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SOME APPLICATIONS OF K-THEORY TO FRAMED BORDISM:

E-IN VARIANT AND TRANSFER

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Introduction

The projective spaces play a special role in topology. The complex projective space, for example, is closely related to spaces like $\text{BU}, \text{U}, \text{MU}$ or $\text{BZ}/m$. Therefore it is not surprising that the stable homotopy groups of $P^\infty_\mathbb{C}$ are important in the study of many interesting geometric and homotopy theoretic problems.

In this paper we study $\pi^S_\ast(P^\infty_\mathbb{C})$ using as the main tools the $e$-invariant of Adams and various transfer maps. The $e$-invariant describes the information about $\pi^S_\ast(X)$ obtainable using $K$-theory and the transfer maps relate $\pi^S_\ast(P^\infty_\mathbb{C})$ to the stable homotopy of spheres and allow us to construct elements in $\pi^S_\ast(P^\infty_\mathbb{C})$.

Although the $e$-invariant was used by Adams with notable success in the study of $\text{im}(J)$ and $\pi^S_\ast(S^0)$, not much is known about the $e$-invariant on other spaces than spheres. The main reason for this is the following. The computation of $\text{im}(e)$ in stable homotopy or stable cohomotopy in positive dimensions differs from the case of stable cohomotopy in negative dimensions in a substantial point. In the latter case, one has the $J$-homomorphism $J: K^{0-i}(X) \rightarrow \pi^{-i+1}_S(X)$ which gives all of $\text{im}(e)$ and the computation of $\text{im}(e)$ reduces to a purely $K$-theoretic problem. In the case of stable homotopy or positive dimensional cohomotopy one still has the restrictions posed on $\text{im}(e)$ by the Adams operations but, in contrast to the case of negative dimensional stable cohomotopy, these are not strong enough to describe $\text{im}(e)$ completely. In stable homotopy and in positive dimensional cohomotopy there is a sort of mixing between $\text{im}(J)$- and $\text{coker}(J)$-phenomena which is not seen by $K$-theory.
A better approximation for \( \text{im}(e) \) is obtained by using BP-homology and its operations. The resulting upper bound for \( \text{im}(e) \) is then the group \( \text{Ext}^{1, 2n}_{BP_* BP}(BP_* BP(\mathbb{P}^\infty), BP(\mathbb{P}^\infty)) \) of the BP-Adams spectral sequence. But even this is not sufficient to compute \( \text{im}(e) \), because there are nonzero differentials starting from these groups.

In the case of real projective space a complete determination of \( \text{im}(e) \) would imply the solution of the Kervaire-invariant problem. So one cannot expect an easy calculation of \( \text{im}(e) \) on the stable homotopy of a given space, but even from incomplete information on \( \text{im}(e) \) one can draw interesting consequences.

The \( S^1 \)-transfer \( t: \pi^S_{n}(\mathbb{P}^\infty) \rightarrow \pi^S_{n+1}(S^0) \) relates the stable homotopy of the complex projective space to the stable homotopy of spheres and in particular relates the elements of \( \pi^S_*(\mathbb{P}^\infty) \) which can be detected by the \( e \)-invariant to elements in \( \pi^S_*(S^0) \) of Adams filtration 2 which are at present known only partially.

One of the main advantages of this transfer is that it raises the degree of the Adams filtration, which is a measure of the complicatedness of elements in stable homotopy. This means that \( t \) maps less complicated elements to more complicated elements and because \( t \) has a good geometric interpretation one retains a geometric description for \( t(x) \) if \( x \) had one.

There is also a transfer \( t: \pi^S_*(\mathbb{P}^\infty \times \mathbb{P}^\infty) \rightarrow \pi^S_*(\mathbb{P}^\infty) \) which allows one to construct and to describe elements in \( \pi^S_*(\mathbb{P}^\infty) \) in geometric terms. Using these transfers several times one can, in a certain sense, resolve a part of \( \pi^S_*(S^0) \), that is lift to filtration 0 and then treat the problem by the methods of rational homology. Of course, the depth of the filtration must be paid for with the extent of the
computations; one has to calculate in large polynomial rings.

