \le \frac{1}{2} - \frac{1}{p},\ s > \frac{n}{2} - \frac{2}{p} - \frac{n}{q}$$

there exists $b < \frac{1}{2}$ so that (54) holds true.

[4]see, e. g., Theorem 3.4.4 in [ST]

The proof follows the same lines as that of Corollary 2.4 and therefore will be omitted.

Remark : Because of $\|f\|_{L^p_t(L^q_x)} = \|\overline{f}\|_{L^p_t(L^q_x)}$ and $\|f\|_{X_{s,b}(-\phi)} = \|\overline{f}\|_{X_{s,b}(\phi)}$ all the results derived in this section so far hold for $X_{s,b}(-\phi)$ instead of $X_{s,b}(\phi)$. Moreover, by Lemma 2.2 they are also valid for the corresponding spaces of nonperiodic functions.

Lemma 2.17 *Assume that for some $1 < p, q < \infty$, $s \geq 0$ and $b \in \mathbf{R}$ the estimate $\|u\|_{L^p_t(L^q_x)} \leq c\|u\|_{X_{s,b}(\phi)}$ is valid. Let B be a ball (or cube) of radius (sidelength) R centered at $\xi_0 \in \mathbf{Z}^n$. Define the projection $P_B u = {\mathcal{F}_x}^{-1}\chi_B \mathcal{F}_x$, where χ_B denotes the characteristic function of B. Then also the estimate*

$$\|P_B u\|_{L^p_t(L^q_x)} \leq cR^s \|u\|_{X_{0,b}(\phi)}$$

holds true.

(cf. [B93], p.143, (5.6) - (5.8))

Proof: If $\xi_0 = 0$, this is obvious. For $\xi_0 \neq 0$ define

$$T_{\xi_0} u(x,t) := \exp\left(-ix\xi_0 - it|\xi_0|^2\right) u(x + 2t\xi_0, t) \ .$$

Then $T_{\xi_0} : L^p_t(L^q_x) \to L^p_t(L^q_x)$ is isometric. For the Fourier transform of $T_{\xi_0} u$ the identity

$$\mathcal{F}T_{\xi_0} u(\xi,\tau) = \mathcal{F}u(\xi + \xi_0, \tau - 2\xi\xi_0 - |\xi_0|^2)$$

is easily checked. Now let B_0 be a ball (or cube) of the same size as B centered at zero. Then we have

$$\begin{aligned}
\mathcal{F}T_{\xi_0} P_B u(\xi,\tau) &= \mathcal{F}P_B u(\xi + \xi_0, \tau - 2\xi\xi_0 - |\xi_0|^2) \\
&= \chi_B(\xi + \xi_0)\mathcal{F}u(\xi + \xi_0, \tau - 2\xi\xi_0 - |\xi_0|^2) \\
&= \chi_{B_0}(\xi)\mathcal{F}T_{\xi_0} u(\xi,\tau) = \mathcal{F}P_{B_0} T_{\xi_0} u(\xi,\tau) \ .
\end{aligned}$$

That is $T_{\xi_0} P_B u = P_{B_0} T_{\xi_0} u$. Moreover, because of

$$\begin{aligned}
\|T_{\xi_0} u\|^2_{X_{0,b}(\phi)} &= \int \mu(d\xi) d\tau \langle \tau + |\xi|^2 \rangle^{2b} |\mathcal{F}u(\xi + \xi_0, \tau - 2\xi\xi_0 - |\xi_0|^2)|^2 \\
&= \int \mu(d\xi) d\tau \langle \tau + |\xi + \xi_0|^2 \rangle^{2b} |\mathcal{F}u(\xi + \xi_0, \tau)|^2 = \|u\|^2_{X_{0,b}(\phi)}
\end{aligned}$$

$T_{\xi_0} : X_{0,b}(\phi) \to X_{0,b}(\phi)$ is also isometric. Now we can conclude

$$\begin{aligned}
\|P_B u\|_{L^p_t(L^q_x)} &= \|T_{\xi_0} P_B u\|_{L^p_t(L^q_x)} \\
&= \|P_{B_0} T_{\xi_0} u\|_{L^p_t(L^q_x)} \\
&\leq cR^s \|T_{\xi_0} u\|_{X_{0,b}(\phi)} = cR^s \|u\|_{X_{0,b}(\phi)}
\end{aligned}$$

□

Remark : If B is a ball centered at ξ_0 and $-B$ is the ball of the same size centered at $-\xi_0$, then a short computation using $\mathcal{F}_x \overline{u}(\xi) = \overline{\mathcal{F}_x u}(-\xi)$ shows that $P_B \overline{u} = \overline{P_{-B} u}$. From this and $\|u\|_{X_{s,b}(-\phi)} = \|\overline{u}\|_{X_{s,b}(\phi)}$ it follows, that Lemma 2.17 remains valid with $X_{s,b}(\phi)$ replaced by $X_{s,b}(-\phi)$. Moreover, as the proof shows, the Lemma is also true in the nonperiodic case.

Part II
Applications: New wellposedness results

In this part we state and prove the wellposedness results, which we obtained by the method described so far. The presentation of these results is divided into three sections:

First we consider a certain class of derivative nonlinear Schrödinger equations, where the nonlinearity depends only on the conjugate wave $\overline{u}$. Due to a rather comfortable algebraic inequality in this case we can prove a very general result being valid in arbitrary space dimensions and for all integer exponents larger than one. Moreover, it covers both the nonperiodic and the periodic case. Here we will rely heavily on the Strichartz type estimates for the Schrödinger equation in the periodic case, and - in order to gain a whole derivative - we will use that variant of the method, where the contracting factor has to come from the nonlinear estimates.

Next we are concerned with nonlinear Schrödinger equations with rough data, that is, they belong to some Sobolev space larger than L^2. This problem has already been studied in part by other authors, who considered the quadratic nonlinearities in one space dimension ([KPV96b]) and in the nonperiodic case in two and three space dimensions ([St97], [CDKS01] respectively [T00]). Here we investigate the cubic and quartic nonlinearities in one space dimension and the quadratic nonlinearities in the periodic case in space dimension two and three.

In the periodic case positive results below L^2 can be achieved only, if some fractional derivatives can be completely controlled by an algebraic inequality. With the only exception of the nonlinearity $N(u) = u^2$ in one space dimension (considered in [KPV96b]) this is the case exactly if the nonlinearity does not depend on u itself. This is worked out here for the nonlinearities $N(u) = \overline{u}^3$ and $N(u) = \overline{u}^4$ in one space dimension (with an optimal result), for the nonlinearity $N(u) = \overline{u}^2$ on $\mathbf{T}^2$ (with an optimal result, thus answering a question raised in [St97][5] affirmatively) and for the latter nonlinearity on $\mathbf{T}^3$ (with a probably improvable result). The use of the Strichartz type inequalities is essential in the derivation of these results.

In the nonperiodic case, due to smoothing, the theory is much richer. For the quadratic nonlinearities we refer here to the above cited literature (cf. also Example 2.3), for the cubic and quartic nonlinearities on the line see Theorems 4.2 and 4.3 below. In the proofs of these theorems certain bi- and trilinear refinements of the onedimensional Strichartz' estimates exhibiting stronger smoothing properties than the linear ones are essential. I believe these estimates are of interest independent of their application here. One of the bilinear refinements is the sharp estimate in Lemma 2.4, leading to Corollary 2.1 due to Bekiranov, Ogawa and Ponce. In order to state and prove the perfect analogue to this estimate in the case of two unbared factors (Lemma 4.2), we introduce the bilinear operator I^s_-, see Definition 4.1.

[5]on top of p. 81

In close analogy to Bourgain's bilinear refinement of Strichartz' inequality in two space dimensions we also have certain trilinear refinements of the onedimensional L^6-Strichartz-estimate. Unfortunately one of these estimates (Lemma 4.3) could not be shown in the whole range of the parameter s, where it was expected, see the problem posed in section 4.2. This leads to the unsatisfactory situation that we cannot say whether or not our results concerning the cubic nonlinearities on the line are optimal, allthough we can go beyond the result being obtained for $N(u) = \overline{u}^3$ by the use of the standard Strichartz' estimate in all three cases in question. Things look better for the quartic nonlinearities, here we can give a complete answer to the problem and in fact for four of the five candidates we can reach all values of s strictly larger than the scaling exponent.

In the last section we use similar arguments to prove local wellposedness of the Cauchy problem for the generalized KdV-equation of order 3 for $s > -\frac{1}{6}$, which is the scaling exponent here. For real valued data the L^2-norm is a conserved quantity, which gives global wellposedness in this case for $s \geq 0$. A central role in the proof of the corresponding nonlinear estimate is played by a bilinear estimate for solutions of the Airy equation involving the operator I_-^s again.

The contents of these three sections were published as preprint, see [Gr00], [Gr01a], [Gr01b].

3 On the Cauchy- and periodic boundary value problem for a certain class of derivative nonlinear Schrödinger equations

In this section we prove local wellposedness of the initial value and periodic boundary value problem for the following class of derivative nonlinear Schrödinger equations

$$u_t - i\Delta u = (\nabla \overline{u})^\beta, \qquad u(0) = u_0 \in H^{s+1}_x.$$

Here the initial value u_0 belongs to the Sobolev space $H^{s+1}_x = H^{s+1}_x(\mathbf{R}^n)$ or $H^{s+1}_x = H^{s+1}_x(\mathbf{T}^n)$, $\beta \in \mathbf{N}_0^n$ is a multiindex of length $|\beta| = m \geq 2$ and we can admit all values of s satisfying

$$s > s_c := \frac{n}{2} - \frac{1}{m-1}, \qquad s \geq 0.$$

The same arguments give local wellposedness for the problem

$$u_t - i\Delta u = \partial_j(\overline{u}^m), \qquad u(0) = u_0 \in H^s_x$$

with the same restrictions on s as above. In the special case of a quadratic nonlinearity in one space dimension (i. e. $m = 2,\ n = 1$) we can reach the value $s = 0$. Employing the conservation of $\|u(t)\|_{L^2_x}$ in this case, we obtain global wellposedness for

$$u_t - i\partial_x^2 u = \partial_x(\overline{u}^2), \qquad u(0) = u_0 \in H^s_x.$$

Throughout this section we will have $\phi : \mathbf{R}^n \to \mathbf{R}$ **or** $\phi : \mathbf{Z}^n \to \mathbf{R}$, $\xi \mapsto -|\xi|^2$.

3.1 The quadratic nonlinearities in one space dimension

Our local wellposedness result here is the following:

Theorem 3.1 *Let $n = 1$ and $s \geq 0$. Then there exists $\delta = \delta(\|u_0\|_{L^2_x}) > 0$, so that there is a solution $u \in X^{\delta}_{s,\frac{1}{2}}(\phi)$ of the initial value (periodic boundary value) problem*

$$u_t - i\partial_x^2 u = \partial_x(\overline{u}^2), \qquad u(0) = u_0 \in H^s_x. \tag{1}$$

This solution is unique in $X^{\delta}_{0,\frac{1}{2}}(\phi)$ and satisfies $u \in C_t((-\delta,\delta), H^s_x)$. Moreover, for any $0 < \delta_0 < \delta$ the mapping data upon solution is locally Lipschitz continuous from H^s_x to $X^{\delta_0}_{s,\frac{1}{2}}(\phi) \cap C_t((-\delta_0,\delta_0), H^s_x)$.

In the same sense the Cauchy and periodic boundary value problem

$$u_t - i\partial_x^2 u = (\partial_x \overline{u})^2, \qquad u(0) = u_0 \in H^{s+1}_x \tag{2}$$

is locally well posed, the solution here belongs to $X^{\delta}_{s+1,\frac{1}{2}}(\phi) \cap C_t((-\delta,\delta), H^{s+1}_x)$ and is unique in $X^{\delta}_{1,\frac{1}{2}}(\phi)$.

Remarks : i) The Cauchy problem in (2) was considered by S. Cohn in [C92]. He obtained local wellposedness for data in H^s_x provided $s \geq 4$ (see Theorem 1 in [C92]).
ii) For the local solutions of (1) guaranteed by Theorem 3.1 the L^2_x-norm is a conserved quantity. To see this assume $u_0 \in H^1_x$ first. Then the corresponding solution u belongs to $C_t((-\delta,\delta), H^1_x)$, which gives $N(u) = \partial_x(\overline{u}^2) \in C_t((-\delta,\delta), L^2_x)$. We can use Proposition 6.1.1 in [CH] to see that

$$\frac{d}{dt}\|u(t)\|^2_{L^2_x} = 2\mathrm{Re}\int \partial_x(\overline{u}^2(t))\overline{u}(t) = \frac{2}{3}\mathrm{Re}\int \partial_x(\overline{u}^3(t)) = 0.$$

Now, since we can rely on continuous dependence, the general case follows by approximation. This gives the following

Corollary 3.1 *The Cauchy- and the periodic boundary value problem (1) is globally well posed for $s \geq 0$ in the sense of Corollary 1.4.*

By the general local existence Theorem, Lemma 1.14, Remark 1.2 and the remark below Lemma 2.9 the proof of Theorem 3.1 reduces to the following estimates:

Theorem 3.2 *Let $n = 1$ and $\theta \in (0, \frac{1}{4})$. Then for all $u_{1,2} \in X_{0,\frac{1}{2}}(\phi)$ supported in $\{(x,t) : |t| \leq \delta\}$ the following estimates are valid:*

i) $\|\overline{u}_1\overline{u}_2\|_{X_{1,-\frac{1}{2}}(\phi)} \leq c\delta^{\theta}\|u_1\|_{X_{0,\frac{1}{2}}(\phi)}\|u_2\|_{X_{0,\frac{1}{2}}(\phi)}$ *and*

ii) $\|\overline{u}_1\overline{u}_2\|_{Y_1(\phi)} \leq c\delta^{\theta}\|u_1\|_{X_{0,\frac{1}{2}}(\phi)}\|u_2\|_{X_{0,\frac{1}{2}}(\phi)}$

Proof: 1. Preparations: Setting $v_i = \overline{u}_i$ the stated inequalities read

$$\|v_1v_2\|_{X_{0,-\frac{1}{2}}(\phi)} \le c\delta^\theta \|v_1\|_{X_{0,\frac{1}{2}}(-\phi)} \|v_2\|_{X_{0,\frac{1}{2}}(-\phi)} \tag{3}$$

and

$$\|v_1v_2\|_{Y_1(\phi)} \le c\delta^\theta \|v_1\|_{X_{0,\frac{1}{2}}(-\phi)} \|v_2\|_{X_{0,\frac{1}{2}}(-\phi)} \ . \tag{4}$$

To show them, we need the following inequality:

$$\begin{array}{rl} & \langle\xi\rangle^2 + \langle\xi_1\rangle^2 + \langle\xi_2\rangle^2 \\ \le & \langle\tau+\xi^2\rangle + \langle\tau_1-\xi_1^2\rangle + \langle\tau_2-\xi_2^2\rangle \\ \le & c(\langle\tau+\xi^2\rangle\chi_A + \langle\tau_1-\xi_1^2\rangle + \langle\tau_2-\xi_2^2\rangle) \ . \end{array} \tag{5}$$

Here A denotes the region, where $\langle\tau+\xi^2\rangle \ge \max_{i=1}^2 \langle\tau_i-\xi_i^2\rangle$. Defining $f_i(\xi,\tau) = \langle\tau-\xi^2\rangle^{\frac{1}{2}}\mathcal{F}v_i(\xi,\tau)$ for $i=1,2$ we have $\|v_i\|_{X_{0,\frac{1}{2}}(-\phi)} = \|f_i\|_{L^2_{\xi,\tau}}$. Now, for given $\theta \in (0,\frac{1}{4})$ we fix $\varepsilon = \frac{1}{4}(\frac{1}{4}-\theta)$.

2. Proof of (3): By Lemma 2.9 and (5) we have:

$$\begin{array}{rl} & \|v_1v_2\|_{X_{1,-\frac{1}{2}}(\phi)} \\ = & c\|\langle\tau+\xi^2\rangle^{-\frac{1}{2}}\langle\xi\rangle \int \mu(d\xi_1)d\tau_1 \prod_{i=1}^2 \langle\tau_i-\xi_i^2\rangle^{-\frac{1}{2}} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ \le & c\sum_{i=1}^3 N_i \end{array}$$

with

$$N_1 = \|\int \mu(d\xi_1)d\tau_1 \prod_{i=1}^2 \langle\tau_i-\xi_i^2\rangle^{-\frac{1}{2}} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}},$$

$$N_2 = \|\langle\tau+\xi^2\rangle^{-\frac{1}{2}} \int \mu(d\xi_1)d\tau_1 \langle\tau_2-\xi_2^2\rangle^{-\frac{1}{2}} \prod_{i=1}^2 f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}$$

and

$$N_3 = \|\langle\tau+\xi^2\rangle^{-\frac{1}{2}} \int \mu(d\xi_1)d\tau_1 \langle\tau_1-\xi_1^2\rangle^{-\frac{1}{2}} \prod_{i=1}^2 f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \ .$$

Lemma 2.9, Hölders inequality, Lemma 2.14 and Lemma 1.10 are now applied to obtain

$$\begin{array}{rcl} N_1 = \|v_1v_2\|_{L^2_{x,t}} & \le & \|v_1\|_{L^4_{x,t}} \|v_2\|_{L^4_{x,t}} \\ & \le & c\|v_1\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)} \|v_2\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)} \\ & = & c\|\psi_{2\delta}v_1\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)} \|\psi_{2\delta}v_2\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)} \\ & \le & c\delta^{\frac{1}{4}-4\varepsilon} \|v_1\|_{X_{0,\frac{1}{2}-\varepsilon}(-\phi)} \|v_2\|_{X_{0,\frac{1}{2}-\varepsilon}(-\phi)} \ . \end{array}$$

Similarly we get

$$\begin{array}{rcl} N_2 = \|(\mathcal{F}^{-1}f_1)v_2\|_{X_{0,-\frac{1}{2}}(\phi)} & \le & \|\psi_{2\delta}(\mathcal{F}^{-1}f_1)v_2\|_{X_{0,-\frac{1}{2}+\varepsilon}(\phi)} \\ & \le & c\delta^{\frac{1}{8}-2\varepsilon} \|(\mathcal{F}^{-1}f_1)v_2\|_{X_{0,-\frac{3}{8}-\varepsilon}(\phi)} \end{array}$$

$$\begin{aligned}
&\leq c\delta^{\frac{1}{8}-2\varepsilon}\|(\mathcal{F}^{-1}f_1)v_2\|_{L^{\frac{4}{3}}_{x,t}}\\
&\leq c\delta^{\frac{1}{8}-2\varepsilon}\|\mathcal{F}^{-1}f_1\|_{L^2_{x,t}}\|v_2\|_{L^4_{x,t}}\\
&\leq c\delta^{\frac{1}{8}-2\varepsilon}\|v_1\|_{X_{0,\frac{1}{2}}(-\phi)}\|\psi_{2\delta}v_2\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)}\\
&\leq c\delta^{\frac{1}{4}-4\varepsilon}\|v_1\|_{X_{0,\frac{1}{2}}(-\phi)}\|v_2\|_{X_{0,\frac{1}{2}}(-\phi)} .
\end{aligned}$$

By exchanging v_1 and v_2 we get the same upper bound for N_3. So, because of $\theta = \frac{1}{4} - 4\varepsilon$, the estimate (3) is proved.

3. Proof of (4): Using Lemma 2.9 and (5) we get

$$\begin{aligned}
&\|v_1v_2\|_{Y_1(\phi)}\\
=\ & c\|\langle\tau+\xi^2\rangle^{-1}\langle\xi\rangle\int\mu(d\xi_1)d\tau_1\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{1}{2}}f_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)}\\
\leq\ & c\sum_{i=1}^{3}N_i ,
\end{aligned}$$

where

$$N_1 = \|\langle\tau+\xi^2\rangle^{-\frac{1}{2}}\int\mu(d\xi_1)d\tau_1\chi_A\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{1}{2}}f_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)} ,$$

$$N_2 = \|\langle\tau+\xi^2\rangle^{-1}\int\mu(d\xi_1)d\tau_1\langle\tau_2-\xi_2^2\rangle^{-\frac{1}{2}}\prod_{i=1}^{2}f_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)}$$

and

$$N_3 = \|\langle\tau+\xi^2\rangle^{-1}\int\mu(d\xi_1)d\tau_1\langle\tau_1-\xi_1^2\rangle^{-\frac{1}{2}}\prod_{i=1}^{2}f_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)} .$$

In order to estimate N_1 we define

$$g_i(\xi,\tau) := \langle\tau-\xi^2\rangle^{\frac{3}{8}+\varepsilon}\mathcal{F}v_i(\xi,\tau) = \langle\tau-\xi^2\rangle^{-\frac{1}{8}+\varepsilon}f_i(\xi,\tau) .$$

Then it is $\|g_i\|_{L^2_{\xi,\tau}} = \|v_i\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)}$ and

$$N_1 = \|\langle\tau+\xi^2\rangle^{-\frac{1}{2}}\int\mu(d\xi_1)d\tau_1\chi_A\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{3}{8}-\varepsilon}g_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)} .$$

Since in A we have $\langle\tau+\xi^2\rangle \geq \max_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle$ as well as $\langle\tau+\xi^2\rangle \geq c\langle\xi_1\rangle^2$, we obtain

$$N_1 \leq c\|\int\mu(d\xi_1)d\tau_1\langle\xi_1\rangle^{-\frac{1}{2}-2\varepsilon}\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{1+\varepsilon}{2}}g_i(\xi_i,\tau_i)\|_{L^2_\xi(L^1_\tau)} ,$$

which we shall now estimate by duality. Therefore let $f_0 \in L^2_\xi$ with $\|f_0\|_{L^2_\xi} = 1$ and $f_0 \geq 0$. Now applying Cauchy-Schwarz' inequality first in the τ- and then in the ξ-variables we get the desired upper bound for N_1:

$$\begin{aligned}
&\int\mu(d\xi d\xi_1)d\tau d\tau_1 f_0(\xi)\langle\xi_1\rangle^{-\frac{1}{2}-2\varepsilon}\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{1+\varepsilon}{2}}g_i(\xi_i,\tau_i)\\
=\ &\int\mu(d\xi_1 d\xi_2)d\tau_1 d\tau_2 f_0(\xi_1+\xi_2)\langle\xi_1\rangle^{-\frac{1}{2}-2\varepsilon}\prod_{i=1}^{2}\langle\tau_i-\xi_i^2\rangle^{-\frac{1+\varepsilon}{2}}g_i(\xi_i,\tau_i)
\end{aligned}$$

$$\leq c \int \mu(d\xi_1 d\xi_2) f_0(\xi_1+\xi_2)\langle\xi_1\rangle^{-\frac{1}{2}-2\varepsilon}\prod_{i=1}^{2}(\int d\tau_i |g_i(\xi_i,\tau_i)|^2)^{\frac{1}{2}}$$

$$\leq c \prod_{i=1}^{2}\|g_i\|_{L^2_{\xi,\tau}} \leq c\prod_{i=1}^{2}\|v_i\|_{X_{0,\frac{3}{8}+\varepsilon}(-\phi)} \leq c\delta^{\frac{1}{4}-4\varepsilon}\prod_{i=1}^{2}\|v_i\|_{X_{0,\frac{1}{2}}(-\phi)} ,$$

where in the last step we have used Lemma 1.10. To estimate N_2 we apply Cauchy-Schwarz on $\int d\tau$:

$$N_2 \leq c\|\langle\tau+\xi^2\rangle^{-\frac{1}{2}+\varepsilon}\int\mu(d\xi_1)d\tau_1\langle\tau_2-\xi_2^2\rangle^{-\frac{1}{2}}\prod_{i=1}^{2} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}$$
$$= \|\psi_{2\delta}(\mathcal{F}^{-1}f_1)v_2\|_{X_{0,-\frac{1}{2}+\varepsilon}(\phi)} .$$

This was already shown to be bounded by

$$c\delta^{\frac{1}{4}-4\varepsilon}\prod_{i=1}^{2}\|v_i\|_{X_{0,\frac{1}{2}}(-\phi)} .$$

The same upper bound for N_3 is obtained by exchanging v_1 and v_2, so the estimate (4) is proved, too. □

3.2 The general case

The local result in the previous section can be extended to higher dimensions and (integer) exponents:

Theorem 3.3 *Let $m,n\in\mathbf{N}$, $m\geq 2$ and $m+n\geq 4$. Then for $s>s_c$ there exists $\delta=\delta(\|u_0\|_{H^s_x})>0$ and a unique solution $u\in X^{\delta}_{s,\frac{1}{2}}(\phi)$ of the initial value (periodic boundary value) problem*

$$u_t - i\Delta u = \partial_j(\overline{u}^m), \qquad u(0)=u_0\in H^s_x.$$

This solution is persistent and for any $0<\delta_0<\delta$ the mapping data upon solution from H^s_x to $X^{\delta_0}_{s,\frac{1}{2}}(\phi)\cap C_t((-\delta_0,\delta_0),H^s_x)$ is locally Lipschitz continuous.
For any $\beta\in\mathbf{N}_0^n$ with $|\beta|=m$ and under the same assumptions on m,n,s the Cauchy problem and the periodic boundary value problem

$$u_t - i\Delta u = (\nabla\overline{u})^{\beta}, \qquad u(0)=u_0\in H^{s+1}_x$$

is locally well posed in the same sense.

Remarks : 1. The special case in Theorem 3.3, where $n=1$, $m=3$ and $s>0$, has already been proved for the nonperiodic case by H. Takaoka, see Thm. 1.2 in [T99].

2. A standard scaling argument suggests, that our result is optimal as long as we are not dealing with the critical case $s=s_c$. In fact, if u is a solution of the first problem in Theorem 3.3 with initial value $u_0\in H^s_x(\mathbf{R}^n)$, then so is u_λ, defined by $u_\lambda(x,t)=\lambda^{\frac{1}{m-1}}u(\lambda x,\lambda^2 t)$, with initial value $u^0_\lambda(x)=u_0(\lambda x)$, and $\|u^0_\lambda\|_{\dot{H}^{s_c}_x}(\mathbf{R}^n)$ is independent of λ.

By the general theory presented in part I the proof of Theorem 3.3 reduces to the following estimates:

Theorem 3.4 *Let $n, m \in \mathbf{N}$ with $m \geq 2$ and $m + n \geq 4$. Assume in addition, that $s > \frac{n}{2} - \frac{1}{m-1}$. Then there exists $\theta > 0$, so that for all $0 < \delta \leq 1$ and for all $u_i \in X_{s,\frac{1}{2}}(\phi)$, $1 \leq i \leq m$, having support in $\{(x,t) : |t| \leq \delta\}$ the estimates*

i) $\| \prod_{i=1}^m \overline{u}_i \|_{X_{s+1,-\frac{1}{2}}(\phi)} \leq c\delta^\theta \prod_{i=1}^m \|u_i\|_{X_{s,\frac{1}{2}}(\phi)}$ *and*

ii) $\| \prod_{i=1}^m \overline{u}_i \|_{Y_{s+1}(\phi)} \leq c\delta^\theta \prod_{i=1}^m \|u_i\|_{X_{s,\frac{1}{2}}(\phi)}$

hold.

To prove Theorem 3.4 we follow the ideas of section 5 in [B93] - essentially we present a simplified version of the proof given there. Here some instructive hints from [G96], section 5, were helpful. In particular, we do use Hilbert space norms instead of Besov-type norms as in [B93]. Perhaps it is worthwile to mention, that for the nonperiodic case there is a much easier proof, using the full strength of the Strichartz estimates in this case. Before we start, we need some preparations:

We shall use the notation introduced in section 2 (before Lemma 2.5), but with χ_M denoting in fact the characteristic function of a set $M \subset \mathbf{R}^n$ or $M \subset \mathbf{Z}^n$, so that the operators $P_M := {\mathcal{F}_x}^{-1} \chi_M \mathcal{F}_x$ become projections. Next we shall fix a couple of Hölder- and Sobolevexponents to be used below:

1. We choose $\frac{1}{p} = \frac{1}{(n+2)(m-1)}$. Then for any $s > \frac{n}{2} - \frac{1}{m-1}$ by Corollaries 2.4 and 2.5, part c), there exists $b < \frac{1}{2}$, so that the following estimate holds:

$$\|u\|_{L^p_{xt}} \leq c\|u\|_{X_{s,b}(\pm\phi)} \tag{6}$$

2. Next we have $\frac{1}{p_0} = \frac{1}{6} + \varepsilon$ for $n = 1$ respectively $\frac{1}{p_0} = \frac{1}{4} + \varepsilon$ for $n \geq 2$ and $s_0 = \varepsilon$ if $n = 1$ respectively $s_0 = (n-2)(\frac{1}{2} - \frac{1}{p_0}) + \varepsilon = \frac{n-2}{4} + (3-n)\varepsilon$ if $n \geq 2$. Then, if $\varepsilon > 0$ is chosen appropriately small, by Corollaries 2.4 and 2.5, part b), and Lemma 2.17 there exists $b < \frac{1}{2}$ for which we have the estimate

$$\|P_B u\|_{L^{p_0}_{xt}} \leq cR^{s_0}\|u\|_{X_{0,b}(\pm\phi)} \ , \tag{7}$$

whenever B is a ball or cube of size R. Dualizing the last inequality, we obtain

$$\|P_B u\|_{X_{0,-b}(\pm\phi)} \leq cR^{s_0}\|u\|_{L^{p_0'}_{xt}} \ , \tag{8}$$

where $\frac{1}{p_0'} = \frac{5}{6} - \varepsilon$ for $n = 1$ respectively $\frac{1}{p_0'} = \frac{3}{4} - \varepsilon$ for $n \geq 2$.

3. We choose $\frac{1}{p_1} = \frac{1}{3} - \varepsilon - \frac{m-2}{3(m-1)}$ for $n = 1$ respectively $\frac{1}{p_1} = \frac{1}{4} - \varepsilon - \frac{m-2}{(n+2)(m-1)}$ for $n \geq 2$ and $s_1 = \frac{n}{2} - \frac{n+2}{p_1} + \varepsilon$. Then it is $s_1 = \frac{1}{2} - \frac{1}{m-1} + 4\varepsilon$ if $n = 1$ respectively $s_1 = \frac{n+2}{4} - \frac{1}{m-1} + (n+3)\varepsilon$ if $n \geq 2$, and by Corollaries 2.4, 2.5, part c), and Lemma 2.17 there exists $b < \frac{1}{2}$ for which

$$\|P_B u\|_{L^{p_1}_{xt}} \leq cR^{s_1}\|u\|_{X_{0,b}(\pm\phi)} \ . \tag{9}$$

Observe that our choice guarantees

$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{m-2}{p} = \frac{1}{2} \qquad \text{resp.} \qquad \frac{1}{p_1} + \frac{1}{2} + \frac{m-2}{p} = \frac{1}{p_0'}$$

(for the Hölder applications) as well as for ε sufficiently small $s_0 + s_1 - s < 0$.

For $m \geq 3$ in addition we shall need the following parameters:

4. Assuming $\frac{s}{n} < \frac{1}{2}$ without loss of generality, we may choose $\frac{1}{q} = \frac{1}{2} - \frac{s}{n} > 0$, so that the Sobolev embedding $H^s_x \subset L^q_x$ holds.

5. In the case of space dimension $n = 1$ we define $\frac{1}{r_0} = \frac{1}{6} - \frac{m-3}{6(m-1)} - \varepsilon$, $\frac{1}{q_0} = s + \frac{1}{6} - \frac{2(m-3)}{3(m-1)} - \varepsilon$ and $\sigma_1 = \varepsilon$, if $m = 3$, as well as $\sigma_1 = \frac{1}{2} - \frac{2}{r_0} - \frac{1}{q_0} + \varepsilon = \frac{m-3}{m-1} - s + 4\varepsilon$ if $m \geq 4$. For $n \geq 2$ let $\frac{1}{r_0} = \frac{1}{4} - \frac{m-3}{(n+2)(m-1)} - 2\varepsilon$, $\frac{1}{q_0} = \frac{s}{n} - \frac{1}{4} - \frac{m-3}{(n+2)(m-1)} - \varepsilon$ and $\sigma_1 = \frac{n}{2} - \frac{2}{r_0} - \frac{n}{q_0} + \varepsilon = \frac{3n}{4} + \frac{1}{2} - \frac{2}{m-1} - s + (n+5)\varepsilon$. Then, for some $b < \frac{1}{2}$, we have the estimate

$$\|P_B u\|_{L^{r_0}_t(L^{q_0}_x)} \leq cR^{\sigma_1} \|u\|_{X_{0,b}(\pm\phi)} \,. \tag{10}$$

In general, this follows from part c) of the Corollaries 2.4, 2.5, except in the case $n = 1$, $m = 3$, where one can use part b) of Corollary 2.4. (Here we assume $s \leq \frac{1}{3}$ in the cases $n = 1$, $m \in \{3, 4\}$.)

6. We close our list of parameters by choosing $\frac{1}{r_1} = \frac{1}{6} - \frac{m-3}{6(m-1)}$, $\frac{1}{q_1} = \frac{1}{2} - \frac{2}{r_1} = \frac{1}{6} + \frac{m-3}{3(m-1)}$ for $n = 1$ respectively $\frac{1}{r_1} = \varepsilon$, $\frac{1}{q_1} = \frac{1}{2}$ for $n \geq 2$. Then, by Corollary 2.4, part c), in the case of space dimension $n = 1$ and by Sobolev embedding in the time variable in the case of $n \geq 2$, we have the estimate

$$\|P_B u\|_{L^{r_1}_t(L^{q_1}_x)} \leq cR^{\varepsilon} \|u\|_{X_{0,b}(\pm\phi)} \tag{11}$$

for some $b < \frac{1}{2}$. Now for the Hölder applications we have

$$\frac{1}{r_0} + \frac{1}{2} + \frac{1}{r_1} + \frac{m-3}{p} = \frac{1}{q_0} + \frac{1}{q} + \frac{1}{q_1} + \frac{m-3}{p} = \frac{1}{p_0'}$$

as well as for ε sufficiently small $s_0 + \sigma_1 + \varepsilon - s < 0$.

Now we derive three praparatory lemmas:

Lemma 3.1 *Let $n, m \in \mathbf{N}$ with $m \geq 2$ and $n + m \geq 4$. Then for $s > \frac{n}{2} - \frac{1}{m-1}$ there exists $b < \frac{1}{2}$, so that for all $v_i \in X_{s,b}(-\phi)$, $1 \leq i, j \leq m$, the following estimate is valid:*

$$\|(J^s v_j) \textstyle\prod_{i=1, i\neq j}^m v_i\|_{L^2_{xt}} \leq c \displaystyle\prod_{i=1}^m \|v_i\|_{X_{s,b}(-\phi)} \,,$$

where $J^s = {\mathcal{F}_x}^{-1} \langle \xi \rangle^s \mathcal{F}_x$.

Proof: Writing

$$\prod_{\substack{i=1\\ i\neq j}}^m v_i = \lim_{l \in \mathbf{N}_0} \prod_{\substack{i=1\\ i\neq j}}^m P_l v_i = \sum_{l \in \mathbf{N}_0} \Big(\prod_{\substack{i=1\\ i\neq j}}^m P_l v_i - \prod_{\substack{i=1\\ i\neq j}}^m P_{l-1} v_i\Big) \,,$$

where

$$\prod_{\substack{i=1\\ i\neq j}}^m P_l v_i - \prod_{\substack{i=1\\ i\neq j}}^m P_{l-1} v_i = \sum_{\substack{k=1\\ k\neq j}}^m \Big(\prod_{\substack{i<k\\ i\neq j}} P_{l-1} v_i\Big) P_{\Delta l} v_k \Big(\prod_{\substack{i>k\\ i\neq j}} P_l v_i\Big) \,,$$

we obtain

$$
\begin{aligned}
& \|(J^s v_j)\prod_{i\neq j} v_i\|_{L^2_{xt}} \\
\leq \quad & \sum_{l\in\mathbf{N}_0}\sum_{\substack{k=1\\k\neq j}}^{m} \|(J^s v_j)(\prod_{\substack{i<k\\i\neq j}} P_{l-1}v_i P_{\Delta l}v_k(\prod_{\substack{i>k\\i\neq j}} P_l v_i))\|_{L^2_{xt}} \qquad (12)\\
\leq \quad & \sum_{l\in\mathbf{N}_0}\sum_{\substack{k=1\\k\neq j}}^{m} \|(J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i)\|_{L^2_{xt}} \ .
\end{aligned}
$$

Next we estimate the contribution for fixed l and k:

$$
\begin{aligned}
& \|(J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i)\|^2_{L^2_{xt}} \\
= \quad & \|\sum_{\alpha\in\mathbf{Z}^n}(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i)\|^2_{L^2_{xt}} \\
= \quad & \sum_{\alpha,\beta\in\mathbf{Z}^n} < (P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i), (P_{Q^l_\beta}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i) >
\end{aligned}
$$

Now the sequence $\{(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq j} P_l v_i)\}_{\alpha\in\mathbf{Z}^n}$ is almost orthogonal in the following sense: The support of $\mathcal{F}(P_{\Delta l}v_k)(\prod_{i\neq j} P_l v_i)$ is contained in $\{(\xi,\tau): |\xi|\leq (m-1)2^l\}$, and thus $\mathcal{F}(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq j} P_l v_i)$ is supported in $C\times\mathbf{R}$, where C is a cube centered at $2^l\alpha$ having the sidelength $m2^l$. So for $|2^l\alpha - 2^l\beta| > c_n 2^l m$, that is for $|\alpha-\beta| > c_n m$, the above expressions are disjointly supported. Thus for these values of α and β we do not get any contribution to the last sum, which we now can estimate by

$$
\begin{aligned}
& \sum_{\alpha\in\mathbf{Z}^n}\sum_{\substack{\beta\in\mathbf{Z}^n\\|\beta|\leq c_n m}} < (P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i), (P_{Q^l_{\alpha+\beta}}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i) > \\
& \qquad \leq c\sum_{\alpha\in\mathbf{Z}^n}\|(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i)\|^2_{L^2_{xt}} \qquad (13)\\
& \qquad \leq c\sum_{\alpha\in\mathbf{Z}^n}\|(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} v_i)\|^2_{L^2_{xt}} \ .
\end{aligned}
$$

Next we use Hölder's inequality, (6), (7) and (9) to get

$$
\begin{aligned}
& \|(P_{Q^l_\alpha}J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} v_i)\|_{L^2_{xt}} \\
\leq \quad & \|P_{Q^l_\alpha}J^s v_j\|_{L^{p_0}_{xt}}\|P_{\Delta l}v_k\|_{L^{p_1}_{xt}}\prod_{i\neq k,j}\|v_i\|_{L^p_{xt}} \qquad (14)\\
\leq \quad & c2^{l(s_0+s_1)}\|P_{Q^l_\alpha}J^s v_j\|_{X_{0,b}(-\phi)}\|P_{\Delta l}v_k\|_{X_{0,b}(-\phi)}\prod_{i\neq k,j}\|v_i\|_{X_{s,b}(-\phi)}
\end{aligned}
$$

for some $b<\frac{1}{2}$. Using $\|P_{\Delta l}v_k\|_{X_{0,b}(-\phi)}\leq c2^{-sl}\|v_k\|_{X_{s,b}(-\phi)}$ we combine (13) and (14) to obtain:

$$
\|(J^s v_j)(P_{\Delta l}v_k)(\prod_{i\neq k,j} P_l v_i)\|^2_{L^2_{xt}}
$$

$$\begin{aligned} &\leq c2^{2l(s_0+s_1-s)} \sum_{\alpha \in \mathbf{Z}^n} \|P_{Q^l_\alpha} J^s v_j\|^2_{X_{0,b}(-\phi)} \prod_{i \neq j} \|v_i\|^2_{X_{s,b}(-\phi)} \\ &= c2^{2l(s_0+s_1-s)} \prod_{i=1}^m \|v_i\|^2_{X_{s,b}(-\phi)} . \end{aligned}$$

Inserting the square root of this into (12) and summing up over k and l we can finish the proof. □

Corollary 3.2 *For n, m and s as in the previous lemma there exists $b < \frac{1}{2}$, so that for all $v_i \in X_{s,\frac{1}{2}}(-\phi)$, $1 \leq i,j \leq m$, the following estimate holds true:*

$$\|(\Lambda^{\frac{1}{2}} J^s v_j) \textstyle\prod_{i=1, i \neq j}^m v_i\|_{X_{0,-b}(\phi)} \leq c\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)} \displaystyle\prod_{\substack{i=1 \\ i \neq j}}^m \|v_i\|_{X_{s,b}(-\phi)} ,$$

where $\Lambda^{\frac{1}{2}} = \mathcal{F}^{-1} \langle \tau - |\xi|^2 \rangle^{\frac{1}{2}} \mathcal{F}$.

Proof: Let the v_i's be fixed for $i \neq j$. Then the previous lemma tells us, that the linear mapping

$$A_j : X_{s,b}(-\phi) \to L^2_{xt}, \qquad f \mapsto (J^s f) \prod_{\substack{i=1 \\ i \neq j}}^m v_i$$

is bounded with norm $\|A_j\| \leq c \prod_{\substack{i=1 \\ i \neq j}}^m \|v_i\|_{X_{s,b}(-\phi)}$. The adjoint mapping A_j^*, given by

$$A_j^* : L^2_{xt} \to X_{-s,-b}(-\phi), \qquad g \mapsto J^s(g \prod_{\substack{i=1 \\ i \neq j}}^m \overline{v_i})$$

then is also bounded with $\|A_j^*\| = \|A_j\|$. From this we get for $g = \overline{\Lambda^{\frac{1}{2}} J^s v_j}$:

$$\begin{aligned} \|(\Lambda^{\frac{1}{2}} J^s v_j) \textstyle\prod_{i=1, i \neq j}^m v_i\|_{X_{0,-b}(\phi)} &= \|J^s(\overline{\Lambda^{\frac{1}{2}} J^s v_j}) \textstyle\prod_{i=1, i \neq j}^m \overline{v_i}\|_{X_{-s,-b}(-\phi)} \\ \leq c\|\Lambda^{\frac{1}{2}} J^s v_j\|_{L^2_{xt}} \prod_{\substack{i=1 \\ i \neq j}}^m \|v_i\|_{X_{s,b}(-\phi)} &= c\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)} \prod_{\substack{i=1 \\ i \neq j}}^m \|v_i\|_{X_{s,b}(-\phi)} \end{aligned}$$

□

Lemma 3.2 *Let $n, m \in \mathbf{N}$ with $m \geq 2$, $n + m \geq 4$ and $s \in (\frac{n}{2} - \frac{1}{m-1}, \frac{n}{2})$. For $n = 1$, $m \in \{3, 4\}$ assume in addition, that $s \leq \frac{1}{3}$. Then there exists $b < \frac{1}{2}$, so that for all $v_i \in X_{s,\frac{1}{2}}(-\phi)$, $1 \leq i,j \leq m$, the following estimate is valid:*

$$\|(J^s v_i)(\Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k=1, k \neq i,j}^m v_k\|_{X_{0,-b}(\phi)} \leq c\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)} \displaystyle\prod_{\substack{k=1 \\ k \neq j}}^m \|v_k\|_{X_{s,b}(-\phi)}$$

Here again we have $\Lambda^{\frac{1}{2}} = \mathcal{F}^{-1} \langle \tau - |\xi|^2 \rangle^{\frac{1}{2}} \mathcal{F}$.

Proof: 1. Similarly as in the proof of the previous lemma we write

$$\Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} v_k = \sum_{l \in \mathbf{N}_0} (P_l \Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} P_l v_k - P_{l-1} \Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} P_{l-1} v_k)$$

with

$$\begin{aligned} & P_l \Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} P_l v_k - P_{l-1} \Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} P_{l-1} v_k \\ = \; & P_{\Delta l} \Lambda^{\frac{1}{2}} v_j \prod_{\substack{k=1 \\ k \neq i,j}}^{m} P_l v_k + P_{l-1} \Lambda^{\frac{1}{2}} v_j \sum_{k \neq i,j} (\prod_{\substack{\nu < k \\ \nu \neq i,j}} P_{l-1} v_\nu) P_{\Delta l} v_k (\prod_{\substack{\nu > k \\ \nu \neq i,j}} P_l v_\nu) \; . \end{aligned}$$

From this we obtain for arbitrary b:

$$\begin{aligned} & \|(J^s v_i)(\Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} v_k\|_{X_{0,-b}(\phi)} \\ \leq \; & \sum_{l \in \mathbf{N}_0} \|(J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|_{X_{0,-b}(\phi)} \\ + \; & \sum_{k \neq i,j} \sum_{l \in \mathbf{N}_0} \|(J^s v_i)(P_l \Lambda^{\frac{1}{2}} v_j)(P_{\Delta l} v_k) \textstyle\prod_{\nu \neq i,j,k} P_l v_\nu\|_{X_{0,-b}(\phi)} \end{aligned} \tag{15}$$

2. Next we show that for some $b < \frac{1}{2}$ the estimate

$$\begin{aligned} & \|(J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|_{X_{0,-b}(\phi)} \\ \leq \; & c 2^{l(s_0+s_1-s)} \|v_j\|_{X_{s,\frac{1}{2}}(-\phi)} \prod_{\substack{i=1 \\ i \neq j}}^{m} \|v_i\|_{X_{s,b}(-\phi)} \end{aligned} \tag{16}$$

holds true. To see this, we start from

$$\begin{aligned} & \|(J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|^2_{X_{0,-b}(\phi)} \\ = \; & \| \textstyle\sum_{\alpha \in \mathbf{Z}^n} (P_{Q^l_\alpha} J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|^2_{X_{0,-b}(\phi)} \\ \leq \; & c \sum_{\alpha \in \mathbf{Z}^n} \|(P_{Q^l_\alpha} J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|^2_{X_{0,-b}(\phi)} \; , \end{aligned}$$

where in the last step we have used the almost orthogonality of the sequence $\{(P_{Q^l_\alpha} J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \prod_{k \neq i,j} P_l v_k\}_{\alpha \in \mathbf{Z}^n}$. Now we use (8), Hölders inequality, (9) and (6) to obtain for some $b < \frac{1}{2}$

$$\begin{aligned} & \|(P_{Q^l_\alpha} J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|_{X_{0,-b}(\phi)} \\ \leq \; & c 2^{l s_0} \|(P_{Q^l_\alpha} J^s v_i)(P_{\Delta l} \Lambda^{\frac{1}{2}} v_j) \textstyle\prod_{k \neq i,j} P_l v_k\|_{L^{p_0'}_{xt}} \\ \leq \; & c 2^{l s_0} \|P_{Q^l_\alpha} J^s v_i\|_{L^{p_1}_{xt}} \|P_{\Delta l} \Lambda^{\frac{1}{2}} v_j\|_{L^2_{xt}} \prod_{k \neq i,j} \|P_l v_k\|_{L^p_{xt}} \\ \leq \; & c 2^{l(s_0+s_1)} \|P_{Q^l_\alpha} J^s v_i\|_{X_{0,b}(-\phi)} \|P_{\Delta l} \Lambda^{\frac{1}{2}} v_j\|_{L^2_{xt}} \prod_{k \neq i,j} \|v_k\|_{X_{s,b}(-\phi)} \; . \end{aligned}$$

Using $\|P_{\Delta l}\Lambda^{\frac{1}{2}}v_j\|_{L^2_{xt}} \le c2^{-ls}\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)}$ we get

$$\begin{aligned} & \|(P_{Q^l_\alpha}J^s v_i)(P_{\Delta l}\Lambda^{\frac{1}{2}}v_j)\textstyle\prod_{k\neq i,j}P_l v_k\|^2_{X_{0,-b}(\phi)} \\ \le\ & c2^{2l(s_0+s_1-s)}\|P_{Q^l_\alpha}J^s v_i\|^2_{X_{0,b}(-\phi)}\|v_j\|^2_{X_{s,\frac{1}{2}}(-\phi)}\prod_{k\neq i,j}\|v_k\|^2_{X_{s,b}(-\phi)} \ . \end{aligned}$$

Now summing up over α we arrive at the square of (16).