Another important property of the transfer is its relation to the J-homomorphism. The usual J-homomorphism can be stabilized to give a map $J' : \pi^S(SO) \to \pi^S(SO)$ of which the complex analogue can be completely described by the transfer.

We begin with the computation of the restrictions imposed on the image of the e-invariant on $\pi^S_*(P_{\infty} C)$ by the Adams operation $\psi^k$. It turns out that this is closely related to the algebraic K-theory of finite fields, constructed by Quillen. Let $A_*(-)$ denote the associated homology theory, which has as coefficients the K-groups of a finite field $F_k$, then the upper bounds given by the $\psi^k$ are essentially the groups $A_*(P_{\infty} C)$. In chapter 1 we compute the order and the number of cyclic summands of these groups and show how to describe elements in $A_*(P_{\infty} C)$ using representation theory. As a first application we determine the number of summands in the group $J(P_{\infty} C)$.

In chapter 2 we introduce transfer maps $t^k : \pi^S_*(P_{\infty} C^{kH}) \to \pi^S_*(S^O)$, where $P_{\infty} C^{kH}$ is the Thom space of $k$ times the Hopf bundle, which generalize the transfer $t : \pi^S_*(P_{\infty} C) \to \pi^S_*(S^O)$ and deduce some of their properties. These transfer maps can be represented by stable maps $\tau^k \in \pi^{2k-1}_*(P_{\infty} C^{kH})$ and we show that the representing map $\tau$ for the transfer $t'$ lies in the image of the J-homomorphism. We identify the cofibre spectrum of $\tau^k$ with a Thom spectrum and give formulas relating the images of the various $t^k$.

Next we discuss the relation of the transfer to the bistable
$J$-homomorphism $J'$: $\pi^S_\ast(SO) \to \pi^S_\ast(S^0)$. As with the transfer the map $J'$ has an important geometric description. If one identifies stable homotopy with reduced framed bordism then $J'$ is described as follows: An element in $\pi^S_\ast(SO)$ is given by a triple $(M, \phi, f)$ where $(M, \phi)$ is a bounding framed manifold and $f: M \to SO$ a map. Using this map one can twist the given framing $\phi$ to obtain a new framing $\phi^f$. Then $J'((M, \phi, f))$ is simply $(M, \phi^f)$. If $J'$ were onto then in each positive dimension $n$ one could find a framed manifold $(M^k, \phi)$ which bounded and gave all other elements in $\pi^S_n(S^0)$ by twisting the given framing.

That $J'$ should be onto was conjectured by G.W. Whitehead and is known to be true at the prime 2. As one of the main results of this paper, we use the relation between $t$ and $J'$ to show that at odd primes larger than 3 $J'$ is not surjective. The method of proving this is to use the cofibre spectrum of the transfer to transform the computation of $t$ from filtration 1 on $\pi^S_\ast(P_\infty C)$ to filtration 2 on $\pi^S_\ast(S^0)$ into a pure filtration 1 problem which can then be treated by means of the $e$-invariant.

The possibility of computing the transfer $t$ from filtration 1 to filtration 2 is then used in chapter 4 to find representing framed manifolds for some of the elements in $\pi^S_\ast(S^0)$. Let $p$ be an odd prime. In some range of dimensions all elements in the $p$-component of $\pi^S_\ast(S^0)$ can be constructed out of three manifolds: the Hopf bundle $\sigma \in \pi^S_2(P_\infty C)$ and 2 elements $x_0, x_1$ in $\pi^S_\ast(P_\infty C \times P_\infty C)$, which can be described by conditions formulated in rational cohomology. For example, for $x_0$ this reads as follows: $x_0$ is given by a framed manifold of dimension $2(p+1)$ and two complex line bundles $\xi_1, \xi_2$ on $M$, such that the Kronecker product $\langle c_1(\xi_1)c_1(\xi_2)^p, [M] \rangle$
is nonzero mod p. Using the Pontrjagin product structure on 
\( \pi_*^{S}(P_* \times P_* \mathbb{C}) \) - which is essentially given by tensor products of complex line bundles - and the transfer map, the elements \( \sigma, x_0, x_1 \) give all of \( \pi_n^{S}(S^0)(p) \) \( (n < 2(p-1)(p^2+1)-4) \) and almost all of \( \pi_n^{S}(P_* \mathbb{C})(p) \).