3. Now we show that there exists $b < \frac{1}{2}$ for which

$$\begin{aligned} & \|(J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\textstyle\prod_{\nu\neq i,j,k}P_l v_\nu\|_{X_{0,-b}(\phi)} \\ \le\ & c2^{l(s_0+\sigma_1+\varepsilon-s)}\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)}\prod_{\substack{i=1\\ i\neq j}}^m\|v_i\|_{X_{s,b}(-\phi)} \ . \end{aligned} \tag{17}$$

Therefore again we write $J^s v_i = \sum_{\alpha\in\mathbf{Z}^n}P_{Q^l_\alpha}J^s v_i$ and use the almost orthogonality of $\{(P_{Q^l_\alpha}J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\prod_{\nu\neq i,j,k}P_l v_\nu\}_{\alpha\in\mathbf{Z}^n}$ to obtain

$$\begin{aligned} & \|(J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\textstyle\prod_{\nu\neq i,j,k}P_l v_\nu\|^2_{X_{0,-b}(\phi)} \\ \le\ & c\sum_{\alpha\in\mathbf{Z}^n}\|(P_{Q^l_\alpha}J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\textstyle\prod_{\nu\neq i,j,k}P_l v_\nu\|^2_{X_{0,-b}(\phi)} \ . \end{aligned}$$

Then we use (8), Hölders inequality, (10), Sobolev embedding in x, (11) and (6) to get for some $b < \frac{1}{2}$:

$$\begin{aligned} & \|(P_{Q^l_\alpha}J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\textstyle\prod_{\nu\neq i,j,k}P_l v_\nu\|_{X_{0,-b}(\phi)} \\ \le\ & c2^{ls_0}\|(P_{Q^l_\alpha}J^s v_i)(P_l\Lambda^{\frac{1}{2}}v_j)(P_{\Delta l}v_k)\textstyle\prod_{\nu\neq i,j,k}P_l v_\nu\|_{L^{p_0'}_{xt}} \\ \le\ & c2^{ls_0}\|P_{Q^l_\alpha}J^s v_i\|_{L^{r_0}_t(L^{q_0}_x)}\|P_l\Lambda^{\frac{1}{2}}v_j\|_{L^2_t(L^q_x)}\|P_{\Delta l}v_k\|_{L^{r_1}_t(L^{q_1}_x)}\prod_{\nu\neq i,j,k}\|P_l v_\nu\|_{L^p_{xt}} \\ \le\ & c2^{l(s_0+\sigma_1+\varepsilon-s)}\|P_{Q^l_\alpha}J^s v_i\|_{X_{0,b}(-\phi)}\|v_j\|_{X_{s,\frac{1}{2}}(-\phi)}\prod_{k\neq i,j}\|v_k\|_{X_{s,b}(-\phi)} \end{aligned}$$

Squaring the last and summing up over α we arrive at the square of (17).

4. Conclusion: Since $s_0 + s_1 - s < 0$ as well as $s_0 + \sigma_1 + \varepsilon - s < 0$ we can now insert (16) and (17) into (15) and finish the proof by summing up over k and l. □

Lemma 3.3 *Let $m, n \in \mathbf{N}$ with $m \ge 2$, $m + n \ge 4$ and $s > \frac{n}{2} - \frac{1}{m-1}$. For $1 \le i, j \le m$ and $v_i \in X_{s,\frac{1}{2}}(-\phi)$ define $f_i(\xi,\tau) = \langle\xi\rangle^s\langle\tau - |\xi|^2\rangle^{\frac{1}{2}}\mathcal{F}v_i(\xi,\tau)$ and*

$$G_{0j}(\xi,\tau) = \langle\tau + |\xi|^2\rangle^{-\frac{1}{2}}\int d\nu\langle\xi_j\rangle^s\chi_A\prod_{i=1}^m\langle\tau_i - |\xi_i|^2\rangle^{-\frac{1}{2}}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i) \ ,$$

where in A the inequality $\langle\tau + |\xi|^2\rangle \ge \max_{i=1}^m\langle\tau_i - |\xi_i|^2\rangle$ holds. Then there exists $b < \frac{1}{2}$ for which the following estimate is valid:

$$\|G_{0j}\|_{L^2_\xi(L^1_\tau)} \le c\prod_{i=1}^m\|v_i\|_{X_{s,b}(-\phi)}$$

Proof: We choose $\varepsilon \in (0, s - \frac{n}{2} + \frac{1}{m-1})$ with $\varepsilon \leq \frac{1}{m-1}$ and define $\delta = \frac{m-1}{2m}\varepsilon$. Observe that, because of

$$\sum_{i=1}^{m} \langle \xi_i \rangle^2 \leq \langle \tau + |\xi|^2 \rangle + \sum_{i=1}^{m} \langle \tau_i - |\xi_i|^2 \rangle$$

in the region A the inequality

$$\langle \tau + |\xi|^2 \rangle \geq c \prod_{i=1}^{m} \langle \tau_i - |\xi_i|^2 \rangle^{2\delta} \prod_{\substack{i=1 \\ i \neq j}}^{m} \langle \xi_i \rangle^{\frac{2}{m-1} - 2\varepsilon}$$

holds. From this we obtain

$$G_{0j}(\xi, \tau) \leq c \int d\nu \prod_{\substack{i=1 \\ i \neq j}}^{m} \langle \xi_i \rangle^{-s - \frac{1}{m-1} + \varepsilon} \prod_{i=1}^{m} \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2} - \delta} f_i(\xi_i, \tau_i) \ .$$

In order to estimate $\|G_{0j}\|_{L^2_\xi(L^1_\tau)}$ by duality let $f_0 \in L^2_\xi$ with $f_0 \geq 0$ and $\|f_0\|_{L^2_\xi} = 1$. By Fubini and Cauchy-Schwarz we get:

$$\begin{aligned}
& \int \mu(d\xi) d\tau d\nu f_0(\xi) G_{0j}(\xi, \tau) \\
\leq \ & c \int \mu(d\xi) d\tau d\nu f_0(\xi) \prod_{i=1}^{m} \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2} - \delta} f_i(\xi_i, \tau_i) \prod_{\substack{i=1 \\ i \neq j}}^{m} \langle \xi_i \rangle^{-s - \frac{1}{m-1} + \varepsilon} \\
= \ & c \int \mu(d\xi_1..d\xi_m) d\tau_1..d\tau_m f_0(\sum_{i=1}^{m} \xi_i) \prod_{i=1}^{m} \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2} - \delta} f_i(\xi_i, \tau_i) \prod_{\substack{i=1 \\ i \neq j}}^{m} \langle \xi_i \rangle^{-s - \frac{1}{m-1} + \varepsilon} \\
\leq \ & c \int \mu(d\xi_1..d\xi_m) f_0(\sum_{i=1}^{m} \xi_i) \prod_{\substack{i=1 \\ i \neq j}}^{m} \langle \xi_i \rangle^{-s - \frac{1}{m-1} + \varepsilon} \prod_{i=1}^{m} (\int d\tau_i f_i(\xi_i, \tau_i)^2 \langle \tau_i - |\xi_i|^2 \rangle^{-\delta})^{\frac{1}{2}} \\
\leq \ & c \prod_{\substack{i=1 \\ i \neq j}}^{m} (\int \mu(d\xi_i) \langle \xi_i \rangle^{-2s - \frac{2}{m-1} + 2\varepsilon})^{\frac{1}{2}} \prod_{i=1}^{m} \|f_i \langle \tau - |\xi|^2 \rangle^{-\frac{\delta}{2}}\|_{L^2_{\xi\tau}} \\
\leq \ & c \prod_{i=1}^{m} \|f_i \langle \tau - |\xi|^2 \rangle^{-\frac{\delta}{2}}\|_{L^2_{\xi\tau}} = c \prod_{i=1}^{m} \|v_i\|_{X_{s, \frac{1-\delta}{2}}(-\phi)} \ .
\end{aligned}$$

From this the statement of the lemma follows for $b = \frac{1-\delta}{2}$. □

Proof of Theorem 3.4: 1. Setting $v_i = \overline{u_i}$ the claimed estimates read

$$\|\textstyle\prod_{i=1}^{m} v_i\|_{X_{s+1, -\frac{1}{2}}(\phi)} \leq c\delta^{\theta} \displaystyle\prod_{i=1}^{m} \|v_i\|_{X_{s, \frac{1}{2}}(-\phi)} \ , \tag{18}$$

$$\|\textstyle\prod_{i=1}^{m} v_i\|_{Y_{s+1}(\phi)} \leq c\delta^{\theta} \displaystyle\prod_{i=1}^{m} \|v_i\|_{X_{s, \frac{1}{2}}(-\phi)} \ . \tag{19}$$

To prove these, we shall assume $s \in (\frac{n}{2} - \frac{1}{m-1}, \frac{n}{2})$ as well as $s \leq \frac{1}{3}$ for $n = 1$ and $m \in \{3, 4\}$. Now for $f_i(\xi, \tau) = \langle \tau - |\xi|^2 \rangle^{\frac{1}{2}} \langle \xi \rangle^s \mathcal{F} v_i(\xi, \tau)$ we have by Lemma 2.9, that the left hand side of (18) is equal to

$$\|\langle \tau + |\xi|^2 \rangle^{-\frac{1}{2}} \langle \xi \rangle^{s+1} \textstyle\int d\nu \prod_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2}} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi\tau}} \leq c \displaystyle\sum_{i=0}^m \|F_i\|_{L^2_{\xi\tau}},$$

where

$$F_0(\xi, \tau) = \langle \xi \rangle^s \int d\nu \prod_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2}} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i)$$

and, for $1 \leq i \leq m$,

$$F_i(\xi, \tau) = \langle \tau + |\xi|^2 \rangle^{-\frac{1}{2}} \langle \xi \rangle^s \int d\nu \langle \tau_i - |\xi_i|^2 \rangle^{\frac{1}{2}} \prod_{k=1}^m \langle \tau_k - |\xi_k|^2 \rangle^{-\frac{1}{2}} \langle \xi_k \rangle^{-s} f_k(\xi_k, \tau_k) \ .$$

Here we have used the inequality

$$\langle \xi \rangle^2 \leq \langle \tau + |\xi|^2 \rangle + \sum_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle \ .$$

Now by $\langle \xi \rangle \leq \sum_{j=1}^m \langle \xi_j \rangle$ it follows, that

$$F_0(\xi, \tau) \leq \sum_{j=1}^m F_{0j}(\xi, \tau), \qquad\qquad F_i(\xi, \tau) \leq \sum_{j=1}^m F_{ij}(\xi, \tau) \ ,$$

where

$$F_{0j}(\xi, \tau) = \int d\nu \langle \xi_j \rangle^s \prod_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2}} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i)$$

and

$$F_{ij}(\xi, \tau) = \langle \tau + |\xi|^2 \rangle^{-\frac{1}{2}} \int d\nu \langle \tau_i - |\xi_i|^2 \rangle^{\frac{1}{2}} \langle \xi_j \rangle^s \prod_{k=1}^m \langle \tau_k - |\xi_k|^2 \rangle^{-\frac{1}{2}} \langle \xi_k \rangle^{-s} f_k(\xi_k, \tau_k) \ .$$

2. To derive the estimate (19) we use the inequality

$$\langle \xi \rangle^2 \leq c(\langle \tau + |\xi|^2 \rangle \chi_A + \sum_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle) \ ,$$

where in the region A we have $\langle \tau + |\xi|^2 \rangle \geq \max_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle$ (cf. Lemma 3.3). Now again by Lemma 2.9 we see that the left hand side of (19) is equal to

$$\|\langle \tau + |\xi|^2 \rangle^{-1} \langle \xi \rangle^{s+1} \textstyle\int d\nu \prod_{i=1}^m \langle \tau_i - |\xi_i|^2 \rangle^{-\frac{1}{2}} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i) \|_{L^2_\xi(L^1_\tau)} \leq c \displaystyle\sum_{i=0}^m \|G_i\|_{L^2_\xi(L^1_\tau)},$$

where now

$$\begin{aligned} G_0(\xi,\tau) &= \langle\tau+|\xi|^2\rangle^{-\frac{1}{2}}\langle\xi\rangle^s \int d\nu\chi_A \prod_{i=1}^m \langle\tau_i-|\xi_i|^2\rangle^{-\frac{1}{2}}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i) \\ &\leq \sum_{j=1}^m G_{0j}(\xi,\tau) \end{aligned}$$

with G_{0j} precisely as in Lemma 3.3, and for $1 \leq i \leq m$

$$G_i(\xi,\tau) = \langle\tau+|\xi|^2\rangle^{-1}\langle\xi\rangle^s \int d\nu \langle\tau_i-|\xi_i|^2\rangle^{\frac{1}{2}} \prod_{k=1}^m \langle\tau_k-|\xi_k|^2\rangle^{-\frac{1}{2}}\langle\xi_k\rangle^{-s} f_k(\xi_k,\tau_k) \ .$$

Using Cauchy-Schwarz' inequality the estimation of G_i, $1 \leq i \leq m$, can easily be reduced to the estimation of F_i, in fact for any $\varepsilon > 0$ we have:

$$\|G_i\|_{L^2_\xi(L^1_\tau)} \leq c_\varepsilon \|\langle\tau+|\xi|^2\rangle^\varepsilon F_i\|_{L^2_{\xi\tau}} \leq \sum_{j=1}^m c_\varepsilon \|\langle\tau+|\xi|^2\rangle^\varepsilon F_{ij}\|_{L^2_{\xi\tau}}$$

3. Using Lemma 2.9 and Lemma 3.1 we have for $1 \leq j \leq m$:

$$\|F_{0j}\|_{L^2_{\xi\tau}} = c\|(J^s v_j)\textstyle\prod_{i=1,i\neq j}^m v_i\|_{L^2_{xt}} \leq c\displaystyle\prod_{i=1}^m \|v_i\|_{X_{s,b}(-\phi)}$$

for some $b < \frac{1}{2}$. Now we use Lemma 1.10 to conclude that

$$\|F_{0j}\|_{L^2_{\xi\tau}} \leq c\delta^\theta \prod_{i=1}^m \|v_i\|_{X_{s,\frac{1}{2}}(-\phi)}$$

for some $\theta > 0$. Similarly, but using Corollary 3.2 (resp. Lemma 3.2) instead of Lemma 3.1, we get the same upper bound for $\|\langle\tau+|\xi|^2\rangle^\varepsilon F_{ij}\|_{L^2_{\xi\tau}}$, provided ε is sufficiently small, for $1 \leq i = j \leq m$ (resp. $1 \leq i \neq j \leq m$). Now the estimate (18) is proved. For the proof of (19), it remains to show that $\|G_{0j}\|_{L^2_\xi(L^1_\tau)}$, $1 \leq j \leq m$, is bounded by the same quantity. But this follows by Lemma 3.3 and Lemma 1.10.

□

4 Some local wellposedness results for nonlinear Schrödinger equations below L^2

4.1 Statement of results

The first local (in time) wellposedness results below L^2 for the initial value problem for nonlinear Schrödinger equations (NLS)

$$u_t - i\Delta u = N(u, \overline{u}), \qquad u(0) = u_0$$

were published in 1996 by Kenig, Ponce and Vega in [KPV96b]. (Here the initial value u_0 is assumed to belong to some Sobolev space $H^s_x = H^s_x(\mathbf{T}^n)$ or $H^s_x = H^s_x(\mathbf{R}^n)$ with $s < 0$.) These authors considered the nonlinearities

$$N_1(u, \overline{u}) = u^2, \qquad N_2(u, \overline{u}) = u\overline{u}, \qquad N_3(u, \overline{u}) = \overline{u}^2$$

in one space dimension. They obtained wellposedness for N_1 and N_3 under the assumptions $u_0 \in H^s_x(\mathbf{R}),\ s > -\frac{3}{4}$ or $u_0 \in H^s_x(\mathbf{T}),\ s > -\frac{1}{2}$ and for N_2, provided that $u_0 \in H^s_x(\mathbf{R}),\ s > -\frac{1}{4}$. Using appropriate counterexamples they also showed that these results are essentially sharp. This was followed in 1997 by Staffilani's paper [St97], where wellposedness for NLS with $N = N_3$ and $u_0 \in H^s_x(\mathbf{R}^2),\ s > -\frac{1}{2}$ was shown.

A standard scaling argument suggests that there are even more possible candidates for the nonlinearity to allow local wellposedness below L^2: The critical Sobolevexponent for NLS with $N(u, \overline{u}) = |u|^\alpha u$ obtained by scaling is $s_c = \frac{n}{2} - \frac{2}{\alpha}$. So, for N_i, $1 \leq i \leq 3$, there might be local wellposedness for some $s < 0$ even for space dimension $n = 3$, and in one space dimension also for cubic and quartic nonlinearities positive results seem to be possible. This conjecture is also suggested by Example 2.1.

Recently new results concerning this question have appeared: In [CDKS01] Colliander, Delort, Kenig and Staffilani could prove that in the nonperiodic setting all the results on N_i, $1 \leq i \leq 3$, carry over from the one- to the twodimensional case (with the same restrictions on s), cf. Example 2.3. Concerning the threedimensional nonperiodic case, Tao has shown wellposedness for NLS with the nonlinearities N_1 and N_3 for $s > -\frac{1}{2}$ and with N_2 for $s > -\frac{1}{4}$ (see [T00], section 11, cf. Example 2.2). So concerning the quadratic nonlinearities in the nonperiodic setting the question is meanwhile completely answered.

Also the following illposedness result should be mentioned: In [KPV01] it was shown that in the continuous case in one space dimension the NLS with nonlinearity $N(u, \overline{u}) = u|u|^2$ is ill posed below L^2 in the sense that the mapping data upon solution is not uniformly continuous, see Thm. 1.1 in [KPV01].

Here the remaining cases are considered, our positive results are gathered in the following three theorems dealing with the periodic case (Theorem 4.1), the cubic nonlinearities in the onedimensional nonperiodic case (Theorem 4.2) respectively with the quartic nonlinearities on the line (Theorem 4.3). Throughout this section we will have $\phi(\xi) = -|\xi|^2$.

Theorem 4.1 *Assume*

i) $n = 1$,	$m = 3$,	$s > -\frac{1}{3}$, *or*
ii) $n = 1$,	$m = 4$,	$s > -\frac{1}{6}$, *or*
iii) $n = 2$,	$m = 2$,	$s > -\frac{1}{2}$, *or*
iv) $n = 3$,	$m = 2$,	$s > -\frac{3}{10}$.

Then there exist $b > \frac{1}{2}$ *and* $\delta = \delta(\|u_0\|_{H^s_x(\mathbf{T}^n)}) > 0$*, so that there is a unique solution* $u \in X^\delta_{s,b}(\phi)$ *of the periodic boundary value problem*

$$u_t - i\Delta u = \overline{u}^m, \qquad u(0) = u_0 \in H^s_x(\mathbf{T}^n).$$

This solution satisfies $u \in C_t((-\delta,\delta), H^s_x(\mathbf{T}^n))$ *and for any* $0 < \delta_0 < \delta$ *the mapping data upon solution is locally Lipschitz continuous from* $H^s_x(\mathbf{T}^n)$ *to* $X^{\delta_0}_{s,b}(\phi)$*.*

The nonlinear estimates leading to this result are contained in Theorems 4.4, 4.5 and 4.8, see sections 4.3 and 4.4 below. For i) and iii) our results are optimal in the framework of the method and up to the endpoint, in fact there are counterexamples showing that the corresponding multilinear estimates fail for lower values of s, see the discussion in section 4.3. For ii) the scaling argument suggests the optimality of our result. The restriction on s in iv) can possibly be lowered down to $-\frac{1}{2}$, cf. the remark below Thm. 4.5. All the following results are restricted to the onedimensional nonperiodic case:

Theorem 4.2 *Assume*

i) $s > -\frac{5}{12}$ *and* $N(u,\overline{u}) = u^3$ *or* $N(u,\overline{u}) = \overline{u}^3$*, or*

ii) $s > -\frac{2}{5}$ *and* $N(u,\overline{u}) = u\overline{u}^2$*.*

Then there exist $b > \frac{1}{2}$ *and* $\delta = \delta(\|u_0\|_{H^s_x(\mathbf{R})}) > 0$*, so that there is a unique solution* $u \in X^\delta_{s,b}(\phi)$ *of the initial value problem*

$$u_t - i\partial_x^2 u = N(u,\overline{u}), \qquad u(0) = u_0 \in H^s_x(\mathbf{R}).$$

This solution is persistent and for any $0 < \delta_0 < \delta$ *the mapping data upon solution is locally Lipschitz continuous from* $H^s_x(\mathbf{R})$ *to* $X^{\delta_0}_{s,b}(\phi)$*.*

For the corresponding trilinear estimates see Theorems 4.6 and 4.7 (and the remark below) in section 4.3. We must leave open the question, whether or not the bound on s in the above Theorem can be lowered down to $-\frac{1}{2}$, which is the scaling exponent in this case. This question is closely related to the problem concerning certain trilinear refinements of Strichartz' estimate posed in section 4.2.

Theorem 4.3 *Let* $s > -\frac{1}{6}$ *and* $N(u,\overline{u}) \in \{u^4, u^3\overline{u}, u\overline{u}^3, \overline{u}^4\}$*. Then there exist* $b > \frac{1}{2}$ *and* $\delta = \delta(\|u_0\|_{H^s_x(\mathbf{R})}) > 0$*, so that there is a unique solution* $u \in X^\delta_{s,b}(\phi)$ *of the initial value problem*

$$u_t - i\partial_x^2 u = N(u,\overline{u}), \qquad u(0) = u_0 \in H^s_x(\mathbf{R}).$$

This solution satisfies $u \in C_t((-\delta,\delta), H^s_x(\mathbf{R}))$ *and for any* $0 < \delta_0 < \delta$ *the mapping data upon solution is locally Lipschitz continuous from* $H^s_x(\mathbf{R})$ *to* $X^{\delta_0}_{s,b}(\phi)$*. The same statement holds true for* $s > -\frac{1}{8}$ *and* $N(u,\overline{u}) = |u|^4$*.*

See Theorems 4.8 and 4.9 as well as Proposition 4.1 in section 4.4 for the crucial nonlinear estimates. The $-\frac{1}{6}$-results should be optimal by scaling, while for the $|u|^4$-nonlinearity the corresponding estimate fails for $s < -\frac{1}{8}$, cf. Example 4.5. Further counterexamples concerned with the periodic case are also given in section 4.4.

4.2 Refinements of Strichartz' inequalities in the onedimensional nonperiodic case

Lemma 4.1 *Let $n = 1$. Then for all $b_0 > \frac{1}{2} \geq s \geq 0$, the following estimates are valid:*

i) $\|u\overline{v}\|_{L^2_t(H^s_x)} \leq c\|v\|_{X_{0,b_0}(\phi)}\|u\|_{X_{0,b}(\phi)}$, *provided* $b > \frac{1}{4} + \frac{s}{2}$,

ii) $\|u\overline{v}\|_{L^p_t(H^s_x)} \leq c\|v\|_{X_{0,b_0}(\phi)}\|u\|_{X_{0,b_0}(\phi)}$, *provided* $\frac{1}{p} = \frac{1}{4} + \frac{s}{2}$,

iii) $\|vw\|_{X_{\sigma,b'}(\phi)} \leq c\|v\|_{X_{\sigma,b_0}(\phi)}\|w\|_{L^2_t(H^{-s-\sigma}_x)}$, *provided* $\sigma \leq 0$, $b' < -\frac{1}{4} - \frac{s}{2}$.

Proof: We start from

$$\|u\overline{v}\|_{L^2_t(\dot{H}^{\frac{1}{2}}_x)} \leq c\|u\|_{X_{0,b}(\phi)}\|v\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{2}$$

(see Corollary 2.1). Combined with

$$\|u\overline{v}\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b}(\phi)}\|v\|_{X_{0,b}(\phi)}, \qquad b > \frac{3}{8},$$

which follows from Strichartz' estimate (cf. Lemma 2.2), this gives

$$\|u\overline{v}\|_{L^2_t(H^{\frac{1}{2}}_x)} \leq c\|v\|_{X_{0,b_0}(\phi)}\|u\|_{X_{0,b}(\phi)}, \qquad b_0, b > \frac{1}{2}. \tag{20}$$

On the other hand, by Hölder and again by Strichartz' estimate we have

$$\|u\overline{v}\|_{L^2_{xt}} \leq c\|v\|_{L^6_{xt}}\|u\|_{L^3_{xt}} \leq c\|v\|_{X_{0,b_0}(\phi)}\|u\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{4}, b_0 > \frac{1}{2}. \tag{21}$$

Now, by interpolation between (20) and (21), we obtain part i). To see part ii), we interpolate (20) with

$$\|u\overline{v}\|_{L^4_t(L^2_x)} \leq \|v\|_{L^8_t(L^4_x)}\|u\|_{L^8_t(L^4_x)} \leq c\|v\|_{X_{0,b_0}(\phi)}\|u\|_{X_{0,b_0}(\phi)}, \qquad b_0 > \frac{1}{2},$$

which follows from the $L^8_t(L^4_x)$-Strichartz-estimate. Next we dualize part i) to obtain part iii) for $\sigma = 0$. For $\sigma < 0$, because of $\langle\xi_1\rangle \leq c\langle\xi\rangle\langle\xi_2\rangle$, we then have

$$\|vw\|_{X_{\sigma,b'}(\phi)} \leq c\|(J^\sigma v)(J^{-\sigma}w)\|_{X_{0,b'}(\phi)} \leq c\|v\|_{X_{\sigma,b_0}(\phi)}\|w\|_{L^2_t(H^{-s-\sigma}_x)}.$$

□

Remark : Taking $\sigma = -\frac{s}{2} \in (-\frac{1}{4}, 0]$ in part iii), we obtain Theorem 1.2 in [KPV96b].

In order to formulate and prove an analogue for Lemma 4.1 in the case of two unbared factors, we introduce some bilinear pseudodifferential operators:

Definition 4.1 *We define $I_-^s(f,g)$ by its Fourier-transform (in the space variable)*

$$\mathcal{F}_x I_-^s(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi} d\xi_1 |\xi_1-\xi_2|^s \mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2).$$

If the expression $|\xi_1-\xi_2|^s$ in the integral is replaced by $\langle\xi_1-\xi_2\rangle^s$, the corresponding operator will be called $J_-^s(f,g)$. Similarly we define $I_+^s(f,g)$ and $J_+^s(f,g)$ by

$$\mathcal{F}_x I_+^s(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi} d\xi_1 |\xi_1+2\xi_2|^s \mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2).$$

Remark (simple properties) :

i) For functions u, v depending on space- and time-variables we have

$$\mathcal{F} I_-^s(u,v)(\xi,\tau) := \int_{\substack{\xi_1+\xi_2=\xi \\ \tau_1+\tau_2=\tau}} d\xi_1 d\tau_1 |\xi_1-\xi_2|^s \mathcal{F}u(\xi_1,\tau_1)\mathcal{F}v(\xi_2,\tau_2)$$

and similar integrals for the other operators.

ii) $I_-^s(f,g)$ always coincides with $I_-^s(g,f)$ (and $J_-^s(f,g)$ with $J_-^s(g,f)$), since we can exchange ξ_1 and ξ_2 in the corresponding integral, while in general we will have $I_+^s(f,g) \neq I_+^s(g,f)$ (and $J_+^s(f,g) \neq J_+^s(g,f)$).

iii) Fixing u and s we define the linear operators M and N by

$$Mv := J_-^s(u,v) \qquad \text{and} \qquad Nw := J_+^s(w,\overline{u}).$$

Then it is easily checked that M and N are formally adjoint with respect to the inner product on L^2_{xt}.

Now we have the following bilinear Strichartz-type estimate:

Lemma 4.2

$$\|I_-^{\frac{1}{2}}(e^{it\partial^2}u_1, e^{it\partial^2}u_2)\|_{L^2_{xt}} \leq c\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}$$

Proof: We will write for short $\hat{u}$ instead of $\mathcal{F}_x u$ and $\int_* d\xi_1$ for $\int_{\xi_1+\xi_2=\xi} d\xi_1$. Then, using Fourier-Plancherel in the space variable we obtain:

$$\begin{aligned}
&\|I_-^{\frac{1}{2}}(e^{it\partial^2}u_1, e^{it\partial^2}u_2)\|^2_{L^2_{xt}} \\
= \; & c\int d\xi dt \left| \int_* d\xi_1 |\xi_1-\xi_2|^{\frac{1}{2}} e^{-it(\xi_1^2+\xi_2^2)} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\right|^2 \\
= \; & c\int d\xi dt \int_* d\xi_1 d\eta_1 e^{-it(\xi_1^2+\xi_2^2-\eta_1^2-\eta_2^2)} (|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
= \; & c\int d\xi \int_* d\xi_1 d\eta_1 \delta(\eta_1^2+\eta_2^2-\xi_1^2-\xi_2^2)(|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
= \; & c\int d\xi \int_* d\xi_1 d\eta_1 \delta(2(\eta_1^2-\xi_1^2+\xi(\xi_1-\eta_1)))(|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)}.
\end{aligned}$$

Now we use $\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n)$, where the sum is taken over all simple zeros of g, in our case:

$$g(x) = 2(x^2 + \xi(\xi_1 - x) - \xi_1^2)$$

with the zeros $x_1 = \xi_1$ and $x_2 = \xi - \xi_1$, hence $g'(x_1) = 2(2\xi_1 - \xi)$ respectively $g'(x_2) = 2(\xi - 2\xi_1)$. So the last expression is equal to

$$\begin{aligned}
& c\int d\xi \int_* d\xi_1 d\eta_1 \frac{1}{|2\xi_1 - \xi|}\delta(\eta_1 - \xi_1)(|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}}\prod_{i=1}^{2}\hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
+ \quad & c\int d\xi \int_* d\xi_1 d\eta_1 \frac{1}{|2\xi_1 - \xi|}\delta(\eta_1 - (\xi-\xi_1))(|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}}\prod_{i=1}^{2}\hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
= \quad & c\int d\xi \int_* d\xi_1 \prod_{i=1}^{2}|\hat{u}_i(\xi_i)|^2 + c\int d\xi\int_* d\xi_1 \hat{u}_1(\xi_1)\overline{\hat{u}_1}(\xi_2)\hat{u}_2(\xi_2)\overline{\hat{u}_2}(\xi_1) \\
\leq \quad & c(\prod_{i=1}^{2}\|u_i\|^2_{L^2_x} + \|\hat{u}_1\hat{u}_2\|^2_{L^1_\xi}) \leq c\prod_{i=1}^{2}\|u_i\|^2_{L^2_x}.
\end{aligned}$$

□

Corollary 4.1 *Let $b_0 > \frac{1}{2}$ and $0 \leq s \leq \frac{1}{2}$. Then the following estimates hold true:*

i) $\|J^s_-(u,v)\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b_0}(\phi)}\|v\|_{X_{0,b}(\phi)}$, *provided* $b > \frac{1}{4} + \frac{s}{2}$,

ii) $\|J^s_+(v,\overline{u})\|_{X_{0,b'}(\phi)} \leq c\|u\|_{X_{0,b_0}(\phi)}\|v\|_{L^2_{xt}}$, *provided* $b' < -\frac{1}{4} - \frac{s}{2}$.

Remark : In i) we may replace $J^s_-(u,v)$ by $J^s_-(\overline{u},\overline{v})$, in fact a short computation shows that $J^s_-(\overline{u},\overline{v}) = \overline{J^s_-(u,v)}$.

Proof: By Lemma 2.1 we obtain from the above estimate

$$\|I^{\frac{1}{2}}_-(u,v)\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b_0}(\phi)}\|v\|_{X_{0,b}(\phi)}, \qquad b, b_0 > \frac{1}{2}.$$

Combining this with

$$\|uv\|_{L^2_{xt}} \leq \|u\|_{L^6_{xt}}\|v\|_{L^3_{xt}} \leq c\|u\|_{X_{0,b_0}(\phi)}\|v\|_{X_{0,b}(\phi)}, \qquad b > \frac{1}{4}, b_0 > \frac{1}{2},$$

we obtain i) for $s = \frac{1}{2}$ and $s = 0$.

To see i) for $0 < s < \frac{1}{2}$, $b > \frac{1}{4} + \frac{s}{2}$, we write $w = \Lambda^b v$, where Λ^b is defined by $\mathcal{F}\Lambda^b v(\xi,\tau) = \langle\tau + \xi^2\rangle^b \mathcal{F}v(\xi,\tau)$. Then we have to show that

$$\|J^s_-(u,\Lambda^{-b}w)\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b_0}(\phi)}\|w\|_{L^2_{xt}}, \tag{22}$$

where

$$\|J^s_-(u,\Lambda^{-b}w)\|_{L^2_{xt}} = \|\int_{\substack{\tau_1+\tau_2=\tau \\ \xi_1+\xi_2=\xi}} \langle\xi_1 - \xi_2\rangle^s \mathcal{F}u(\xi_1,\tau_1)\langle\tau_2 + \xi_2^2\rangle^{-b}\mathcal{F}w(\xi_2,\tau_2)\|_{L^2_{\xi\tau}}.$$

Notice that, by the preceding, (22) is already known in the limiting cases $(s,b) = (0, \frac{1}{4}+\varepsilon)$ and $(s,b) = (\frac{1}{2}, \frac{1}{2}+\varepsilon)$, $\varepsilon > 0$. Choosing $\varepsilon = b - \frac{1}{4} - \frac{s}{2}$ we have

$$\langle \xi_1 - \xi_2 \rangle^s \langle \tau_2 + \xi_2^2 \rangle^{-b} \leq \langle \tau_2 + \xi_2^2 \rangle^{-\frac{1}{4}-\varepsilon} + \langle \xi_1 - \xi_2 \rangle^{\frac{1}{2}} \langle \tau_2 + \xi_2^2 \rangle^{-\frac{1}{2}-\varepsilon}$$

and hence

$$\|J_-^s(u, \Lambda^{-b} w)\|_{L^2_{xt}} \leq \|u(\Lambda^{-\frac{1}{4}-\varepsilon} w)\|_{L^2_{xt}} + \|J_-^{\frac{1}{2}}(u, \Lambda^{-\frac{1}{2}-\varepsilon} w)\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b_0}(\phi)} \|w\|_{L^2_{xt}}.$$

Finally, ii) follows from i) by duality (cf. part iii) of the remark on simple properties of J_-^s). □

In view on Bourgain's bilinear refinement of the L^4_{xt}-Strichartz-estimate (Lemma 2.5 and Corollary 2.2) and on the fact that the exponent in the onedimensional Strichartz' estimate is 6 the question for trilinear refinements of this estimate comes up naturally. Here we give a partial answer to this question, starting with the following application of Kato's smoothing effect:

Lemma 4.3 *Let* $0 \leq s \leq \frac{1}{4}$, $b > \frac{1}{2}$. *Then the estimate*

$$\|u_1 u_2 u_3\|_{L^2_{xt}} \leq c\|u_1\|_{X_{s,b}(\phi)} \|u_2\|_{X_{-s,b}(\phi)} \|u_3\|_{X_{0,b}(\phi)}$$

holds true.

Proof: For $s = 0$ this follows from standard Strichartz' estimate, for $s = \frac{1}{4}$ we argue as follows: Interpolation between the L^6-estimate and the Kato smoothing effect (part i) of Lemma 2.3) with $\theta = \frac{1}{2}$ yields

$$\|u_2\|_{L^{12}_x(L^3_t)} \leq c\|u_2\|_{X_{-\frac{1}{4},b}(\phi)}, \quad b > \frac{1}{2}.$$

On the other hand we have the maximal function estimate

$$\|u_1\|_{L^4_x(L^\infty_t)} \leq c\|u_1\|_{X_{\frac{1}{4},b}(\phi)}, \quad b > \frac{1}{2},$$

see part ii) of Lemma 2.3. Combining this with Hölder's inequality and standard Strichartz we obtain

$$\begin{aligned} \|u_1 u_2 u_3\|_{L^2_{xt}} &\leq c\|u_1\|_{L^4_x(L^\infty_t)} \|u_2\|_{L^{12}_x(L^3_t)} \|u_3\|_{L^6_{xt}} \\ &\leq c\|u_1\|_{X_{\frac{1}{4},b}(\phi)} \|u_2\|_{X_{-\frac{1}{4},b}(\phi)} \|u_3\|_{X_{0,b}(\phi)}, \end{aligned}$$

which is the claim for $s = \frac{1}{4}$. For $0 < s < \frac{1}{4}$ the result then follows by multilinear interpolation, see Thm. 4.4.1 in [BL]. □

Remark : An alternative proof of Lemma 4.3 (up to ε's) not using the Kato effect is given in Appendix A1.

Problem: Does the above estimate hold for $\frac{1}{4} < s < \frac{1}{2}$?

Corollary 4.2 *Assume $0 \le s \le \frac{1}{4}$ and $b > \frac{1}{2}$. Let $\tilde{u}$ denote u or $\overline{u}$. Then the following estimates are valid:*

i) $\|\tilde{u}_1\tilde{u}_2\tilde{u}_3\|_{L^2_{xt}} \le c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{-s,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$,

ii) $\|\tilde{u}_1\tilde{u}_2\tilde{u}_3\|_{X_{-s,-b}(\phi)} \le c\|u_1\|_{L^2_{xt}}\|u_2\|_{X_{-s,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$,

iii) $\|\tilde{u}_1\tilde{u}_2\tilde{u}_3\|_{L^2_t(H^s_x)} \le c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{0,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$,

iv) $\|\tilde{u}_1\tilde{u}_2\tilde{u}_3\|_{X_{-s,-b}(\phi)} \le c\|u_1\|_{L^2_t(H^{-s}_x)}\|u_2\|_{X_{0,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$.

Proof: Clearly, in $\|u_1u_2u_3\|_{L^2_{xt}}$ any factor u_i may be replaced by $\overline{u}_i$. This gives i). From this we obtain ii) by duality. Writing $\langle\xi\rangle \le \langle\xi_1\rangle + \langle\xi_2\rangle + \langle\xi_3\rangle$ and applying i) twice (plus standard Strichartz), part iii) can be seen. Dualizing again, part iv) follows. □

In some cases, using the bilinear estimates in Lemma 4.1 and in Corollary 4.1, we can prove better $L^2_t(H^s_x)$-estimates:

Lemma 4.4 *i) For $|s| < \frac{1}{2} < b$ the following estimate holds:*

$$\|u_1\overline{u}_2u_3\|_{L^2_t(H^s_x)} \le c\|u_1\|_{X_{0,b}(\phi)}\|u_2\|_{X_{0,b}(\phi)}\|u_3\|_{X_{s,b}(\phi)}$$

ii) For $-\frac{1}{2} < s \le 0$, $b > \frac{1}{2}$ the following is valid:

$$\|u_1\overline{u}_2u_3\|_{L^2_t(H^s_x)} \le c\|u_1\|_{X_{0,b}(\phi)}\|u_2\|_{X_{s,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$$

Remark : Using multilinear interpolation (Thm. 4.4.1 in [BL]) we obtain

$$\|u_1\overline{u}_2u_3\|_{L^2_t(H^s_x)} \le c\|u_1\|_{X_{s_1,b}(\phi)}\|u_2\|_{X_{s_2,b}(\phi)}\|u_3\|_{X_{s_3,b}(\phi)},$$

provided $-\frac{1}{2} < s \le 0$, $b > \frac{1}{2}$, $s_{1,2,3} \le 0$ and $s_1 + s_2 + s_3 = s$. Moreover, we may replace $u_1\overline{u}_2u_3$ on the left hand side by $\overline{u}_1u_2\overline{u}_3$.

Proof: First we show i) for $s > 0$. From $\langle\xi\rangle \le c(\langle\xi_1+\xi_2\rangle + \langle\xi_3\rangle)$ it follows that

$$\|u_1\overline{u}_2u_3\|_{L^2_t(H^s_x)} \le c\|J^s(u_1\overline{u}_2)u_3\|_{L^2_{xt}} + \|u_1\overline{u}_2J^su_3\|_{L^2_{xt}} =: c(N_1+N_2).$$

Using the standard L^6_{xt}-Strichartz-estimate we see that N_2 is bounded by the right hand side of i). For N_1 we have with $s = \frac{1}{p}$, $\frac{1}{2} - s = \frac{1}{q}$ ($\Rightarrow H^s \subset L^q$, $H^{\frac{1}{2}} \subset H^{s,p}$):

$$\begin{aligned} N_1 &\le c\|J^s(u_1\overline{u}_2)\|_{L^2_t(L^p_x)}\|u_3\|_{L^\infty_t(L^q_x)} \\ &\le c\|u_1\overline{u}_2\|_{L^2_t(H^{\frac{1}{2}}_x)}\|u_3\|_{L^\infty_t(H^s_x)} \\ &\le c\|u_1\|_{X_{0,b}(\phi)}\|u_2\|_{X_{0,b}(\phi)}\|u_3\|_{X_{s,b}(\phi)} \end{aligned}$$

by Lemma 4.1, part i), and the Sobolev embedding in the time variable.

Next we consider i) for $s < 0$. Writing $\langle \xi_3 \rangle \leq c(\langle \xi \rangle + \langle \xi_1 + \xi_2 \rangle)$, we obtain

$$\|u_1 \overline{u}_2 u_3\|_{L^2_t(H^s_x)} \leq c\|u_1 \overline{u}_2 J^s u_3\|_{L^2_{xt}} + \|J^{-s}(u_1 \overline{u}_2) J^s u_3\|_{L^2_t(H^s_x)} =: c(N_1 + N_2).$$

To estimate N_1 we use again the standard L^6_{xt}-Strichartz estimate. For N_2 we use the embedding $L^q \subset H^s$, $s - \frac{1}{2} = -\frac{1}{q}$ and Hölder's inequality:

$$\begin{aligned} N_2 &\leq c\|J^{-s}(u_1 \overline{u}_2) J^s u_3\|_{L^2_t(L^q_x)} \\ &\leq c\|J^{-s}(u_1 \overline{u}_2)\|_{L^2_t(L^p_x)} \|u_3\|_{L^\infty_t(H^s_x)}, \end{aligned}$$

where $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$. The second factor is bounded by $c\|u_3\|_{X_{s,b}(\phi)}$ because of Sobolev's embedding Theorem in the time variable. For the first factor we use the embedding $H^{\frac{1}{2}} \subset H^{-s,p}$ (observe that $s = -\frac{1}{p}$) and again Lemma 4.1, i).

We conclude the proof by showing ii): Here we have $\xi = (\xi_1 + \xi_2) + (\xi_3 + \xi_2) - \xi_2$ respectively $\langle \xi_2 \rangle \leq c(\langle \xi \rangle + \langle \xi_1 + \xi_2 \rangle + \langle \xi_3 + \xi_2 \rangle)$ and thus

$$\|u_1 \overline{u}_2 u_3\|_{L^2_t(H^s_x)} \leq c(N_1 + N_2 + N_3)$$

with

$$N_1 = \|u_1 (J^s \overline{u}_2) u_3\|_{L^2_{xt}} \leq c\|u_1\|_{X_{0,b}(\phi)} \|u_2\|_{X_{s,b}(\phi)} \|u_3\|_{X_{0,b}(\phi)}$$

(by standard Strichartz) and

$$N_2 = \|J^{-s}(u_1 J^s \overline{u}_2) u_3\|_{L^2_t(H^s_x)}, \quad N_3 = \|u_1 J^{-s}((J^s \overline{u}_2) u_3)\|_{L^2_t(H^s_x)}.$$

By symmetry between u_1 and u_3 it is now sufficient to estimate N_2: Using the embedding $L^q \subset H^s$, $s - \frac{1}{2} = -\frac{1}{q}$, Hölder's inequality and the embedding $H^{\frac{1}{2}} \subset H^{-s,p}$, $-s = \frac{1}{p}$ we obtain

$$\begin{aligned} N_2 &\leq c\|J^{-s}(u_1 J^s \overline{u}_2) u_3\|_{L^2_t(L^q_x)} \\ &\leq c\|J^{-s}(u_1 J^s \overline{u}_2)\|_{L^2_t(L^p_x)} \|u_3\|_{L^\infty_t(L^2_x)} \\ &\leq c\|J^{\frac{1}{2}}(u_1 J^s \overline{u}_2)\|_{L^2_{xt}} \|u_3\|_{L^\infty_t(L^2_x)}. \end{aligned}$$

Again, Lemma 4.1, i) and the Sobolev embedding in t give the desired bound. □

Lemma 4.5 *For $-\frac{1}{2} < s \leq 0$, $b > \frac{1}{2}$ the following holds true:*

$$\|u_1 u_2 u_3\|_{L^2_t(H^s_x)} \leq c\|u_1\|_{X_{s,b}(\phi)} \|u_2\|_{X_{0,b}(\phi)} \|u_3\|_{X_{0,b}(\phi)}$$

Remark : Again we may use multilinear interpolation to get

$$\|u_1 u_2 u_3\|_{L^2_t(H^s_x)} \leq c\|u_1\|_{X_{s_1,b}(\phi)} \|u_2\|_{X_{s_2,b}(\phi)} \|u_3\|_{X_{s_3,b}(\phi)}$$

for $-\frac{1}{2} < s \leq 0$, $b > \frac{1}{2}$, $s_{1,2,3} \leq 0$ and $s_1 + s_2 + s_3 = s$. The same holds true with $u_1 u_2 u_3$ replaced by $\overline{u}_1 \overline{u}_2 \overline{u}_3$.