It is even possible to use this method to construct some new elements in \( \pi_*^{S}(S^0) \); we have done this only in one case, namely we have constructed the element \( \beta_9 \) for \( p = 3 \).

In chapter 6 we discuss methods for finding elements in the co-kernel of \( e: \pi_*^{S}(P_* \mathbb{C}) \to A_*^{S}(P_* \mathbb{C}) \). Elements in \( \text{coker}(e) \) are closely related to values of the transfer maps mentioned above. If the \( e \)-invariant mapped surjective by onto \( A_*^{S}(P_* \mathbb{C}) \) then all these transfers would have to map \( F^1 \pi_*^{S}(P_* \mathbb{C}^{kH}) \) to higher filtration in \( \pi_*^{S} \). Every element of \( F^2 \pi_*^{S}(S^0) \) which is in the image of some \( t^S \) gives rise to an infinite series of elements in \( \text{coker}(e) \). By the methods of § 3 we can determine \( \text{im}(t^S) \) in a certain range of dimensions and use this to compute \( \text{im}(e) \) and thereby \( \pi_*^{S}(P_* \mathbb{C})(p) \) in the same range.

The computation of the \( e \)-invariant on \( \pi_*^{S}(P_* \mathbb{C}) \) is connected with the problem of determining the image of the Hurewicz map

\[ h_m : \Omega_{2n-1}^{fr}(\mathbb{Z}/m) \to \Omega_{2n-1}^{U}(\mathbb{Z}/m) \]

from framed bordism to complex bordism or \( K \)-theory for the classifying space of a finite cyclic group. The image of \( h_m \) essentially describes the values which the \( a \)-invariant, an invariant derived from the equivariant signature, can take on free equivariantly framed \( \mathbb{Z}/m \)-manifolds. For \( m \) a prime, this is known. We compute as an application the image of \( h_m \) for \( m = p^2 \) and \( n \neq 0(p) \).
In the last chapter we deal with the problem of computing some James numbers. The James number $U(n,k)$ describes how far away the projection $\pi: W_{n,k} \to S^{2n-1}$ of the complex Stiefel manifold to the sphere is from having a section. There is a conjecture that these numbers are determined by $K$-theory. Using the properties of the transfer maps and the $e$-invariant we verify this conjecture in some cases, precisely we compute $U(n+M_k,k)$ where $M_k$ is the order of the Hopf bundle in $J(P_{k-1}C)$ and $n=1,2,k-1,k-2$ and $k+2$ for all $k$.

The numbers $U(n,k)$ are closely related to the values of the transfer maps $t^k$ on $F^0\pi_*^s(P_{\infty}C^kH)$. As an application it is shown that the elements $u_r$ in the 2-component of $\pi_*^s(S^0)$ are in the image of the transfers $t^2$ and $t(-2)$.

Another application of the computation of $U(n,k)$ is the construction of infinite families of elements in the metastable homotopy groups $\pi_{2m+2t}(U(2m))$ of the unitary group.
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§7 Some James numbers
§ 1 The algebraic K-groups of $P_\infty$.  

In this chapter we define the e-invariant on $\pi^S_*(P_\infty)$ and show how it is related to the Hurewicz map $h: \pi^S_*(P_\infty) \to A_*(P_\infty)$, where $A_*(X)$ is the homology theory defined by the $K$-theory of a finite field. We then compute the order and the number of cyclic summands of $A_*(P_\infty)$ and show how to describe the elements of $A_*(P_\infty)$. As an application we determine the number of cyclic summands in the group $J(P_n)$. 
The e-invariant of a stable map \( f: S^{m+k} \rightarrow S^k X \) is defined if \( f^* = 0 \) in K-theory and classifies the extension

\[
0 \rightarrow \tilde{K}^*(S^{m+k}) \rightarrow \tilde{K}^*(C_f) \rightarrow \tilde{K}^*(S^k X) \rightarrow 0
\]

induced by the cofibre sequence of \( f \) (\( C_f = \text{cofibre of } f \)).