Proof: It is easily checked that for $\rho, \lambda \geq 0$ the inequality

$$\langle \xi_1 \rangle^\rho \leq c(\langle \xi \rangle^\rho + \frac{\langle \xi_1 - \xi_2 \rangle^{\rho+\lambda}}{\langle \xi_1 + \xi_2 \rangle^\lambda} + \frac{\langle \xi_1 - \xi_3 \rangle^{\rho+\lambda}}{\langle \xi_1 + \xi_3 \rangle^\lambda})$$

is valid, if $\xi = \xi_1 + \xi_2 + \xi_3$. Choosing $\rho = -s$ and $\lambda = s + \frac{1}{2}$ it follows, that

$$\|u_1 u_2 u_3\|_{L^2_t(H^s_x)} \leq c(N_1 + N_2 + N_3),$$

where

$$N_1 = \|(J^s u_1) u_2 u_3\|_{L^2_{xt}} \leq c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{0,b}(\phi)}\|u_3\|_{X_{0,b}(\phi)}$$

(by standard Strichartz) and

$$N_2 = \|(J^{-\lambda} J_-^{\frac{1}{2}}(J^s u_1, u_2)) u_3\|_{L^2_t(H^s_x)}, \quad N_3 = \|(J^{-\lambda} J_-^{\frac{1}{2}}(J^s u_1, u_3)) u_2\|_{L^2_t(H^s_x)}.$$

Now, by symmetry between u_2 and u_3, it is sufficient to estimate N_2. Using the embedding $L^q \subset H^s$, $(s - \frac{1}{2} = -\frac{1}{q})$ and Hölder we get

$$\begin{aligned} N_2 &\leq c\|J^{-\lambda} J_-^{\frac{1}{2}}(J^s u_1, u_2) u_3\|_{L^2_t(L^q_x)} \\ &\leq c\|J^{-\lambda} J_-^{\frac{1}{2}}(J^s u_1, u_2)\|_{L^2_t(L^p_x)}\|u_3\|_{L^\infty_t(L^2_x)} \end{aligned}$$

with $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$. The second factor is bounded by $c\|u_3\|_{X_{0,b}(\phi)}$. For the first factor we observe that $L^2 \subset H^{-\lambda,p}$, so it can be estimated by

$$\|J_-^{\frac{1}{2}}(J^s u_1, u_2)\|_{L^2_{xt}} \leq c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{0,b}(\phi)},$$

where in the last step we have used Corollary 4.1, i). □

4.3 Estimates on quadratic and cubic nonlinearities

Theorem 4.4 *Let $n = 1, m = 3$ or $n = 2, m = 2$. Assume $0 \geq s > -\frac{1}{m}$ and $-\frac{1}{2} < b' < \frac{ms}{2}$. Then in the periodic and nonperiodic case for all $b > \frac{1}{2}$ the estimate*

$$\|\prod_{i=1}^m \overline{u}_i\|_{X_{0,b'}(\phi)} \leq c\prod_{i=1}^m \|u_i\|_{X_{s,b}(\phi)}$$

holds true.

Proof: Defining $f_i(\xi,\tau) = \langle \tau - |\xi|^2\rangle^b \langle \xi\rangle^s \mathcal{F}\overline{u}_i(\xi,\tau)$, $1 \leq i \leq m$, we have

$$\|\prod_{i=1}^m \overline{u}_i\|_{X_{0,b'}(\phi)} = c\|\langle \tau + |\xi|^2\rangle^{b'} \int d\nu \prod_{i=1}^m \langle \tau_i - |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}.$$

Because of

$$\tau + |\xi|^2 - \sum_{i=1}^m (\tau_i - |\xi_i|^2) = |\xi|^2 + \sum_{i=1}^m |\xi_i|^2$$

there is the inequality

$$\begin{aligned} \langle \xi\rangle^2 + \sum_{i=1}^m \langle \xi_i\rangle^2 &\leq \langle \tau + |\xi|^2\rangle + \sum_{i=1}^m \langle \tau_i - |\xi_i|^2\rangle \\ &\leq c(\langle \tau + |\xi|^2\rangle + \sum_{i=1}^m \langle \tau_i - |\xi_i|^2\rangle \chi_{A_i}), \end{aligned}$$

where in A_i we have $\langle \tau_i - |\xi_i|^2\rangle \geq \langle \tau + |\xi|^2\rangle$. Since $b' < \frac{ms}{2}$ is assumed, it follows

$$\langle \xi\rangle^{\varepsilon}\prod_{i=1}^{m}\langle \xi_i\rangle^{-s+\varepsilon} \leq c(\langle \tau + |\xi|^2\rangle^{-b'} + \sum_{i=1}^{m}\langle \tau_i - |\xi_i|^2\rangle^{-b'}\chi_{A_i})$$

for some $\varepsilon > 0$. From this we conclude that

$$\| \textstyle\prod_{i=1}^{m} \overline{u}_i\|_{X_{0,b'}(\phi)} \leq c\sum_{j=0}^{m}\|I_j\|_{L^2_{\xi,\tau}},$$

with

$$I_0(\xi,\tau) = \langle \xi\rangle^{-\varepsilon}\int d\nu \prod_{i=1}^{m}\langle \tau_i - |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i)$$

and, for $1 \leq j \leq m$,

$$\begin{aligned} I_j(\xi,\tau) &= \langle \xi\rangle^{-\varepsilon}\langle \tau + |\xi|^2\rangle^{b'}\int d\nu \langle \tau_j - |\xi_j|^2\rangle^{-b'}\prod_{i=1}^{m}\langle \tau_i - |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i)\chi_{A_j} \\ &\leq \langle \xi\rangle^{-\varepsilon}\langle \tau + |\xi|^2\rangle^{-b}\int d\nu \langle \tau_j - |\xi_j|^2\rangle^{b}\prod_{i=1}^{m}\langle \tau_i - |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i). \end{aligned}$$

To estimate I_0 we use Hölders inequality and Lemma 2.15 respectively Lemma 2.16:

$$\begin{aligned} \|I_0\|_{L^2_{\xi,\tau}} &\leq \| \textstyle\int d\nu \prod_{i=1}^{m}\langle \tau_i - |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ &= c\|\textstyle\prod_{i=1}^{m} J^{s-\varepsilon}\overline{u}_i\|_{L^2_{x,t}} \leq c\displaystyle\prod_{i=1}^{m}\|J^{s-\varepsilon}\overline{u}_i\|_{L^{2m}_{x,t}} \\ &\leq c\prod_{i=1}^{m}\|J^{s}\overline{u}_i\|_{X_{0,b}(-\phi)} = c\prod_{i=1}^{m}\|\overline{u}_i\|_{X_{s,b}(-\phi)}. \end{aligned}$$

To estimate I_j, $1 \leq j \leq m$, we define $p = 2m$ and p' by $\frac{1}{p} + \frac{1}{p'} = 1$. Then we use the dual versions of Lemma 2.15 respectively 2.16, Hölders inequality and the Lemmas themselves to obtain:

$$\begin{aligned} \|I_j\|_{L^2_{\xi,\tau}} &\leq c\|(\textstyle\prod_{\substack{i=1\\ i\neq j}}^{m} J^{s-\varepsilon}\overline{u}_i)(J^{-\varepsilon}\mathcal{F}^{-1}f_j)\|_{X_{-\varepsilon,-b}(\phi)} \\ &\leq c\|(\textstyle\prod_{\substack{i=1\\ i\neq j}}^{m} J^{s-\varepsilon}\overline{u}_i)(J^{-\varepsilon}\mathcal{F}^{-1}f_j)\|_{L^{p'}_{x,t}} \\ &\leq c\|J^{-\varepsilon}\mathcal{F}^{-1}f_j\|_{L^2_{x,t}}\prod_{\substack{i=1\\ i\neq j}}^{m}\|J^{s-\varepsilon}\overline{u}_i\|_{L^p_{x,t}} \\ &\leq c\|f_j\|_{L^2_{\xi,\tau}}\prod_{\substack{i=1\\ i\neq j}}^{m}\|J^{s}\overline{u}_i\|_{X_{0,b}(-\phi)} = c\prod_{i=1}^{m}\|\overline{u}_i\|_{X_{s,b}(-\phi)} \end{aligned}$$

□

Theorem 4.5 *Let $n = 3$ and assume $0 \geq s > -\frac{3}{10}$, $-\frac{1}{2} < b' < \frac{s}{2} - \frac{7}{20}$ and $b > \frac{1}{2}$. Then in the periodic case the estimate*

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{s,b'}(\phi)} \leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}$$

holds true.

Proof: Writing $f_i(\xi, \tau) = \langle \tau - |\xi|^2 \rangle^b \langle \xi \rangle^s \mathcal{F}\overline{u}_i(\xi, \tau)$, $1 \leq i \leq 2$, we have

$$\| \prod_{i=1}^{2} \overline{u}_i \|_{X_{s,b'}(\phi)} = c \| \langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^{b'} \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} \langle \xi_i \rangle^{-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}}.$$

By the expressions $\langle \tau + |\xi|^2 \rangle$ and $\langle \tau_i - |\xi_i|^2 \rangle$, $i = 1, 2$, the quantity $\langle \xi \rangle^2 + \langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2$ can be controlled. So we split the domain of integration into $A_0 + A_1 + A_2$, where in A_0 we have $\langle \tau + |\xi|^2 \rangle = \max(\langle \tau + |\xi|^2 \rangle, \langle \tau_1 - |\xi_1|^2 \rangle, \langle \tau_2 - |\xi_2|^2 \rangle)$ and in A_j, $j = 1, 2$, it should hold that $\langle \tau_j - |\xi_j|^2 \rangle = \max(\langle \tau + |\xi|^2 \rangle, \langle \tau_1 - |\xi_1|^2 \rangle, \langle \tau_2 - |\xi_2|^2 \rangle)$. First we consider the region A_0: Here we use that for $\varepsilon > 0$ sufficiently small

$$\langle \xi \rangle^{\frac{3}{10}+s} \prod_{i=1}^{2} \langle \xi_i \rangle^{-s+\frac{1}{5}+\varepsilon} \leq c \langle \tau + |\xi|^2 \rangle^{-b'}.$$

This gives the upper bound

$$\begin{aligned} &\| \langle \xi \rangle^{-\frac{3}{10}} \int d\nu \prod_{i=1}^{2} \langle \tau_i - |\xi_i|^2 \rangle^{-b} \langle \xi_i \rangle^{-\frac{1}{5}-\varepsilon} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ &\quad = c \| \prod_{i=1}^{2} J^{s-\frac{1}{5}-\varepsilon} \overline{u}_i \|_{L^2_t(H_x^{-\frac{3}{10}})}. \end{aligned}$$

Now, using the embedding $L^q_x \subset H_x^{-\frac{3}{10}}$, $\frac{1}{q} = \frac{3}{5}$, Hölder's inequality and Corollary 2.5, part b) (with $p = 4$, $q = \frac{10}{3}$, $s > \frac{1}{5}$ and $b > \frac{9}{20}$), we get the following chain of inequalities:

$$\begin{aligned} \| \prod_{i=1}^{2} J^{s-\frac{1}{5}-\varepsilon} \overline{u}_i \|_{L^2_t(H_x^{-\frac{3}{10}})} &\leq c \| \prod_{i=1}^{2} J^{s-\frac{1}{5}-\varepsilon} \overline{u}_i \|_{L^2_t(L^q_x)} \\ &\leq c \| J^{s-\frac{1}{5}-\varepsilon} u_1 \|_{L^4_t(L^{2q}_x)} \| J^{s-\frac{1}{5}-\varepsilon} u_2 \|_{L^4_t(L^{2q}_x)} \\ &\leq c \prod_{i=1}^{2} \|u_i\|_{X_{s,b}(\phi)}. \end{aligned}$$

Now, by symmetry, it only remains to show the estimate for the region A_1: Here we use

$$\langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^{b+b'} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s+\frac{1}{4}+\varepsilon} \leq c \langle \xi \rangle^{-\frac{1}{4}-\varepsilon} \langle \tau_1 - |\xi_1|^2 \rangle^b$$

to obtain the upper bound

$$\begin{aligned} &\| \langle \xi \rangle^{-\frac{1}{4}-\varepsilon} \langle \tau + |\xi|^2 \rangle^{-b} \int d\nu f_1(\xi_1, \tau_1) \langle \xi_2 \rangle^{-\frac{1}{4}-\varepsilon} \langle \tau_2 - |\xi_2|^2 \rangle^{-b} f_2(\xi_2, \tau_2) \|_{L^2_{\xi,\tau}} \\ &\quad = c \| (\mathcal{F}^{-1} f_1)(J^{s-\frac{1}{4}-\varepsilon} u_2) \|_{X_{-\frac{1}{4}-\varepsilon,-b}(\phi)}, \end{aligned}$$

where $\|f_1\|_{L^2_{\xi,\tau}} = \|\mathcal{F}^{-1}f_1\|_{L^2_{x,t}} = \|u_1\|_{X_{s,b}(\phi)}$. Now we use the dual form of Lemma 2.16, Hölder's inequality and the Lemma itself to obtain

$$\begin{aligned}
\|\mathcal{F}^{-1}f_1 J^{s-\frac{1}{4}-\varepsilon}u_2\|_{X_{-\frac{1}{4}-\varepsilon,-b}(\phi)} &\leq c\|\mathcal{F}^{-1}f_1 J^{s-\frac{1}{4}-\varepsilon}u_2\|_{L^{\frac{4}{3}}_{xt}} \\
&\leq c\|\mathcal{F}^{-1}f_1\|_{L^2_{xt}}\|J^{s-\frac{1}{4}-\varepsilon}u_2\|_{L^4_{xt}} \\
&\leq c\prod_{i=1}^{2}\|u_i\|_{X_{s,b}(\phi)}.
\end{aligned}$$

□

Remark : In the nonperiodic case we can combine the argument given above with the $L^4_t(L^3_x)$-Strichartz-estimate to obtain the estimate in question whenever $s > -\frac{1}{2}$, $b' < \frac{s}{2} - \frac{1}{4}$, $b > \frac{1}{2}$, see Example 2.2. As far as I know, it is still an open question, whether or not the analogue of this Strichartz-estimate, that is

$$X_{\varepsilon,b}(\phi) \subset L^4_t(\mathbf{R}, L^3_x(\mathbf{T}^3)), \qquad b > \frac{1}{2},\ \varepsilon > 0$$

holds in the periodic case. This, of course, could be used to lower the bound on s in the above theorem down to $-\frac{1}{2} + \varepsilon$.

Before we turn to the cubic nonlinearities in the continuous case, let us briefly discuss some counterexamples concerning the periodic case: The examples given by Kenig, Ponce and Vega connected with the onedimensional periodic case (see the proof of Thm 1.10, parts (ii) and (iii) in [KPV96b]) show that the estimate

$$\|u_1\overline{u}_2\|_{X_{s,b'}(\phi)} \leq c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{s,b}(\phi)}$$

fails for all $s < 0$, $b, b' \in \mathbf{R}$, and that the estimate

$$\|\overline{u}_1\overline{u}_2\|_{X_{s,b'}(\phi)} \leq c\|u_1\|_{X_{s,b}(\phi)}\|u_2\|_{X_{s,b}(\phi)}$$

fails for all $s < -\frac{1}{2}$, if $b - b' \leq 1$. From this we can conclude by the method of descent, that these estimates also fail in higher dimensions. So our estimate on $\overline{u}_1\overline{u}_2$ is sharp (up to the endpoint), while in three dimensions the estimate might be improved (as indicated above), and for $u_1\overline{u}_2$ no results with $s < 0$ can be achieved by the method. For the bilinear form $B(u_1, u_2) = u_1u_2$ in the two- and threedimensional periodic setting we have the following counterexample exhibiting a significant difference between the periodic and nonperiodic case (cf. the results in [CDKS01] and [T00] mentioned in 4.1):

Example 4.1 *In the periodic case in space dimension $d \geq 2$ the estimate*

$$\|\textstyle\prod_{i=1}^{2} u_i\|_{X_{s,b'}(\phi)} \leq c\displaystyle\prod_{i=1}^{2}\|u_i\|_{X_{s,b}(\phi)}$$

fails for all $s < 0$, $b, b' \in \mathbf{R}$.

Proof: The above estimate implies

$$\|\langle \tau + |\xi|^2\rangle^{b'}\langle \xi\rangle^s \int d\nu \prod_{i=1}^2 \langle \tau_i + |\xi_i|^2\rangle^{-b}\langle \xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c\prod_{i=1}^{2}\|f_i\|_{L^2_{\xi,\tau}}.$$

Choosing two orthonormal vectors e_1 and e_2 in $\mathbf{R}^d$ and defining for $n \in \mathbf{N}$

$$f_1^{(n)}(\xi,\tau) = \delta_{\xi,ne_1}\chi(\tau+n^2), \quad f_2^{(n)}(\xi,\tau) = \delta_{\xi,ne_2}\chi(\tau+n^2),$$

where χ is the characteristic function of $[-1,1]$, we have $\|f_i^{(n)}\|_{L^2_{\xi,\tau}} = c$ and it would follow that

$$n^{-2s}\|\langle \tau + |\xi|^2\rangle^{b'}\langle \xi\rangle^s \int d\nu \prod_{i=1}^2 f_i^{(n)}(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c. \tag{23}$$

Now a simple computation shows that

$$\int d\nu \prod_{i=1}^{2} f_i^{(n)}(\xi_i,\tau_i) \geq \delta_{\xi,n(e_1+e_2)}\chi(\tau+2n^2),$$

which inserted into (23) gives $n^{-s} \leq c$. This is a contradiction for all $s<0$.

□

The next example shows that our estimate on $\overline{u}_1\overline{u}_2\overline{u}_3$ is essentially sharp:

Example 4.2 *In the periodic case in one space dimension the estimate*

$$\|\prod_{i=1}^3 \overline{u}_i\|_{X_{s,b'}(\phi)} \leq c\prod_{i=1}^{3}\|u_i\|_{X_{s,b}(\phi)}$$

fails for all $s < -\frac{1}{3}$, *if* $b - b' \leq 1$.

Proof: From the above estimate we obtain

$$\|\langle \tau + \xi^2\rangle^{b'}\langle \xi\rangle^s \int d\nu \prod_{i=1}^3 \langle \tau_i - \xi_i^2\rangle^{-b}\langle \xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c\prod_{i=1}^{3}\|f_i\|_{L^2_{\xi,\tau}}.$$

Then for $n \in \mathbf{N}$ we define

$$f_{1,2}^{(n)}(\xi,\tau) = \delta_{\xi,n}\chi(\tau-n^2), \quad f_3^{(n)}(\xi,\tau) = \delta_{\xi,-2n}\chi(\tau-4n^2),$$

with χ as in the previous example. Again we have $\|f_i^{(n)}\|_{L^2_{\xi,\tau}} = c$ and

$$n^{-3s}\|\langle \tau + \xi^2\rangle^{b'}\langle \xi\rangle^s \int d\nu \prod_{i=1}^3 f_i^{(n)}(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c.$$

Now it can be easily checked that

$$\int d\nu \prod_{i=1}^{3} f_i^{(n)}(\xi_i,\tau_i) \geq \delta_{\xi,0}\chi(\tau-6n^2).$$

This leads to $n^{-3s+2b'} \leq c$ respectively to $\frac{2}{3}b' \leq s$. Consider next the following sequences of functions

$$g_1^{(n)}(\xi,\tau) = \delta_{\xi,n}\chi(\tau+5n^2), \quad g_2^{(n)}(\xi,\tau) = \delta_{\xi,n}\chi(\tau-n^2), \quad g_3^{(n)}(\xi,\tau) = \delta_{\xi,-2n}\chi(\tau-4n^2).$$

Arguing as before we are lead to the restriction $-\frac{2}{3}b \leq s$. Adding up these two restrictions and taking into account that $b - b' \leq 1$ we arrive at $s \geq -\frac{1}{3}$. □

For all the other cubic nonlinearities the corresponding estimates fail for $s < 0$, $b, b' \in \mathbf{R}$, see the examples 4.3 and 4.4 in the next section as well as the remarks below. Next we consider the cubic nonlinearities in the continuous case:

Theorem 4.6 *In the nonperiodic case in one space dimension the estimates*

$$\| \textstyle\prod_{i=1}^3 \overline{u}_i \|_{X_{\sigma,b'}(\phi)} \leq c \displaystyle\prod_{i=1}^3 \|u_i\|_{X_{s,b}(\phi)} \tag{24}$$

and

$$\| \textstyle\prod_{i=1}^3 u_i \|_{X_{\sigma,b'}(\phi)} \leq c \displaystyle\prod_{i=1}^3 \|u_i\|_{X_{s,b}(\phi)} \tag{25}$$

hold, provided $0 \geq s > -\frac{5}{12}$, $-\frac{1}{2} < b' < \frac{1}{2}(\frac{1}{4} + 3s)$, $\sigma < \min(0, 3s - 2b')$, $b' \leq s$ *and* $b > \frac{1}{2}$.

Proof: 1. To show (24), we write $f_i(\xi,\tau) = \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s \mathcal{F}u_i(\xi,\tau)$, $1 \leq i \leq 3$. Then we have

$$\begin{aligned} & \| \textstyle\prod_{i=1}^3 \overline{u}_i \|_{X_{\sigma,b'}(\phi)} = \| \prod_{i=1}^3 u_i \|_{X_{\sigma,b'}(-\phi)} \\ = \; & \| \langle \tau - \xi^2 \rangle^{b'} \langle \xi \rangle^\sigma \textstyle\int d\nu \prod_{i=1}^3 \langle \tau_i + \xi_i^2 \rangle^{-b} \langle \xi_i \rangle^{-s} f_i(\xi_i,\tau_i) \|_{L^2_{\xi,\tau}}. \end{aligned}$$

For $0 \leq \alpha, \beta, \gamma$ with $\alpha + \beta + \gamma = 2$ we have the inequality

$$\langle \xi_1 \rangle^\alpha \langle \xi_2 \rangle^\beta \langle \xi_3 \rangle^\gamma \leq \langle \xi \rangle^2 + \sum_{i=1}^3 \langle \xi_i \rangle^2 \leq c(\langle \tau - \xi^2 \rangle + \sum_{i=1}^3 \langle \tau_i + \xi_i^2 \rangle \chi_{A_i}),$$

where in A_i the expression $\langle \tau_i + \xi_i^2 \rangle$ is dominant. Hence

$$\| \textstyle\prod_{i=1}^3 \overline{u}_i \|_{X_{\sigma,b'}(\phi)} \leq c \displaystyle\sum_{k=0}^3 N_k$$

with

$$\begin{aligned} N_0 \; = \; & \| \langle \xi \rangle^\sigma \textstyle\int d\nu \prod_{i=1}^3 \langle \tau_i + \xi_i^2 \rangle^{-b} \langle \xi_i \rangle^{\frac{2b'}{3} - s} f_i(\xi_i,\tau_i) \|_{L^2_{\xi,\tau}} \\ = \; & c \| \textstyle\prod_{i=1}^3 J^{\frac{2b'}{3}} u_i \|_{L^2_t(H^\sigma_x)} \leq c \displaystyle\prod_{i=1}^3 \| J^{\frac{2b'}{3}} u_i \|_{X_{\frac{\sigma}{3},b}(\phi)} \leq c \prod_{i=1}^3 \|u_i\|_{X_{s,b}(\phi)}, \end{aligned}$$

where we have used Lemma 4.5 and the assumption $\sigma \leq 3s - 2b'$. Next we estimate N_1 by

$$\begin{aligned} & \|\langle\tau - \xi^2\rangle^{b'}\langle\xi\rangle^{\sigma}\int d\nu \prod_{i=1}^3\langle\tau_i + \xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\chi_{A_1}\|_{L^2_{\xi,\tau}} \\ \leq\ & c\|\langle\tau - \xi^2\rangle^{-b}\langle\xi\rangle^{\sigma}\int d\nu\langle\xi_1\rangle^{2b'-3s}f_1(\xi_1,\tau_1)\prod_{i=2}^3\langle\tau_i + \xi_i^2\rangle^{-b}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ =\ & c\|(\Lambda^b J^{2b'-2s}u_1)(J^s u_2)(J^s u_3)\|_{X_{\sigma,-b}(-\phi)}, \end{aligned}$$

where $\Lambda^b = \mathcal{F}^{-1}\langle\tau + \xi^2\rangle^b\mathcal{F}$. By part iv) of Corollary 4.2 this is bounded by

$$\begin{aligned} & c\|\Lambda^b J^{2b'-2s}u_1\|_{L^2_t(H^\sigma_x)}\|u_2\|_{X_{s,b}(\phi)}\|u_3\|_{X_{s,b}(\phi)} \\ =\ & c\|u_1\|_{X_{2b'-2s+\sigma,b}(\phi)}\|u_2\|_{X_{s,b}(\phi)}\|u_3\|_{X_{s,b}(\phi)} \leq c\prod_{i=1}^3\|u_i\|_{X_{s,b}(\phi)}, \end{aligned}$$

since $2b' - 2s + \sigma \leq s$. To estimate N_k for $k = 2, 3$ one only has to exchange the indices 1 and k. Now (24) is shown.