In the case of a space \( X \) with torsion-free homology groups, this definition can be reformulated using the functional Chern character (see \([2] \)). The Thom-Pontrjagin construction identifies stable homotopy \( \pi_n^s(X^+) \) with framed bordism \( \Omega_n^{fr}(X) \), and so an element in \( \pi_n^s(X) \) may be given by a framed manifold rather than by a stable map. For this situation a third definition is appropriate. The Hurewicz map \( h: \Omega_n^{fr}(X) \rightarrow K_n(X) \) can be interpreted directly in terms of framed manifolds. The following diagram

\[
\begin{array}{cccc}
\Omega_n^{fr}(X; \mathbb{Q}) & \xrightarrow{\beta} & \Omega_n^{fr}(X; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{r} & \Omega_n^{fr}(X; \mathbb{Q}) \\
\downarrow h_{\mathbb{Q}/\mathbb{Z}} & & \downarrow h & & \downarrow h \\
K_n(X; \mathbb{Q}) & \xrightarrow{j} & K_n(X; \mathbb{Q}/\mathbb{Z}) & \rightarrow & K_1(X)
\end{array}
\]

defines the functional Hurewicz map \( h \) as a map from the subgroup \( \ker(r) \cap \ker(h) \) of \( \Omega_n^{fr}(X) \) into \( K_2n(X; \mathbb{Q})/(K_2n(X; \mathbb{Z})+H_2n(X; \mathbb{Q})) \).

For \( X \) with \( H_n(X; \mathbb{Z}) \) torsion-free, the last group can be identified via the K-theory Kronecker product with the range of the functional Chern character. A proof that \( h \) is the same as the functional Chern character and so gives the e-invariant can be found in \([21] \).

Stable operations act trivially on classes in the image of \( h \), so they give upper bounds for the image of \( e \). The Adams operations \( \psi^k \) are not
compatible with Bott periodicity and so not stable. To get a stable operation from the Adams operation $\psi^k$, one has to introduce coefficients $R$, in which $k$ is invertible. The stable operation

$$\psi^k_n: K^n(X; R) \to K^n(X; R)$$

is then defined for even values of $n$ by $\psi^k_{2m} = k^{-m} \psi^k$ and for $n$ odd via suspension. Because of the stability of the operations $\psi^k_n$, they induce operations in the $K$-homology groups, also denoted by $\psi^k_n$.

Immediately from the definition of the Hurewicz map one sees that $(\psi^k_n - 1)$ vanishes on $\text{im}(h: \pi^n_n(X) \to K^n_n(X))$. So the subgroups $\ker(\psi^k_n - 1) \subset K_n(X; Q/\mathbb{Z})(p)$ for $k \neq 0(p)$ constitute upper bounds for the $p$-component of $\text{im}(e)$ in dimension $n$. Our principal tool for calculating these upper bounds is the algebraic $K$-theory introduced by Quillen: Seymour [36] has constructed a multiplicative cohomology theory $A^n_{a}(X)_R$ (a a prime, $1/a \in R$, $R = \mathbb{Q}$) which fits into the exact sequence

$$\rightarrow A^n_{a}(X)_R \xrightarrow{k} K^n(X; R) \xrightarrow{\psi^a_{n-1}} K^n(X; R) \xrightarrow{j} A^{n+1}_{a}(X)_R \rightarrow (1.1)$$

$A^0_{a}(X)$ is based on vector bundles invariant under the Adams operation $\psi^a$.

In the following we choose a fixed odd prime $p$ and a prime number $k$ generating the group of units $(\mathbb{Z}/p^2\mathbb{Z})^*$. We set $R = \mathbb{Z}(p)$, the integers localised at $p$, and denote $A^n_{a}(X)_R$ simply by $A^n_{a}(X)$. The coefficient groups $A^n_{a}(\mathbb{X})$ are computed directly from (1.1) using

$$v_p(k^t - 1) = \begin{cases} 0 & \text{if } t \not\equiv 0(p-1) \\ 1 + v_p(t) & \text{if } t \equiv 0(p-1) \end{cases}$$

(1.2)

where $v_p(n)$ is the power of $p$ in $n$ (see [11]).

To the cohomology theory $A^n_{a}$ there correspond a homology theory $A^n_{a}$
and connected versions, denoted by $A^n$ and $A^\omega_n$. The groups $A^n$ are isomorphic to the higher algebraic $K$-groups for the finite field $F_k$ constructed by Quillen [32]. For $p = 2$ it is possible to choose a better $A^\omega$, see [36].

The Hurewicz map $h: \pi_n(X^+) \to K_n(X)$ factorizes through $A_n(X)$, but also the $e$-invariant is determined by $h_A: \pi_n^e(X) \to A^\omega_n(X)$.

**Lemma 1.3:** Let $H(X;\mathbb{Z}(p))$ be torsion-free, then the $p$-component of the $e$-invariant on the torsion subgroup of $\pi_n^e(X)$ is determined by the Hurewicz map $h_A: \pi_n^e(X) \to A_{2n-1}(X)$.

**Proof:** We have a commuting diagram of Bockstein sequences

\[ \begin{array}{ccccccccc}
\pi_n^s(X;\mathbb{Q}) & \longrightarrow & \pi_n^s(X;\mathbb{Z}(p)) & \beta & \longrightarrow & \pi_n^s(X;\mathbb{Z}(p)) & \longrightarrow & \pi_n^s(X;\mathbb{Q}) \\
\downarrow & & \downarrow h_{\mathbb{Z}} & & \downarrow h & & \downarrow h_A & \\
Ad_2^n(X;\mathbb{Q}) & \longrightarrow & Ad_2^n(X;\mathbb{Z}(p)) & \longrightarrow & Ad_2^n(X;\mathbb{Z}(p)) & \longrightarrow & K_1(X;\mathbb{Z}(p)) \\
K_2^n(X;\mathbb{Q}) & \longrightarrow & K_2^n(X;\mathbb{Z}(p)) & \longrightarrow & K_1(X;\mathbb{Z}(p)) \\
\end{array} \]

which shows, that $h_r = i^{-1}h_{\mathbb{Q}/\mathbb{Z}}s^{-1}$ factorizes through $Ad_2^n(X;\mathbb{Z}(p))/im Ad_2^n(X;\mathbb{Q}) \cong ker(r_A)$ and is thus determined by $h_{Ad}$; clearly $h_{Ad}$ factorizes through $h_A$.

Because of (1.3) we will also use the name $e$-invariant for the map $h_A: \pi_n^s(\mathbb{P}^\infty) \longrightarrow A_{2n-1}(\mathbb{P}^\infty)$.

A first upper bound for $im(e)$ on $\pi_n^s(\mathbb{P}^\infty)(p)$ is thus given by the group $A_{2n-1}(\mathbb{P}^\infty)$. Let $\alpha_p(n)$ denote the sum of the coefficients in the $p$-adic representation of the number $n$. 
Proposition 1.4: Let $n = t(p-1) + s$ with $0 < s < p-1$, then the order of $\tilde{A}_{2n-1}(P_\infty)$ is $p^a$, where

$$a = \frac{t}{p-1} \left( \sum_{i=1}^{n} \nu_p(i) + \frac{\alpha_p(n-s)}{p-1} \right)$$

Proof: We first calculate $\text{Ad}_{2n-1}(P_{n-1})$. As a basis of $K_0(P_{n-1})$ we choose the elements $b_i$ (i$\neq$0) dual to $(L-1)^i \in K^0(P_{n-1})$, where $L$ is the universal line bundle. Via the Kronecker product the operation $\psi_n^k$ induces a dual operation $c(\psi_n^k)$ in K-homology:

$$< y, c\psi_n^k(x) > = < \psi_n^k(y), x >$$

for all $y \in K^0(P_{n-1})$. It is easy to see that $c\psi_n^k = (\psi_n^k)^{-1}$. To calculate kernel or cokernel of $\psi_{2n}^k - 1$ we can therefore equally well use $c(\psi_{2n}^k)^{-1}$. Using $\psi_0^k(x^i) = (L-1)^i = ((x+1)^k-1)^i$, one sees that $c\psi_{2n}^k$ is given by an upper triangular matrix $B$ with respect to the basis $b_1, ..., b_{n-1}$. The diagonal entries are $k^{i-n-1}$. The determinant of $B$ gives the order of $\text{coker}(B) = \text{Ad}_{2n-1}(P_{n-1})$. By (1.2) we have

$$\nu_p(\det B) = \frac{n-1}{p-1} \left( \sum_{i=1}^{n-1} (1+\nu_p(i)) \right).$$