2. Now we prove the second estimate: With f_i as above we have

$$\|\prod_{i=1}^3 u_i\|_{X_{\sigma,b'}(\phi)} = c\|\langle\tau + \xi^2\rangle^{b'}\langle\xi\rangle^{\sigma}\int d\nu\prod_{i=1}^3\langle\tau_i + \xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}.$$

Here the quantity, which can be controlled by the expressions $\langle\tau + \xi^2\rangle$, $\langle\tau_i + \xi_i^2\rangle$, $1 \leq i \leq 3$, is

$$c.q. := |\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi^2|.$$

So we divide the domain of integration into two parts A and A^c, where in A it should hold that

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi^2 \leq c\ \ c.q.\ .$$

Then concerning this region we can argue precisely as in the first part of this proof. For the region A^c we may assume by symmetry that $\xi_1^2 \geq \xi_2^2 \geq \xi_3^2$. Then it is easily checked that in A^c we have

$$1.\ \xi^2 \geq \frac{1}{2}\xi_1^2 \geq \frac{1}{2}\xi_2^2 \qquad \text{and} \qquad 2.\ \xi_3^2 \leq \xi_1^2 \leq c(\xi_1 \pm \xi_3)^2.$$

From this it follows

$$\prod_{i=1}^3\langle\xi_i\rangle^{-s} \leq c\langle\xi\rangle^{-\sigma}\langle\xi_1 + \xi_3\rangle^{-s_0}\langle\xi_1 - \xi_3\rangle^{\frac{1}{2}}$$

for $s_0 = \frac{1}{2} + 2b' + \varepsilon$, so that $-3s \leq -\sigma - s_0 + \frac{1}{2} = -\sigma - 2b' - \varepsilon$ for ε sufficiently small. Hence

$$\begin{aligned} & \|\langle\tau + \xi^2\rangle^{b'}\langle\xi\rangle^{\sigma}\int d\nu\prod_{i=1}^3\langle\tau_i + \xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\chi_{A^c}\|_{L^2_{\xi,\tau}} \\ \leq\ & c\|\langle\tau + \xi^2\rangle^{b'}\int d\nu\langle\xi_1 + \xi_3\rangle^{-s_0}\langle\xi_1 - \xi_3\rangle^{\frac{1}{2}}\prod_{i=1}^3\langle\tau_i + \xi_i^2\rangle^{-b}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ =\ & c\|(J^s u_2)J^{-s_0}J_-^{\frac{1}{2}}(J^s u_1, J^s u_3)\|_{X_{0,b'}(\phi)}. \end{aligned}$$

Using part iii) of Lemma 4.1 (observe that $b' < -\frac{1}{4} + \frac{s_0}{2}$) and part i) of Corollary 4.1 this can be estimated by

$$\begin{aligned} & c\|J^s u_2\|_{X_{0,b}(\phi)}\|J^{-s_0}J_-^{\frac{1}{2}}(J^s u_1, J^s u_3)\|_{L_t^2(H_x^{s_0})} \\ \leq \quad & c\|u_2\|_{X_{s,b}(\phi)}\|J_-^{\frac{1}{2}}(J^s u_1, J^s u_3)\|_{L_{xt}^2} \leq c\prod_{i=1}^{3}\|u_i\|_{X_{s,b}(\phi)}. \end{aligned}$$

□

Theorem 4.7 *In the nonperiodic case in one space dimension the estimate*

$$\|u_1 \textstyle\prod_{i=2}^3 \overline{u}_i\|_{X_{s,b'}(\phi)} \leq c\displaystyle\prod_{i=1}^{3}\|u_i\|_{X_{s,b}(\phi)} \tag{26}$$

holds, provided $-\frac{1}{4} \geq s > -\frac{2}{5}$, $-\frac{1}{2} < b' < \min\left(s - \frac{1}{10}, -\frac{1}{4} + \frac{s}{2}\right)$ *and* $b > \frac{1}{2}$.

Proof: We write $f_1(\xi,\tau) = \langle \tau + \xi^2\rangle^b \langle \xi \rangle^s \mathcal{F}u_1(\xi,\tau)$ and, for $i = 2,3$, $f_i(\xi,\tau) = \langle \tau - \xi^2\rangle^b \langle \xi\rangle^s \mathcal{F}\overline{u}_i(\xi,\tau)$. Then, using the abbreviations $\sigma_0 = \tau + \xi^2$, $\sigma_1 = \tau_1 + \xi_1^2$ and, for $i = 2,3$, $\sigma_i = \tau_i - \xi_i^2$, we have

$$\|u_1 \textstyle\prod_{i=2}^3 \overline{u}_i\|_{X_{s,b'}(\phi)} = c\|\langle\sigma_0\rangle^{b'}\langle\xi\rangle^s \int d\nu \prod_{i=1}^3 \langle\sigma_i\rangle^{-b}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}.$$

Here the quantity

$$c.q. := |\xi^2 + \xi_2^2 + \xi_3^2 - \xi_1^2| = 2|\xi_2\xi_3 - \xi(\xi_2+\xi_3)|$$

can be controlled by the expressions $\langle\sigma_i\rangle$, $0 \leq i \leq 3$. Thus we divide the domain of integration into $A + A^c$, where in A it should hold that $c.q. \geq c\langle\xi_2\rangle\langle\xi_3\rangle$.

First we consider the region A^c. Here we have

$$\begin{aligned} & 1.\langle\xi_2\rangle \leq c\langle\xi\rangle \qquad \text{or} \qquad \langle\xi_3\rangle \leq c\langle\xi\rangle \\ \text{and} \qquad & 2.\langle\xi_{2,3}\rangle \leq c\langle\xi_2 \pm \xi_3\rangle \quad \text{or} \quad \langle\xi_{2,3}\rangle \leq c\langle\xi \pm \xi_{2,3}\rangle. \end{aligned}$$

Writing $A^c = B_1 + B_2$, where in B_1 we assume $\langle\xi_2\rangle \leq \langle\xi_3\rangle$ and in B_2, consequently, $\langle\xi_2\rangle \geq \langle\xi_3\rangle$, it will be sufficient by symmetry to consider the subregion B_1. Now B_1 is splitted again into B_{11} and B_{12}, where in B_{11} we assume $\langle\xi_{2,3}\rangle \leq c\langle\xi_2 \pm \xi_3\rangle$ and in B_{12} it should hold that $\langle\xi_{2,3}\rangle \leq c\langle\xi \pm \xi_{2,3}\rangle$.

Subregion B_{11}: Here it holds that $\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle \leq c\langle\xi\rangle\langle\xi_2 - \xi_3\rangle\langle\xi_2+\xi_3\rangle$, giving the upper bound

$$\begin{aligned} & \|\langle\sigma_0\rangle^{b'}\int d\nu \langle\xi_2+\xi_3\rangle^{-s}\langle\xi_2-\xi_3\rangle^{-s}\textstyle\prod_{i=1}^3\langle\sigma_i\rangle^{-b} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \\ = \quad & c\|(J^s u_1)J^{-s}J_-^{-s}(J^s\overline{u}_2, J^s\overline{u}_3)\|_{X_{0,b'}(\phi)} \\ \leq \quad & c\|u_1\|_{X_{s,b}(\phi)}\|J_-^{-s}(J^s\overline{u}_2, J^s\overline{u}_3)\|_{L^2_{xt}} \leq c\prod_{i=1}^{3}\|u_i\|_{X_{s,b}(\phi)}, \end{aligned}$$

where we have used part iii) of Lemma 4.1 (demanding for $b' < -\frac{1}{4} + \frac{s}{2}$) and part i) of Corollary 4.1.

Subregion B_{12}: Here we have
$\langle \xi_1 \rangle \langle \xi_2 \rangle \langle \xi_3 \rangle \leq c \langle \xi \rangle \langle \xi - \xi_3 \rangle \langle \xi + \xi_3 \rangle$, leading to the upper bound

$$\begin{aligned} & \| \langle \sigma_0 \rangle^{b'} \textstyle\int d\nu \langle \xi_1 + \xi_2 + 2\xi_3 \rangle^{-s} \langle \xi_1 + \xi_2 \rangle^{-s} \prod_{i=1}^{3} \langle \sigma_i \rangle^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \; & c \| J_+^{-s} (J^{-s}((J^s u_1)(J^s \overline{u}_2)), J^s \overline{u}_3) \|_{X_{0,b'}(\phi)} \\ \leq \; & c \| u_3 \|_{X_{s,b}(\phi)} \| J^{-s}((J^s u_1)(J^s \overline{u}_2)) \|_{L^2_{xt}} \leq c \prod_{i=1}^{3} \| u_i \|_{X_{s,b}(\phi)}. \end{aligned}$$

Here we have used part ii) of Corollary 4.1 (leading again to the restriction $b' < -\frac{1}{4} + \frac{s}{2}$) and part i) of Lemma 4.1. By this the discussion for the region A^c is completed.

Next we consider the region $A = \sum_{j=0}^{3} A_j$, where in A_j the expression $\langle \sigma_j \rangle$ is assumed to be dominant. By symmetry between the second and third factor (also in the exceptional region A^c) it will be sufficient to show the estimate for the subregions A_0, A_1 and A_2.

Subregion A_0: Here we can use $\langle \xi_2 \rangle \langle \xi_3 \rangle \leq c \langle \sigma_0 \rangle$ to obtain the upper bound

$$\begin{aligned} & \| \langle \xi \rangle^s \textstyle\int d\nu \langle \sigma_1 \rangle^{-b} \langle \xi_1 \rangle^{-s} f_1(\xi_1, \tau_1) \prod_{i=2}^{3} \langle \sigma_i \rangle^{-b} \langle \xi_i \rangle^{b'-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \; & c \| u_1 J^{b'} \overline{u}_2 J^{b'} \overline{u}_3 \|_{L^2_t(H^s_x)} \leq c \prod_{i=1}^{3} \| u_i \|_{X_{s,b}(\phi)} \end{aligned}$$

by part ii) of Lemma 4.4, provided $s > -\frac{1}{2}$ (in the last step we have also used $s \geq b'$).

Subregion A_1: Here we have $\langle \xi_2 \rangle \langle \xi_3 \rangle \leq c \langle \sigma_1 \rangle$ and $\langle \sigma_0 \rangle \leq \langle \sigma_1 \rangle$. Subdivide A_1 again into A_{11} and A_{12} with $\langle \xi_1 \rangle \leq c \langle \xi \rangle$ in A_{11} and, consequently, $\langle \xi_1 \rangle \approx \langle \xi_2 + \xi_3 \rangle$ in A_{12}. Then for A_{11} we have the upper bound

$$\begin{aligned} & \| \langle \sigma_0 \rangle^{-b} \textstyle\int d\nu f_1(\xi_1, \tau_1) \prod_{i=2}^{3} \langle \sigma_i \rangle^{-b} \langle \xi_i \rangle^{b'-s} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \; & c \| (\mathcal{F}^{-1} f_1)(J^{b'} \overline{u}_2)(J^{b'} \overline{u}_3) \|_{X_{0,-b}(\phi)} \leq c \| (\mathcal{F}^{-1} f_1)(J^{b'} \overline{u}_2)(J^{b'} \overline{u}_3) \|_{L^1_t(L^2_x)} \end{aligned}$$

by Sobolev's embedding theorem (plus duality) in the time variable. Now using Hölder's inequality and the $L^4_t(L^\infty_x)$-Strichartz estimate this can be controlled by

$$\| \mathcal{F}^{-1} f_1 \|_{L^2_{xt}} \| J^{b'} u_2 \|_{L^4_t(L^\infty_x)} \| J^{b'} u_3 \|_{L^4_t(L^\infty_x)} \leq c \prod_{i=1}^{3} \| u_i \|_{X_{s,b}(\phi)},$$

provided $b' \leq s$.

Now A_{12} is splitted again into A_{121}, where we assume $\langle \xi_2 + \xi_3 \rangle \leq c \langle \xi_2 - \xi_3 \rangle$, implying that also $\langle \xi_1 \rangle \leq c \langle \xi_2 - \xi_3 \rangle$, and A_{122}, where $\langle \xi_2 \rangle \approx \langle \xi_3 \rangle$. Consider the subregion A_{121} first: Using $\langle\xi_1\rangle^{-s} \leq c\langle\xi_2-\xi_3\rangle^{\frac{1}{2}}\langle\xi_2+\xi_3\rangle^{-s-\frac{1}{2}}$, for this region we obtain the upper bound

$$
\begin{aligned}
&\|\langle\sigma_0\rangle^{-b}\langle\xi\rangle^s \int d\nu f_1(\xi_1,\tau_1)\langle\xi_2-\xi_3\rangle^{\frac{1}{2}}\langle\xi_2+\xi_3\rangle^{-s-\frac{1}{2}}\textstyle\prod_{i=2}^3\langle\sigma_i\rangle^{-b}\langle\xi_i\rangle^{b'-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}\\
= \;& c\|(\mathcal{F}^{-1}f_1)J^{-s-\frac{1}{2}}J_-^{\frac{1}{2}}(J^{b'}\overline{u}_2,J^{b'}\overline{u}_3)\|_{X_{s,-b}(\phi)}\\
\leq \;& c\|(\mathcal{F}^{-1}f_1)J^{-s-\frac{1}{2}}J_-^{\frac{1}{2}}(J^{b'}\overline{u}_2,J^{b'}\overline{u}_3)\|_{L^1_t(L^p_x)} \qquad (s-\frac{1}{2}=-\frac{1}{p})\\
\leq \;& c\|\mathcal{F}^{-1}f_1\|_{L^2_{xt}}\|J^{-s-\frac{1}{2}}J_-^{\frac{1}{2}}(J^{b'}\overline{u}_2,J^{b'}\overline{u}_3)\|_{L^2_t(L^q_x)} \qquad (\frac{1}{p}=\frac{1}{2}+\frac{1}{q})\\
\leq \;& c\|u_1\|_{X_{s,b}(\phi)}\|J_-^{\frac{1}{2}}(J^{b'}\overline{u}_2,J^{b'}\overline{u}_3)\|_{L^2_{xt}} \leq c\prod_{i=1}^3\|u_i\|_{X_{s,b}(\phi)}.
\end{aligned}
$$

Next we consider the subregion A_{122}, where $\langle\xi_2\rangle \approx \langle\xi_3\rangle \geq c\langle\xi_1\rangle$. Here we get the upper bound

$$
\begin{aligned}
&\|\langle\sigma_0\rangle^{-b}\langle\xi\rangle^s \int d\nu f_1(\xi_1,\tau_1)\langle\xi_1\rangle^{-s-\frac{1}{6}}\textstyle\prod_{i=2}^3\langle\sigma_i\rangle^{-b}\langle\xi_i\rangle^{b'-s+\frac{1}{12}}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}\\
= \;& c\|(\Lambda^b J^{-\frac{1}{6}}u_1)(J^{b'+\frac{1}{12}}\overline{u}_2)(J^{b'+\frac{1}{12}}\overline{u}_3)\|_{X_{s,-b}(\phi)}, \qquad (\Lambda^b=\mathcal{F}^{-1}\langle\tau+\xi^2\rangle^b\mathcal{F})\\
\leq \;& c\|\Lambda^b u_1\|_{L^2_t(H_x^{-\frac{1}{4}-\frac{1}{6}})}\|J^{b'+\frac{1}{12}}u_2\|_{X_{0,b}(\phi)}\|J^{b'+\frac{1}{12}}u_3\|_{X_{0,b}(\phi)},
\end{aligned}
$$

where we have used $s \leq -\frac{1}{4}$ and part iv) of Corollary 4.2. The latter is bounded by $c\prod_{i=1}^3\|u_i\|_{X_{s,b}(\phi)}$, provided $s \geq -\frac{5}{12}$ and $s \geq b'+\frac{1}{12}$. Thus the discussion for the region A_1 is complete.

Subregion A_2: First we write $A_2 = A_{21}+A_{22}$, where in A_{21} it should hold that $\langle\xi_1\rangle \leq c\langle\xi\rangle$. Then this subregion can be treated precisely as the subregion A_{11}, leading to the bound $s > -\frac{1}{2}$. For the remaining subregion A_{22} it holds that

$$\langle\xi_2\rangle\langle\xi_3\rangle \leq c\langle\sigma_2\rangle \qquad \text{and} \qquad \langle\xi_1\rangle \leq c\langle\xi_2+\xi_3\rangle.$$

Now A_{22} is splitted again into A_{221}, where we assume $\langle\xi_1\rangle \leq c\langle\xi_2\rangle$, and into A_{222}, where we then have $\langle\xi_2\rangle \ll \langle\xi_1\rangle$. The upper bound for A_{221} is

$$
\begin{aligned}
&\|\langle\sigma_0\rangle^{-b}\langle\xi\rangle^s \int d\nu f_2(\xi_2,\tau_2)\langle\xi_2\rangle^{-s}\textstyle\prod_{i\neq 2}\langle\sigma_i\rangle^{-b}\langle\xi_i\rangle^{b'-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}\\
\leq \;& c\|(\Lambda_-^b\overline{u}_2)(J^{b'}u_1)(J^{b'}\overline{u}_3)\|_{X_{s,-b}(\phi)} \qquad (\Lambda_-^b=\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^b\mathcal{F})\\
\leq \;& c\|\Lambda_-^b\overline{u}_2\|_{L^2_t(H^s_x)}\|u_1\|_{X_{b',b}(\phi)}\|u_3\|_{X_{b',b}(\phi)} \leq c\prod_{i=1}^3\|u_i\|_{X_{s,b}(\phi)}.
\end{aligned}
$$

Here we have used part i) of Lemma 4.4 (dualized version) and the assumption $s \geq b'$.

For the subregion A_{222} the argument is a bit more complicated and it is here, where the strongest restrictions on s occur: Subdivide A_{222} again into A_{2221} and A_{2222} with $\langle\xi_2\rangle^2 \leq \langle\xi_1\rangle$ in A_{2221}. Then in A_{2221} it holds that

$$(\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle)^{\frac{2}{5}} \leq c\langle\xi_1\rangle \leq c\langle\xi_3\rangle \leq c\langle\xi_2 \pm \xi_3\rangle,$$

hence, for $\varepsilon = 1 + \frac{5}{2}s$ (> 0),

$$\prod_{i=1}^{3} \langle \xi_i \rangle^{-s} \leq c \langle \xi_2 - \xi_3 \rangle^{\frac{1}{2}} \langle \xi_2 + \xi_3 \rangle^{\frac{1}{2} - \varepsilon}.$$

Then, throwing away the $\langle \xi \rangle^s$-factor, we obtain the upper bound

$$\begin{aligned} & \| \langle \sigma_0 \rangle^{b'} \langle \xi_2 - \xi_3 \rangle^{\frac{1}{2}} \langle \xi_2 + \xi_3 \rangle^{\frac{1}{2} - \varepsilon} \textstyle\prod_{i=1}^{3} \langle \sigma_i \rangle^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \quad & c \| (J^s u_1) J^{\frac{1}{2} - \varepsilon} J_-^{\frac{1}{2}} (J^s \overline{u}_2, J^s \overline{u}_3) \|_{X_{0,b'}(\phi)} \\ \leq \quad & c \| u_1 \|_{X_{s,b}(\phi)} \| J_-^{\frac{1}{2}} (J^s \overline{u}_2, J^s \overline{u}_3) \|_{L^2_{xt}} \leq c \prod_{i=1}^{3} \| u_i \|_{X_{s,b}(\phi)} \end{aligned}$$

by Lemma 4.1, part iii), and Corollary 4.1, part i) (and the remark below), leading to the restriction $b' < \frac{5}{4}s$, which - in the allowed range for s - is in fact weaker than $b' < s - \frac{1}{10}$. Finally we consider the subregion A_{2222}, where we have $\langle \xi_1 \rangle^{\frac{1}{2}} \leq \langle \xi_2 \rangle \ll \langle \xi_1 \rangle \approx \langle \xi_3 \rangle$, implying that

$$\langle \xi_1 \rangle^{\frac{3}{20}} \leq c (\langle \xi_2 \rangle \langle \xi_3 \rangle)^{\frac{1}{10}}.$$

This gives the upper bound

$$\begin{aligned} & \| \langle \sigma_0 \rangle^{-b} \langle \xi \rangle^s \textstyle\int d\nu \langle \xi_1 \rangle^{-s - \frac{3}{20}} \langle \sigma_1 \rangle^{-b} f_1(\xi_1, \tau_1) \prod_{i=2}^{3} \langle \xi_i \rangle^{b' - s + \frac{1}{10}} f_i(\xi_i, \tau_i) \langle \sigma_3 \rangle^{-b} \|_{L^2_{\xi,\tau}} \\ & \quad \leq c \| (J^{-\frac{3}{20}} u_1)(\Lambda_-^b J^{b' + \frac{1}{10}} \overline{u}_2)(J^{b' + \frac{1}{10}} \overline{u}_3) \|_{X_{s,b}(\phi)}. \end{aligned}$$

Now using $s \leq -\frac{1}{4}$ again and part ii) of Corollary 4.2 this can be estimated by

$$c \| u_1 \|_{X_{-\frac{3}{20} - \frac{1}{4}, b}(\phi)} \| \Lambda_-^b J^{b' + \frac{1}{10}} \overline{u}_2 \|_{L^2_{xt}} \| u_3 \|_{X_{b' + \frac{1}{10}, b}(\phi)} \leq c \prod_{i=1}^{3} \| u_i \|_{X_{s,b}(\phi)},$$

since $s > -\frac{2}{5}$ and $s > b' + \frac{1}{10}$ as assumed. □

Remark: The estimate (26) also holds under the assumption $s \geq -\frac{1}{4}$, $b' < -\frac{3}{8}$ and $b > \frac{1}{2}$. For $s = -\frac{1}{4}$ this is contained in the above theorem, and for $s > -\frac{1}{4}$ this follows from $\langle \xi \rangle \leq c \prod_{i=1}^{3} \langle \xi_i \rangle$.

4.4 Estimates on quartic nonlinearities

Theorem 4.8 *Let $n = 1$. Assume $0 \geq s > -\frac{1}{6}$ and $-\frac{1}{2} < b' < \frac{3s}{2} - \frac{1}{4}$. Then in the periodic and nonperiodic case for all $b > \frac{1}{2}$ the estimate*

$$\| \textstyle\prod_{i=1}^{4} \overline{u}_i \|_{X_{s,b'}(\phi)} \leq c \displaystyle\prod_{i=1}^{4} \| u_i \|_{X_{s,b}(\phi)}$$

holds true.

Proof: Again we write $f_i(\xi,\tau) = \langle\tau-\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}\overline{u}_i(\xi,\tau)$, so that

$$\|\prod_{i=1}^4 \overline{u}_i\|_{X_{s,b'}(\phi)} = c\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s\int d\nu\prod_{i=1}^4\langle\tau_i-\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}.$$

Now we can use the inequality

$$\langle\xi\rangle^2+\sum_{i=1}^4\langle\xi_i\rangle^2 \leq \langle\tau+\xi^2\rangle+\sum_{i=1}^4\langle\tau_i-\xi_i^2\rangle$$

and the assumption $b' < \frac{3s}{2}-\frac{1}{4}$ to obtain

$$\langle\xi\rangle^{s+\frac{1}{2}-\varepsilon}\prod_{i=1}^4\langle\xi_i\rangle^{-s+\varepsilon} \leq c(\langle\tau+\xi^2\rangle^{-b'}+\sum_{i=1}^4\langle\tau_i-\xi_i^2\rangle^{-b'}\chi_{A_i})$$

for some $\varepsilon > 0$. (Again in A_i we assume $\langle\tau_i-\xi_i^2\rangle \geq \langle\tau+\xi^2\rangle$.) From this it follows that

$$\|\prod_{i=1}^4 \overline{u}_i\|_{X_{s,b'}(\phi)} \leq c\sum_{j=0}^4\|I_j\|_{L^2_{\xi,\tau}},$$

with

$$I_0(\xi,\tau) = \langle\xi\rangle^{-\frac{1}{2}+\varepsilon}\int d\nu\prod_{i=1}^4\langle\tau_i-\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i)$$

and, for $1 \leq j \leq m$,

$$\begin{aligned}
I_j(\xi,\tau) &= \langle\xi\rangle^{-\frac{1}{2}+\varepsilon}\langle\tau+\xi^2\rangle^{b'}\int d\nu\langle\tau_j-\xi_j^2\rangle^{-b'}\prod_{i=1}^4\langle\tau_i-\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i)\chi_{A_j}\\
&\leq \langle\xi\rangle^{-\frac{1}{2}+\varepsilon}\langle\tau+\xi^2\rangle^{-b}\int d\nu\langle\tau_j-\xi_j^2\rangle^{b}\prod_{i=1}^4\langle\tau_i-\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-\varepsilon}f_i(\xi_i,\tau_i).
\end{aligned}$$

Next we estimate I_0 using first Sobolev's embedding theorem, then Hölder's inequality, again Sobolev and finally part a) of Corollary 2.4 (with $p = 8$ and $q = 4$). Here ε', ε'' denote suitable small, positive numbers.

$$\begin{aligned}
\|I_0\|_{L^2_{\xi,\tau}} &= \|\prod_{i=1}^4 J^{s-\varepsilon}\overline{u}_i\|_{L^2_t(H_x^{-\frac{1}{2}+\varepsilon})} \leq c\|\prod_{i=1}^4 J^{s-\varepsilon}\overline{u}_i\|_{L^2_t(L_x^{1+\varepsilon'})}\\
&\leq c\prod_{i=1}^4\|J^{s-\varepsilon}\overline{u}_i\|_{L^8_t(L_x^{4+4\varepsilon'})} \leq c\prod_{i=1}^4\|J^{s-\varepsilon''}\overline{u}_i\|_{L^8_t(L^4_x)}\\
&\leq c\prod_{i=1}^4\|J^s\overline{u}_i\|_{X_{0,b}(-\phi)} = c\prod_{i=1}^4\|\overline{u}_i\|_{X_{s,b}(-\phi)}
\end{aligned}$$

To estimate I_j, $1 \leq j \leq 4$, we use Sobolev (in both variables) plus duality, Hölder, again Sobolev (in the space variable) and Lemma 2.15. Again we need suitable small, positive numbers ε', ε'' and ε'''.

$$
\begin{aligned}
\|I_j\|_{L^2_{\xi,\tau}} &\leq c\|(\textstyle\prod_{\substack{i=1\\i\neq j}}^4 J^{s-\varepsilon}\overline{u}_i)(J^{-\varepsilon}\mathcal{F}^{-1}f_j)\|_{X_{-\frac{1}{2}+\varepsilon,-b}(\phi)} \\
&\leq c\|(\textstyle\prod_{\substack{i=1\\i\neq j}}^4 J^{s-\varepsilon}\overline{u}_i)(J^{-\varepsilon}\mathcal{F}^{-1}f_j)\|_{L^1_t(L^{1+\varepsilon'}_x)} \\
&\leq c\|J^{-\varepsilon}\mathcal{F}^{-1}f_j\|_{L^2_{x,t}}\prod_{\substack{i=1\\i\neq j}}^4\|J^{s-\varepsilon}\overline{u}_i\|_{L^6_t(L^{6+\varepsilon''}_x)} \\
&\leq c\|J^{-\varepsilon}\mathcal{F}^{-1}f_j\|_{L^2_{x,t}}\prod_{\substack{i=1\\i\neq j}}^4\|J^{s-\varepsilon'''}\overline{u}_i\|_{L^6_{xt}} \\
&\leq c\|f_j\|_{L^2_{\xi,\tau}}\prod_{\substack{i=1\\i\neq j}}^4\|J^s\overline{u}_i\|_{X_{0,b}(-\phi)} = c\prod_{i=1}^4\|\overline{u}_i\|_{X_{s,b}(-\phi)}
\end{aligned}
$$

□

In the periodic case the following examples show, that for all the other quartic nonlinearities (u^4, $u^3\overline{u}$, ..., $u\overline{u}^3$) the corresponding estimates fail for all $s<0$. The argument is essentially that given in the proof of Thm 1.10 in [KPV96b].

Example 4.3 *In the periodic case in one space dimension the estimate*

$$\|\textstyle\prod_{i=1}^4 u_i\|_{X_{s,b'}(\phi)} \leq c\displaystyle\prod_{i=1}^4\|u_i\|_{X_{s,b}(\phi)}$$

fails for all $s<0$, $b,b'\in\mathbf{R}$.

Proof: The above estimate implies

$$\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s\textstyle\int d\nu\prod_{i=1}^4\langle\tau_i+\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c\displaystyle\prod_{i=1}^4\|f_i\|_{L^2_{\xi,\tau}}.$$

Defining for $n\in\mathbf{N}$

$$f_{1,2}^{(n)}(\xi,\tau)=\delta_{\xi,2n}\chi(\tau+\xi^2),\quad f_3^{(n)}(\xi,\tau)=\delta_{\xi,-n}\chi(\tau+\xi^2),\quad f_4^{(n)}(\xi,\tau)=\delta_{\xi,0}\chi(\tau+\xi^2),$$

where χ is the characteristic function of $[-1,1]$, we have $\|f_i^{(n)}\|_{L^2_{\xi,\tau}}=c$ and it would follow that

$$n^{-3s}\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s\textstyle\int d\nu\prod_{i=1}^4 f_i^{(n)}(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}\leq c. \tag{27}$$

Now a simple computation shows that

$$\int d\nu\prod_{i=1}^4 f_i^{(n)}(\xi_i,\tau_i)\geq\delta_{\xi,3n}\chi(\tau+\xi^2).$$

Inserting this into (27) we obtain $n^{-2s}\leq c$, which is a contradiction for any $s<0$.

□

Remark : Using only the sequences $f_i^{(n)}$, $1 \le i \le 3$, from the above proof, the same calculation shows that in the periodic case the estimate

$$\| \prod_{i=1}^{3} u_i \|_{X_{s,b'}(\phi)} \le c \prod_{i=1}^{3} \|u_i\|_{X_{s,b}(\phi)}$$

fails for all $s < 0$, $b, b' \in \mathbf{R}$.

Example 4.4 *In the periodic case in one space dimension the estimates*

$$\|u_1 \overline{u}_2 \tilde{u}_3 \tilde{u}_4\|_{X_{s,b'}(\phi)} \le c \prod_{i=1}^{4} \|u_i\|_{X_{s,b}(\phi)},$$

where $\tilde{u} = u$ *or* $\tilde{u} = \overline{u}$, *fail for all* $s < 0$, $b, b' \in \mathbf{R}$.

Proof: We define

$$f_1^{(n)}(\xi,\tau) = \delta_{\xi,n}\chi(\tau+\xi^2) \quad , \quad f_2^{(n)}(\xi,\tau) = \delta_{\xi,-n}\chi(\tau-\xi^2)$$
$$f_{3,4}^{(n)}(\xi,\tau) = \delta_{\xi,0}\chi(\tau \pm \xi^2) \qquad (+ \text{ for } \tilde{u}_{3,4} = u_{3,4}, \ - \text{ for } \tilde{u}_{3,4} = \overline{u}_{3,4}).$$

Then the above estimate would imply

$$n^{-2s} \| \langle \tau + \xi^2 \rangle^{b'} \langle \xi \rangle^s \int d\nu \prod_{i=1}^{4} f_i^{(n)}(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \le c. \tag{28}$$

Now

$$\int d\nu \prod_{i=1}^{4} f_i^{(n)}(\xi_i, \tau_i) \ge \delta_{\xi,0}\chi(\tau),$$

which inserted into (28) again leads to $n^{-2s} \le c$. □

Remark : Using only the sequences $f_i^{(n)}$, $1 \le i \le 3$, from the above proof, we see that in the periodic case the estimates

$$\|u_1 \overline{u}_2 \tilde{u}_3\|_{X_{s,b'}(\phi)} \le c \prod_{i=1}^{3} \|u_i\|_{X_{s,b}(\phi)}$$

fail for all $s < 0$, $b, b' \in \mathbf{R}$.

Now we turn to discuss the continuous case, where we can use the bi- and trilinear inequalities of section 4.2 in order to prove the relevant estimates for some $s < 0$. We start with the following

Proposition 4.1 *Let* $0 \ge s > -\frac{1}{8}$, $-\frac{1}{2} < b' < -\frac{1}{4} + 2s$. *Then in the continuous case in one space dimension for any* $b > \frac{1}{2}$ *the estimate*

$$\|u_1 u_2 \overline{u}_3 \overline{u}_4\|_{X_{s,b'}(\phi)} \le c \prod_{i=1}^{4} \|u_i\|_{X_{s,b}(\phi)}$$

holds true.

Proof: Apply part iii) of Lemma 4.1 to obtain

$$\|u_1u_2\overline{u}_3\overline{u}_4\|_{X_{s,b'}(\phi)} \leq c\|u_1\|_{X_{s,b}(\phi)}\|u_2\overline{u}_3\overline{u}_4\|_{L^2_t(H^{\sigma-s})},$$

provided that $s \leq 0$, $-\frac{1}{2} \leq \sigma \leq 0$, $b' < -\frac{1}{4} + \frac{\sigma}{2}$. This is fulfilled for $\sigma = 4s$ and the second factor is equal to

$$\|u_2\overline{u}_3\overline{u}_4\|_{L^2_t(H^{3s})} \leq c\prod_{i=2}^4 \|u_i\|_{X_{s,b}(\phi)}$$

by Lemma 4.4 and the remark below. □

To show that this proposition is essentially (up to the endpoint) sharp, we present the following counterexample (cf. Thm 1.4 in [KPV96b]):

Example 4.5 *In the nonperiodic case in one space dimension the estimate*

$$\|u_1u_2\overline{u}_3\overline{u}_4\|_{X_{s,b'}(\phi)} \leq c\prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}$$

fails for all $s < -\frac{1}{8}$, $b, b' \in \mathbf{R}$.

Proof: The above estimate implies

$$\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s \textstyle\int d\nu \prod_{i=1}^4 \langle\sigma_i\rangle^{-b}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c\displaystyle\prod_{i=1}^4 \|f_i\|_{L^2_{\xi,\tau}},$$

where $\langle\sigma_{1,2}\rangle = \langle\tau_{1,2}+\xi^2_{1,2}\rangle$ and $\langle\sigma_{3,4}\rangle = \langle\tau_{3,4}-\xi^2_{3,4}\rangle$. Choosing

$$f^{(n)}_{1,2}(\xi,\tau) = \chi(\xi-n)\chi(\tau+\xi^2), \qquad f^{(n)}_{3,4}(\xi,\tau) = \chi(\xi+n)\chi(\tau-\xi^2)$$

we arrive at

$$n^{-4s}\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s \textstyle\int d\nu \prod_{i=1}^4 f^{(n)}_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}} \leq c. \tag{29}$$

Now an elementary computation gives

$$\int d\nu \prod_{i=1}^4 f^{(n)}_i(\xi_i,\tau_i) \geq c\chi_c(2n\xi)\chi_c(\tau),$$

where χ_c is the characteristic function of $[-c,c]$. Inserting this into (29) we get $n^{-4s-\frac{1}{2}} \leq c$, which is a contradiction for any $s < -\frac{1}{8}$. □

Finally we consider the remaining nonlinearities u^4, $u^3\overline{u}$ and $u\overline{u}^3$, for which we can lower the bound on s down to $-\frac{1}{6}+\varepsilon$:

Theorem 4.9 *Let* $n = 1$. *Assume* $0 \geq s > -\frac{1}{6}$, $-\frac{1}{2} < b' < \frac{3s}{2} - \frac{1}{4}$ *and* $b > \frac{1}{2}$. *Then in the nonperiodic case the estimates*

$$\|N(u_1,u_2,u_3,u_4)\|_{X_{s,b'}(\phi)} \leq c\prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}$$

hold true for $N(u_1,u_2,u_3,u_4) = \prod_{i=1}^4 u_i$, $= (\prod_{i=1}^3 u_i)\overline{u}_4$ *or* $= (\prod_{i=1}^3 \overline{u}_i)u_4$.

Proof: 1. We begin with the nonlinearity $N(u_1,u_2,u_3,u_4) = \prod_{i=1}^4 u_i$. Writing $f_i(\xi,\tau) = \langle \tau+\xi^2\rangle^b \langle\xi\rangle^s \mathcal{F}u_i(\xi,\tau)$ we have

$$\|\prod_{i=1}^4 u_i\|_{X_{s,b'}(\phi)} = c\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s \int d\nu \prod_{i=1}^4 \langle\tau_i+\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}}.$$

The quantity controlled by the expressions $\langle\tau+\xi^2\rangle$, $\langle\tau_i+\xi_i^2\rangle$, $1\le i\le 4$, is $|\sum_{i=1}^4 \xi_i^2 - \xi^2|$. We divide the domain of integration into A and A^c, where in A we assume $\xi^2 \le \frac{\xi_1^2}{2}$ and thus

$$|\sum_{i=1}^4 \xi_i^2 - \xi^2| \ge c(\sum_{i=1}^4 \xi_i^2 + \xi^2).$$

So concerning this region we may refer to the proof of Theorem 4.8. For the region A^c, where $\xi_1^2 \le 2\xi^2$, we have the upper bound

$$c\|(J^s u_1)\prod_{i=2}^4 u_i\|_{X_{0,b'}(\phi)} \le c\|u_1\|_{X_{s,b}(\phi)} \|\prod_{i=2}^4 u_i\|_{L^2_t(H^{3s}_x)} \le c\prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}$$

by Lemma 4.1, part iii), which requires $b' < \frac{3s}{2} - \frac{1}{4}$, $s \ge -\frac{1}{6}$, and by Lemma 4.5 (and the remark below), which demands $s > -\frac{1}{6}$.

2. Next we consider $N(u_1,u_2,u_3,u_4) = (\prod_{i=1}^3 u_i)\overline{u}_4$. For $1\le i\le 3$ we choose the f_i as in the first part of this proof and $f_4(\xi,\tau) = \langle\tau-\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}\overline{u}_4(\xi,\tau)$. Then the left hand side of the claimed estimate is equal to

$$c\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s \int d\nu \prod_{i=1}^3 \langle\tau_i+\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\langle\tau_4-\xi_4^2\rangle^{-b}\langle\xi_4\rangle^{-s} f_4(\xi_4,\tau_4)\|_{L^2_{\xi,\tau}}.$$

Now the quantity controlled by the expressions $\langle\tau+\xi^2\rangle$, $\langle\tau_i+\xi_i^2\rangle$, $1\le i\le 3$, and $\langle\tau_4-\xi_4^2\rangle$ is

$$c.q. := |\xi_1^2+\xi_2^2+\xi_3^2-\xi_4^2-\xi^2|.$$

We divide the domain of integration into the regions A, B and $C=(A+B)^c$, where in A it should hold that

$$c.q. \;\; \ge c(\sum_{i=1}^4 \xi_i^2 + \xi^2).$$

Again, concerning this region we may refer to the proof of Thm. 4.8.

Next we write $B = \bigcup_{i=1}^3 B_i$, where in B_i we assume $\xi_i^2 \le c\xi^2$ for some large constant c. By symmetry it is sufficient to consider the subregion B_1, where we obtain the upper bound

$$c\|(J^s u_1)u_2u_3\overline{u}_4\|_{X_{0,b'}(\phi)} \le c\|u_1\|_{X_{s,b}(\phi)}\|u_2u_3\overline{u}_4\|_{L^2_t(H^{3s}_x)} \le c\prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}$$

by Lemma 4.1, part iii), demanding for $b' < \frac{3s}{2} - \frac{1}{4}$, $3s \ge -\frac{1}{2}$, and Lemma 4.4 (resp. the remark below), where $s > -\frac{1}{6}$ is required.

Considering the region C we may assume by symmetry between the first three factors that $\xi_1^2 \ge \xi_2^2 \ge \xi_3^2$. Then for this region it is easily checked that

1. $\xi^2 \ll \xi_3^2$, 2. $\xi_4^2 \geq \frac{3}{2}\xi_2^2$, hence $\xi_4^2 \leq c(\xi_4+\xi_2)^2$, and 3. $\xi_1^2 \leq c(\xi_1-\xi_3)^2$.

This implies

1. $\langle\xi\rangle^{-2s}\langle\xi_4\rangle^{-s} \leq c\langle\xi_4+\xi_2\rangle^{-3s}$ and
2. $\langle\xi\rangle^{\frac{1}{2}+3s}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s} \leq c\langle\xi_1-\xi_3\rangle^{\frac{1}{2}}$,

leading to the upper bound

$$\begin{aligned} & \|J_-^{\frac{1}{2}}(J^su_1, J^su_3)J^{-3s}(J^su_2J^s\overline{u}_4)\|_{X_{-\frac{1}{2},b'}(\phi)} \\ \leq\ & c\|J_-^{\frac{1}{2}}(J^su_1, J^su_3)J^{-3s}(J^su_2J^s\overline{u}_4)\|_{L_t^p(L_x^{1+\varepsilon})} && (b'-\frac{1}{2}=-\frac{1}{p}) \\ \leq\ & c\|J_-^{\frac{1}{2}}(J^su_1, J^su_3)\|_{L_{xt}^2}\|J^{-3s}(J^su_2J^s\overline{u}_4)\|_{L_t^q(L_x^{2+\varepsilon'})} && (\frac{1}{q}=\frac{1}{p}-\frac{1}{2}=-b'). \end{aligned}$$

Using Corollary 4.1 the first factor can be estimated by

$$c\|u_1\|_{X_{s,b}(\phi)}\|u_3\|_{X_{s,b}(\phi)},$$

while for the second we can use Sobolev's embedding Theorem and part ii) of Lemma 4.1 to obtain the bound

$$c\|J^su_2J^s\overline{u}_4\|_{L_t^q(H_x^{-3s+\varepsilon''})} \leq c\|u_2\|_{X_{s,b}(\phi)}\|u_4\|_{X_{s,b}(\phi)}.$$

Here the restriction $b' < \frac{3s}{2}-\frac{1}{4}$ is required again.

3. Finally we consider $N(u_1,u_2,u_3,u_4) = (\prod_{i=1}^3 \overline{u}_i)u_4$. With $f_i(\xi,\tau) = \langle\tau-\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}\overline{u}_i(\xi,\tau)$, $1 \leq i \leq 3$ and $f_4(\xi,\tau) = \langle\tau+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}u_4(\xi,\tau)$ the norm on the left hand side is equal to

$$c\|\langle\tau+\xi^2\rangle^{b'}\langle\xi\rangle^s\int d\nu\prod_{i=1}^3\langle\tau_i-\xi_i^2\rangle^{-b}\langle\xi_i\rangle^{-s}f_i(\xi_i,\tau_i)\langle\tau_4+\xi_4^2\rangle^{-b}\langle\xi_4\rangle^{-s}f_4(\xi_4,\tau_4)\|_{L^2_{\xi,\tau}}.$$

The controlled quantity here is

$$c.q. := |\xi_1^2+\xi_2^2+\xi_3^2-\xi_4^2+\xi^2|.$$

Divide the area of integration into A, B and $C=(A+B)^c$, where in A we assume again

$$c.q. \geq c(\sum_{i=1}^4\xi_i^2+\xi^2)$$

in order to refer to the proof of Theorem 4.8. In B we assume $\xi_4^2 \leq c\xi^2$, so that for this region we have the bound

$$c\|\overline{u}_1\overline{u}_2\overline{u}_3(J^su_4)\|_{X_{0,b'}(\phi)} \leq c\|u_4\|_{X_{s,b}(\phi)}\|u_1u_2u_3\|_{L_t^2(H_x^{3s})} \leq c\prod_{i=1}^4\|u_i\|_{X_{s,b}(\phi)}$$

by Lemma 4.1, part iii), and Lemma 4.5 and the remark below. Here $b' < \frac{3s}{2}-\frac{1}{4}$ and $s > -\frac{1}{6}$ is required.

For the region C we shall assume again that $\xi_1^2 \geq \xi_2^2 \geq \xi_3^2$. Then it is easily checked that in C

1. $\xi^2 \ll \xi_4^2$, 2. $\xi_4^2 \geq \frac{3}{2}\xi_2^2$, hence $\xi_4^2 \leq c(\xi_4+\xi_2)^2$, and 3. $\xi_1^2 \leq c(\xi_1-\xi_3)^2$.