The group $A_{2n-1}(P_\infty)$ is defined as the image of $\text{Ad}_{2n-1}(P_{n-1})$ in $A_{2n-1}(P_n)$ and is therefore isomorphic to

$$\text{Ad}_{2n-1}(P_{n-1})/\text{im}(\partial: A_{2n}(P_n;P_{n-1}) \rightarrow \text{Ad}_{2n-1}(P_{n-1}))$$

Because of $\text{Ad}_{2n}(P_n;P_{n-1}) = \text{Ad}_{2n}(S^{2n}) \cong Z(p)$ we only need to calculate the image of $\text{Ad}_{2n}(P_n) \rightarrow \text{Ad}_{2n}(P_n;P_{n-1})$. For this we need $\text{Ad}_{2n}(P_n)$. The multiplication map of $P_\infty$ makes the groups $\text{Ad}_*(P_\infty)$, $A_*(P_\infty)$ and $K_*(P_\infty)$ into Pontrjagin rings.

Lemma 1.5: The $n^{th}$-power of $b_1$ in the Pontrjagin product generates
I. For low values of $n$, one can use the theory of elementary divisors to calculate the structure of $\text{coker } c(\psi_{2n}^k - 1)$, which is $\text{Ad}_{2n}(\mathbb{P}_\infty \mathcal{C})$. The exact sequence of $(\mathbb{P}_n \mathcal{C}, \mathbb{P}_{n-1} \mathcal{C})$ then shows, that $\text{A}_{2n-1}(\mathbb{P}_\infty \mathcal{C})$ is the torsion subgroup of $\text{Ad}_{2n-1}(\mathbb{P}_n \mathcal{C})$.

The Chern character maps the eigenspace of $c \psi_0^k$ for the eigenvalue $k^n$ in $K_*(\mathbb{P}_\infty \mathcal{C}; \mathbb{Q})$ bijectively onto $H_{2n}(\mathbb{P}_\infty \mathcal{C}; \mathbb{Q})$. So every element in $\ker(c \psi_0^k - 1)$ is a rational multiple of $b^n_1$. An elementary calculation gives $b^n_1$ as a linear combination of the elements $b_1$:

$$b^n_1 = \sum_{k=1}^{n} k! S(n, k) b_k$$

where

$$S(n, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k+j} \binom{k}{j} j^n$$

is a Stirling number of the second kind. Since $S(n, 1) = 1$, there is no element $z$ in $K_0(\mathbb{P}_\infty \mathcal{C})$ with $b \cdot z = b^n_1$ and $b \neq 2$ and $b \neq 1$ and $b^n_1$ must generate $\text{Ad}_{2n}(\mathbb{P}_\infty \mathcal{C})$.

We now can finish the proof of (1.4):

Since $S(n, n) = 1$, we have $b^n_1 = n! \cdot b_n + \ldots$ and because $\text{Ad}_{2n}(\mathbb{P}_n \mathcal{C}, \mathbb{P}_{n-1} \mathcal{C})$ is generated by the image of $b^n_1$, the map $\text{Ad}_{2n}(\mathbb{P}_n \mathcal{C}) \to \text{Ad}_{2n}(\mathbb{P}_n \mathcal{C}, \mathbb{P}_{n-1} \mathcal{C})$ is multiplication by $n!$. Im $\theta$ is therefore isomorphic to $\mathbb{Z}/n! \mathbb{Z}$ and the number of elements in $\text{Ad}_{2n-1}(\mathbb{P}_\infty \mathcal{C})$ is $p^a$ where

$$a = \sum_{i=1}^{t} \nu_p(i) - \nu_p(n!)$$

Applying the well known formula for $\nu_p(n!)$, gives then (1.4).