Then for C we have the estimate

$$\begin{aligned} & \|J_-^{\frac{1}{2}}(J^s\overline{u}_1, J^s\overline{u}_3)J^{-3s}(J^s u_4 J^s \overline{u}_2)\|_{X_{-\frac{1}{2},b'}(\phi)} \\ \leq \quad & c\|J_-^{\frac{1}{2}}(J^s u_1, J^s u_3)\|_{L^2_{xt}}\|J^{-3s}(J^s u_2 J^s \overline{u}_4)\|_{L^q_t(L^{2+\varepsilon}_x)} \end{aligned}$$

with $\frac{1}{q} = -b'$, cf. the corresponding part of step 2. of this proof. Again we can use Corollary 4.1 and part ii) of Lemma 4.1 to obtain the desired bound. □

5 A bilinear Airy-estimate with application to gKdV-3

In the last section we could prove an optimal and exhaustive result concerning NLS with quartic nonlinearities on the line (see Theorem 4.3). It turned out - which is somewhat surprising - that on the line all the quartic nonlinearities are better behaved than the cubic one $N(u,\overline{u}) = u|u|^2$. The situation is similar in the case of the generalized Korteweg-deVries-equation of order k (for short gKdV-k), that is

$$u_t + u_{xxx} + (u^{k+1})_x = 0,$$

the phase function here is $\phi(\xi) = \xi^3$. For $k = 1$ this is the KdV-equation, for $k = 2$ this is usually called the modified KdV-equation. Concerning the latter local wellposedness on the line is known for $s \geq \frac{1}{4}$ (see Theorem 2.4 in [KPV93a]) and it was shown in [KPV01] that the Cauchy problem for this equation is ill posed for data in H^s_x, $s < \frac{1}{4}$, in the sense that the mapping data upon solution is not uniformly continuous, see Theorems 1.2 and 1.3 in [KPV01]. Using similar arguments as in the previous section we can show here that the Cauchy problem for gKdV-3 is locally well posed in H^s_x for $s > -\frac{1}{6}$, which is the scaling exponent in this case:

Theorem 5.1 *Let $s > -\frac{1}{6}$. Then there exist $b > \frac{1}{2}$ and $\delta = \delta(\|u_0\|_{H^s_x(\mathbf{R})}) > 0$, so that there is a unique solution $u \in X^\delta_{s,b}(\phi)$ of the Cauchy problem*

$$u_t + u_{xxx} + (u^4)_x = 0, \qquad u(0) = u_0 \in H^s_x(\mathbf{R}). \tag{30}$$

This solution is persistent and for any $0 < \delta_0 < \delta$ the mapping data upon solution is locally Lipschitz continuous from $H^s_x(\mathbf{R})$ to $X^{\delta_0}_{s,b}(\phi)$.

Remarks : i) So far, local wellposedness of this problem is known for data $u_0 \in H^s_x(\mathbf{R})$, $s \geq \frac{1}{12}$. This was shown by Kenig, Ponce and Vega in 1993, see Theorem 2.6 in [KPV93a].

ii) For real valued data u_0 the solution guaranteed by Theorem 5.1 remains real valued. In fact, if $u_0 = \overline{u}_0$ and if u is a solution of (30), then so is $\overline{u}$, so that by uniqueness we have $u = \overline{u}$. In this case, if $u_0 \in H^s_x(\mathbf{R})$ for $s \geq 0$, the L^2_x-norm of the solution is a conserved quantity (cf. the argument in remark ii) below Theorem 3.1), and we obtain the following

Corollary 5.1 *For real valued data $u_0 \in H^s_x(\mathbf{R})$, $s \geq 0$ the Cauchy problem (30) is globally well posed in the sense of Corollary 1.4.*

By the general theory the proof of Theorem 5.1 reduces to the following estimate:

Theorem 5.2 *For $0 \geq s > -\frac{1}{6}$, $-\frac{1}{2} < b' < s - \frac{1}{3}$ and $b > \frac{1}{2}$ the estimate*

$$\|\partial_x \textstyle\prod_{i=1}^4 u_i\|_{X_{s,b'}(\phi)} \leq c \displaystyle\prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}$$

is valid.

The main new tool for the proof of Theorem 5.2 is the bilinear Airy-estimate below. Here I^s denotes the Riesz potential of order $-s$ and I^s_- is the bilinear operator introduced in section 4.2:

Lemma 5.1

$$\|I^{\frac{1}{2}}I_-^{\frac{1}{2}}(e^{-t\partial^3}u_1, e^{-t\partial^3}u_2)\|_{L^2_{xt}} \le c\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}$$

Proof: Replacing the phase function $\phi(\xi) = -\xi^2$ by $\phi(\xi) = \xi^3$ in the proof of Lemma 4.2 we obtain

$$\begin{aligned} & \|I^{\frac{1}{2}}I_-^{\frac{1}{2}}(e^{-t\partial^3}u_1, e^{-t\partial^3}u_2)\|^2_{L^2_{xt}} \\ = \quad & c\int d\xi|\xi| \int_* d\xi_1 d\eta_1 \delta(3\xi(\eta_1^2 - \xi_1^2 + \xi(\xi_1 - \eta_1)))..\\ \times \quad & ..(|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^{2} \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)}. \end{aligned}$$

Now we use $\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n)$, where the sum is taken over all simple zeros of g, in our case:

$$g(x) = 3\xi(x^2 + \xi(\xi_1 - x) - \xi_1^2)$$

with the zeros $x_1 = \xi_1$ and $x_2 = \xi - \xi_1$, hence $g'(x_1) = 3\xi(2\xi_1 - \xi)$ respectively $g'(x_2) = 3\xi(\xi - 2\xi_1)$. As in the proof of Lemma 4.2 we see that the last expression is equal to

$$\begin{aligned} & c\int d\xi \int_* d\xi_1 \prod_{i=1}^{2} |\hat{u}_i(\xi_i)|^2 + c\int d\xi \int_* d\xi_1 \hat{u}_1(\xi_1)\overline{\hat{u}_1}(\xi_2)\hat{u}_2(\xi_2)\overline{\hat{u}_2}(\xi_1) \\ \le \quad & c(\prod_{i=1}^{2} \|u_i\|^2_{L^2_x} + \|\hat{u}_1\hat{u}_2\|^2_{L^1_\xi}) \le c\prod_{i=1}^{2}\|u_i\|^2_{L^2_x}. \end{aligned}$$

□

By Lemma 2.1 we get the following

Corollary 5.2 *Let $b > \frac{1}{2}$. Then the following estimate holds true:*

$$\|I^{\frac{1}{2}}I_-^{\frac{1}{2}}(u,v)\|_{L^2_{xt}} \le c\|u\|_{X_{0,b}(\phi)}\|v\|_{X_{0,b}(\phi)}$$

Together with the Strichartz type inequalities for the Airy equation (see Lemma 2.7) this will be sufficient to prove the crucial nonlinear estimate:

Proof of Theorem 5.2: Writing $f_i(\xi,\tau) = \langle\tau - \xi^3\rangle^b\langle\xi\rangle^s\mathcal{F}u_i(\xi,\tau)$, $1 \le i \le 4$, we have

$$\|\partial_x \textstyle\prod_{i=1}^4 u_i\|_{X_{s,b'}(\phi)} = c\|\langle\tau - \xi^3\rangle^{b'}\langle\xi\rangle^s|\xi| \int d\nu \prod_{i=1}^4 \langle\tau_i - \xi_i^3\rangle^{-b}\langle\xi_i\rangle^{-s} f_i(\xi_i,\tau_i)\|_{L^2_{\xi,\tau}},$$

where $d\nu = d\xi_1..d\xi_3 d\tau_1..d\tau_3$ and $\sum_{i=1}^4(\xi_i,\tau_i) = (\xi,\tau)$. Now the domain of integration is divided into the regions A, B and $C = (A \cup B)^c$, where in A we assume

$|\xi_{max}| \leq c$. (Here ξ_{max} is defined by $|\xi_{max}| = \max_{i=1}^4 |\xi_i|$, similarly ξ_{min}.) Then for the region A we have the upper bound

$$\begin{aligned} & c\| \int d\nu \prod_{i=1}^4 \langle \tau_i - \xi_i^3 \rangle^{-b} f_i(\xi_i, \tau_i)\|_{L^2_{\xi,\tau}} \\ = \; & c\| \prod_{i=1}^4 J^s u_i\|_{L^2_{x,t}} \leq c \prod_{i=1}^4 \|J^s u_i\|_{L^8_{x,t}} \leq c \prod_{i=1}^4 \|u_i\|_{X_{s,b}(\phi)}, \end{aligned}$$

where in the last step Lemma 2.7 , part ii), with $p = q = 8$ was applied.

Besides $|\xi_{max}| \geq c$ ($\Rightarrow \langle \xi_{max} \rangle \leq c|\xi_{max}|$) we shall assume for the region B that

i) $|\xi_{min}| \leq 0.99|\xi_{max}|$ or

ii) $|\xi_{min}| > 0.99|\xi_{max}|$, and there are exactly two indices $i \in \{1, 2, 3, 4\}$ with $\xi_i > 0$.

Then the region B can be splitted again into a finite number of subregions, so that for any of these subregions there exists a permutation π of $\{1, 2, 3, 4\}$ with

$$|\xi|\langle\xi\rangle^s \prod_{i=1}^4 \langle \xi_i \rangle^{-s} \leq c|\xi_{\pi(1)} + \xi_{\pi(2)}|^{\frac{1}{2}} |\xi_{\pi(1)} - \xi_{\pi(2)}|^{\frac{1}{2}} \langle \xi_{\pi(3)} \rangle^{-\frac{3s}{2}} \langle \xi_{\pi(4)} \rangle^{-\frac{3s}{2}}.$$

Assume $\pi = id$ for the sake of simplicity now. Then we get the upper bound

$$\begin{aligned} & \|\langle \tau - \xi^3 \rangle^{b'} \int d\nu |\xi_1 + \xi_2|^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}} \langle \xi_3 \rangle^{-\frac{3s}{2}} \langle \xi_4 \rangle^{-\frac{3s}{2}} \prod_{i=1}^4 \langle \tau_i - \xi_i^3 \rangle^{-b} f_i(\xi_i, \tau_i)\|_{L^2_{\xi,\tau}} \\ & = c\|(I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4)\|_{X_{0,b'}(\phi)}. \end{aligned}$$

To estimate the latter expression, we fix some Sobolev- and Hölderexponents:

i) $\frac{1}{q_0} = \frac{1}{2} - b'$, so that $L_t^{q_0}(L_x^2) \subset X_{0,b'}(\phi)$,

ii) $\frac{2}{p} = \frac{1}{q_0} - \frac{1}{2} = -b'$,

iii) $\frac{1}{q} = \frac{1}{2} - \frac{2}{p} = \frac{1}{2} + b'$, so that by Lemma 2.7 $\|J^{\frac{1}{p}} u\|_{L_t^p(L_x^q)} \leq c\|u\|_{X_{0,b}(\phi)}$,

iv) $\varepsilon = \frac{1}{p} + \frac{3s}{2} > \frac{1}{q}$ (since $s > \frac{1}{3} + b'$), so that $H_x^{\varepsilon,q} \subset L_x^\infty$.

Now we have

$$\begin{aligned} & \|(I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4)\|_{X_{0,b'}(\phi)} \\ \leq \; & c\|(I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4)\|_{L_t^{q_0}(L_x^2)} \\ \leq \; & c\|I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^s u_1, J^s u_2)\|_{L^2_{xt}} \|J^{-\frac{s}{2}} u_3\|_{L_t^p(L_x^\infty)} \|J^{-\frac{s}{2}} u_4\|_{L_t^p(L_x^\infty)}. \end{aligned}$$

Now by Corollary 5.2 the first factor can be controlled by $c\|u_1\|_{X_{s,b}(\phi)} \|u_2\|_{X_{s,b}(\phi)}$, while for the second we have the upper bound

$$c\|J^{-\frac{3s}{2}+\varepsilon} J^s u_3\|_{L_t^p(L_x^q)} = c\|J^{\frac{1}{p}} J^s u_3\|_{L_t^p(L_x^q)} \leq c\|u_3\|_{X_{s,b}(\phi)}.$$

The third factor can be treated in precisely the same way. So for the contributions of the region B we have obtained the desired bound.

Finally we consider the remaining region C: Here the $|\xi_i|$, $1 \le i \le 4$, are all very close together and $\ge c\langle \xi_i \rangle$. Moreover, at least three of the variables ξ_i have the same sign. Thus for the quantity $c.q.$ controlled by the expressions $\langle \tau - \xi^3 \rangle$, $\langle \tau_i - \xi_i^3 \rangle$, $1 \le i \le 4$, we have in this region:

$$c.q. := |\xi^3 - \sum_{i=1}^{4} \xi_i^3| \ge c \sum_{i=1}^{4} \langle \xi_i \rangle^3 \ge c \langle \xi \rangle^3$$

and hence, since $s > \frac{1}{3} + b'$ is assumed,

$$|\xi| \langle \xi \rangle^s \prod_{i=1}^{4} \langle \xi_i \rangle^{-s} \le c(\langle \tau - \xi^3 \rangle^{-b'} + \sum_{i=1}^{4} \langle \tau_i - \xi_i^3 \rangle^{-b'} \chi_{C_i}),$$

where in the subregion C_i, $1 \le i \le 4$, the expression $\langle \tau_i - \xi_i^3 \rangle$ is dominant. The first contribution can be estimated by

$$\begin{aligned} & c \| \textstyle\int d\nu \prod_{i=1}^{4} \langle \tau_i - \xi_i^3 \rangle^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ = \;& c \| \textstyle\prod_{i=1}^{4} J^s u_i \|_{L^2_{x,t}} \le c \displaystyle\prod_{i=1}^{4} \| J^s u_i \|_{L^8_{x,t}} \le c \prod_{i=1}^{4} \| u_i \|_{X_{s,b}(\phi)}, \end{aligned}$$

where we have used Lemma 2.7, part ii). For the contribution of the subregion C_1 we take into account that $\langle \tau_1 - \xi_1^3 \rangle = \max\{\langle \tau - \xi^3 \rangle, \langle \tau_i - \xi_i^3 \rangle, 1 \le i \le 4\}$, which gives

$$\langle \tau - \xi^3 \rangle^{b+b'} |\xi| \langle \xi \rangle^s \prod_{i=1}^{4} \langle \xi_i \rangle^{-s} \le c \langle \tau_1 - \xi_1^3 \rangle^b.$$

So, for this contribution we get the upper bound

$$\begin{aligned} & c \| \langle \tau - \xi^3 \rangle^{-b} \textstyle\int d\nu \langle \tau_1 - \xi_1^3 \rangle^b \prod_{i=1}^{4} \langle \tau_i - \xi_i^3 \rangle^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}} \\ \le \;& c \| \mathcal{F}^{-1} f_1 \textstyle\prod_{i=2}^{4} J^s u_i \|_{X_{0,-b}(\phi)} \le c \| \mathcal{F}^{-1} f_1 \prod_{i=2}^{4} J^s u_i \|_{L^{\frac{8}{7}}_{xt}} \\ \le \;& c \| \mathcal{F}^{-1} f_1 \|_{L^2_{xt}} \displaystyle\prod_{i=2}^{4} \| J^s u_i \|_{L^8_{x,t}} \le c \prod_{i=1}^{4} \| u_i \|_{X_{s,b}(\phi)}. \end{aligned}$$

Here we have used the dual version of the L^8-Strichartz estimate, Hölder and the estimate itself. For the remaining subregions C_i the same argument applies. □

A Appendix

A.1 Alternative proof of Lemma 4.3 (up to ε's)

Lemma A.1 [6] *Let $l \geq m$. Then in the onedimensional nonperiodic case the following trilinear refinement of Strichartz' inequality is valid:*

$$\|e^{it\partial^2}u_1 e^{it\partial^2}P_{\Delta l}u_2 e^{it\partial^2}P_{\Delta m}u_3\|_{L^2_{xt}} \leq c2^{\frac{m-l}{4}}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}\|u_3\|_{L^2_x}$$

Proof: By the standard Strichartz' estimate we may assume $m \ll l$. Arguing as in the proof of Lemma 2.4 we obtain

$$\begin{aligned} & \|e^{it\partial^2}u_1 e^{it\partial^2}P_{\Delta l}u_2 e^{it\partial^2}P_{\Delta m}u_3\|^2_{L^2_{xt}} \\ = \; & c\int d\xi \int_* d\xi_1 d\xi_2 d\eta_1 d\eta_2 \delta(\sum_{i=1}^3 \xi_i^2 - \eta_i^2)\prod_{i=1}^3 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)}.. \\ \times \; & ..\chi_{\Delta l}(\xi_2)\chi_{\Delta m}(\xi_3)\chi_{\Delta l}(\eta_2)\chi_{\Delta m}(\eta_3) \leq cI_1, \end{aligned}$$

with

$$I_1 = \int d\xi \int_* d\xi_1 d\xi_2 \prod_{i=1}^3 |\hat{u}_i(\xi_i)|^2 \int_* d\eta_1 d\eta_2 \delta(\sum_{i=1}^3 \xi_i^2 - \eta_i^2)\chi_{\Delta l}(\eta_2)\chi_{\Delta m}(\eta_3).$$

For the inner integral $I = I(\xi, \xi_1, \xi_2)$ we use the change of variable

$$y_1 = \eta_1 + \eta_2 - \frac{2\xi}{3} \qquad\qquad y_2 = \eta_1 - \eta_2$$

respectively

$$\eta_1 = \frac{1}{2}(y_1 + y_2) + \frac{\xi}{3} \qquad \eta_2 = \frac{1}{2}(y_1 - y_2) + \frac{\xi}{3} \qquad \eta_3 = \frac{\xi}{3} - y_1,$$

giving

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = \frac{1}{2}(3y_1^2 + y_2^2) + \frac{\xi^2}{3},$$

to obtain

$$I(\xi, \xi_1, \xi_2) = \int_{P(y_1,y_2)=0} \frac{dS_{(y_1,y_2)}}{|\nabla P(y_1,y_2)|}\chi_{\Delta l}(\frac{1}{2}(y_1 - y_2) + \frac{\xi}{3})\chi_{\Delta m}(\frac{\xi}{3} - y_1),$$

where $P(y_1, y_2) = \frac{1}{2}(3y_1^2 + y_2^2) - \sum_{i=1}^3 \xi_i^2 + \frac{\xi^2}{3}$ and $|\nabla P(y_1, y_2)| = (9y_1^2 + y_2^2)^{\frac{1}{2}}$. Writing $a^2 = \sum_{i=1}^3 \xi_i^2 - \frac{\xi^2}{3}$ we have $2^l \leq ca \leq c|\nabla P(y_1, y_2)|$ and omitting the $\chi_{\Delta l}$-factor we can estimate

$$I(\xi, \xi_1, \xi_2) \leq ca^{-1}\int_{3y_1^2+y_2^2=2a^2} dS_{(y_1,y_2)}\chi_{\Delta m}(y_1 - \frac{\xi}{3}).$$

[6] Notation as introduced before Lemma 2.5

The remaining line integral is the length of the intersection of the ellipse of dimension a with the strip of size 2^m around $\frac{\xi}{3}$. Elementary geometric considerations show that this can be estimated by $c2^{\frac{m}{2}}a^{\frac{1}{2}}$, which gives

$$I(\xi,\xi_1,\xi_2) \leq c2^{\frac{m}{2}}a^{-\frac{1}{2}} \leq c2^{\frac{m-l}{2}},$$

respectively

$$\|e^{it\partial^2}u_1 e^{it\partial^2}P_{\Delta l}u_2 e^{it\partial^2}P_{\Delta m}u_3\|^2_{L^2_{xt}} \leq c2^{\frac{m-l}{2}}\prod_{k=1}^3\|u_k\|^2_{L^2_x}.$$

□

Using the dyadic decomposition and Lemma 2.1 we get similarly as in the proof of Corollary 2.2

Corollary A.1 *Let $n = 1$, $\varepsilon > 0$ and $0 < s < \frac{1}{4}$ and $b > \frac{1}{2}$. Then, in the nonperiodic case the estimates*

i) $\|\prod_{k=1}^3 e^{it\partial^2}u_k\|_{L^2_{xt}} \leq c\|u_1\|_{L^2_x}\|u_2\|_{H^{-s}_x}\|u_3\|_{H^{s+\varepsilon}_x}$,

ii) $\|\prod_{k=1}^3 u_k\|_{L^2_{xt}} \leq c\|u_1\|_{X_{0,b}(\phi)}\|u_2\|_{X_{-s,b}(\phi)}\|u_3\|_{X_{s+\varepsilon,b}(\phi)}$

hold true for $\phi(\xi) = -\xi^2$.

A.2 Remark on $\delta(P)$

Let $P \in C^2(\mathbf{R}^n)$, $f \in C^0_0(\mathbf{R}^n)$ and $(J_\varepsilon)_{\varepsilon>0}$ a smooth approximate identity. Then we define $\delta(P)$ by

$$\int \delta(P(x))f(x)dx := \lim_{\varepsilon\to 0}\int J_\varepsilon(P(x))f(x)dx,$$

whenever the limit exists and is independent of $(J_\varepsilon)_{\varepsilon>0}$. Consider the integral

$$I := \int_{\mathbf{R}} dt \int e^{-itP(x)}f(x)dx,$$

where the inner integral is known to be nonnegative. Choosing $(J_\varepsilon)_{\varepsilon>0}$ even with $\mathcal{F}_t J_\varepsilon \nearrow \frac{1}{\sqrt{2\pi}}$ we obtain by the Beppo Levi and Fubini theorems that

$$I = 2\pi\int \delta(P(x))f(x)dx.$$

Under appropriate assumptions on P and f this can be expressed as a surface integral:

Lemma A.2 *Assume that $|\nabla P| \neq 0$ on* Supp $(f)\cap U$, where U is a neighbourhood of $\{P=0\}$, and that $f|_U \in C^1(U)$. Then

$$\int \delta(P(x))f(x)dx = \int_{P=0}\frac{f(x)}{|\nabla P(x)|}dS_x.$$

Proof: We can write $f = \sum_{k=0}^{n} f_k$, where f_0 is supported away from $\{P = 0\}$, and with $\frac{\partial P}{\partial x_k} \neq 0$ on Supp (f_k) for $1 \leq k \leq n$. Then $\int \delta(P(x)) f_0(x) dx = 0$, and for $1 \leq k \leq n$ we have with $\Phi_\varepsilon(x) = \int_{-\infty}^{x} J_\varepsilon(t) dt$:

$$\begin{aligned}
\int J_\varepsilon(P(x)) f_k(x) dx &= \int \left(\frac{\partial}{\partial x_k} \Phi_\varepsilon(P(x)) \right) \frac{f_k(x)}{\frac{\partial P}{\partial x_k}(x)} dx \\
&= -\int \Phi_\varepsilon(P(x)) \left(\frac{\partial}{\partial x_k} \frac{f_k(x)}{\frac{\partial P}{\partial x_k}(x)} \right) dx \\
&\overset{(\varepsilon \to 0)}{\longrightarrow} -\int_{P \geq 0} \left(\frac{\partial}{\partial x_k} \frac{f_k(x)}{\frac{\partial P}{\partial x_k}(x)} \right) dx = \int_{P=0} \frac{f_k(x)}{|\nabla P(x)|} dS_x,
\end{aligned}$$

where in the last step we have used the divergence theorem. □

Remarks : i) The surface integral in the above Lemma is essentially the <u>definition</u> of $\delta(P)$ given in [GS], chap. III, §1.

ii) In the onedimensional case the above formula reduces to

$$\int \delta(P(x)) f(x) dx = \sum_{x_n} \frac{f(x_n)}{|P'(x_n)|},$$

where the sum is taken over all simple zeros of P. Also this is given as <u>definition</u> of $\delta(P)$ in [GS], p. 180.

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