Remarks:

1. For low values of $n$, one can use the theory of elementary divisors to calculate the structure of $\text{coker } c(\psi_{2n}^k - 1)$, which is $\text{Ad}_{2n-1}(\mathbb{P}_n \mathcal{C})$. The exact sequence of $(\mathbb{P}_n \mathcal{C}, \mathbb{P}_{n-1} \mathcal{C})$ then shows, that $\text{A}_{2n-1}(\mathbb{P}_\infty \mathcal{C})$ is the torsion subgroup of $\text{Ad}_{2n-1}(\mathbb{P}_n \mathcal{C})$.

2. To calculate in $\text{A}_{2n-1}(\mathbb{P}_\infty \mathcal{C})$ one thinks of $\text{A}_{2n-1}(\mathbb{P}_\infty \mathcal{C})$ as
\( \text{Ad}_{2n} (\mathbb{P}_n \mathbb{C}; \mathbb{Q}/\mathbb{Z})/\text{im} \text{Ad}_{2n} (\mathbb{P}_n \mathbb{C}; \mathbb{Q}) \) (The group \( \text{Ad}_{2n} (\mathbb{P}_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) is the kernel of \( c\psi^k \cdot k^n \) on \( K_0 (\mathbb{P}_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) and \( \text{im} \text{Ad}_{2n} (\mathbb{P}_\infty \mathbb{C}; \mathbb{Q}) = b_1^n \cdot \mathbb{Q}/\mathbb{Z} \))

that is, one calculates in \( \ker (c\psi^k \cdot k^n) \subset K_0 (\mathbb{P}_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) modulo \( b_1^n \cdot \mathbb{Q}/\mathbb{Z} \).

To describe elements in \( \text{A}_{2n-1} (\mathbb{P}_\infty \mathbb{C}) \) explicitly and to avoid the theory of elementary divisors, we study the natural map

\[ \pi_* : \text{A}_{2n-1} (\mathbb{B}_2/\mathbb{P}^r) \to \text{A}_{2n-1} (\mathbb{P}_\infty \mathbb{C}) \].

For fixed \( n \) and sufficiently high value of \( r \) \( \pi_* \) becomes surjective. This is also true for \( \pi_*^S \) as shown in [21]. The same proof applies for \( A_n \). There is a description of \( K_1 (\mathbb{B}_2/\mathbb{P}^r) \) which leads to an easy computation of

\[ \ker (c\psi^k \cdot 2n-1) = \text{Ad}_{2n-1} (\mathbb{B}_2/\mathbb{P}^r) \].

This is done in [22], we summarize here results from [22], which we shall need.

Let \( R (\mathbb{Z}/\mathbb{P}^r) = \mathbb{Z}[\rho]/(\rho^p - 1) \) denote the representation ring of \( \mathbb{Z}/\mathbb{P}^r \), where \( \rho \) is a primitive one-dimensional complex representation.

In [45] and [22] it is shown that \( K_1 (\mathbb{B}_2/\mathbb{P}^r) \) is isomorphic to \( R (\mathbb{Z}/\mathbb{P}^r) \otimes \mathbb{Z}/\mathbb{P}^\infty \), where \( R (\mathbb{Z}/\mathbb{P}^r) \) is the quotient of \( R (\mathbb{Z}/\mathbb{P}^r) \) by the ideal given by the regular representation of \( \mathbb{Z}/\mathbb{P}^r \).

We define the element \( x(i, k) \) to be \( p^{-k} \sum s^n \cdot \rho^s \cdot p^i \) where the sum is taken over all \( s, 1 \leq s \leq p^{r-i} \) (i.e. with \( s \neq 0(p) \)).

**Lemma 1.6 ([22]):** \( \text{Ad}_{2n-1} (\mathbb{B}_2/\mathbb{P}^r) \) is isomorphic to

\[ \bigoplus_{i=0}^{r-1} \mathbb{Z}/\mathbb{P}^{r-i} \cdot p^i (n) \]

and the \( i \text{th} \)-summand is generated by \( x(i, r-i) \).

For the rest of this section we identify the groups \( K_1 (\mathbb{B}_2/\mathbb{P}^r), K_0 (\mathbb{B}_2/\mathbb{P}^r ; \mathbb{Q}/\mathbb{Z}) \) and \( R (\mathbb{Z}/\mathbb{P}^r) \otimes \mathbb{Z}/\mathbb{P}^\infty \).
The map $\pi^*: K_*^{0} (B\mathbb{Z} / p^r; \mathbb{Q} / \mathbb{Z}) \to K_*^{0} (P_{\infty} \mathbb{C}; \mathbb{Q} / \mathbb{Z})$ can be calculated as follows: we choose $\rho$, so that $\pi^* (L) = \rho$. If we write

$$\pi^* (x(i,k)) = p^{-k} \sum_j a_j b_j,$$

we have

$$a_j / p^k = < \pi^* (L - 1)^j, x(i,k) >,$$

where $<$, $>$ is the Kronecker product between $\tilde{K}_0^0 (X)$ and $\tilde{K}_0^1 (X; \mathbb{Q} / \mathbb{Z})$ with values in $\mathbb{Q} / \mathbb{Z}$.

For $X = BG$ this is calculated in [22]. One has

$$< (L - 1), \rho^k / p^s > = \begin{cases} 1 / p^s & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

By the binomial theorem

$$a_j / p^k = \sum_{l \leq m < p^r - i} \binom{j}{mp_i} (-1)^j m^n p^k$$

This formula shows that the element $y_o = \sum_i x(i, k - n)$ is mapped onto $p^{-k} b_1$ in $K_*^{0} (P_{\infty} \mathbb{C}; \mathbb{Q} / \mathbb{Z})$. To compute $A_{2n-1} (B\mathbb{Z} / p^r)$ it is only necessary to find out which linear combinations of $x(i, j)$ $(i > 0, j \in 2n - n)$ are in

$$\text{im} (\text{Ad}_{2n-1} (B\mathbb{Z} / p^r) (2n-1) \to A_{2n-1} (B\mathbb{Z} / p^r) (2n)) = A_{2n-1} (B\mathbb{Z} / p^r).$$

The next proposition allows one to calculate $A_{2n-1} (B\mathbb{Z} / p^r)$.

**Proposition 1.8:** Let $\omega = \sum_{i > 0} d_i x(i, j)$ be an element of $\text{Ad}_{2n-1} (B\mathbb{Z} / p^r)$ $\cong \text{Ad}_{2n} (B\mathbb{Z} / p^r; \mathbb{Q} / \mathbb{Z})$ and

$$\pi^* (\omega) = \sum_j c_j b_j$$

it's image in $\text{Ad}_{2n} (P_{\infty} \mathbb{C}; \mathbb{Q} / \mathbb{Z})$.

Then $\omega \in A_{2n-1} (B\mathbb{Z} / p^r)$ iff $c_j = 0$ for $j > n$.

**Proof:** If $\omega \in \text{Ad}_{2n} (B\mathbb{Z} / p^r; \mathbb{Q} / \mathbb{Z})$ has $\pi^* (\omega) = \sum_{j=0}^{n} c_j b_j$, then

$$\pi^* (\omega) \in \text{Ad}_{2n} (P_{\infty} \mathbb{C}; \mathbb{Q} / \mathbb{Z}).$$

We will show that $\pi^* (\omega)$ lies in the image.
of \( \text{Ad}_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to \text{Ad}_{2n}(P_n \mathbb{C};\mathbb{Q}/\mathbb{Z}) \). The sphere bundle of \( L^r_p \to P_n \mathbb{C} \) is the \((2n+1)\)-skeleton of \( B\mathbb{Z}/p^r \). We look at the Gysin sequence of \( L^r_p \) in \( K \)-homology with \( \mathbb{Q}/\mathbb{Z} \)-coefficients:

\[
0 \to K_0(S(L^r_p);\mathbb{Q}/\mathbb{Z}) \xrightarrow{\pi_*} K_0(P_n \mathbb{C};\mathbb{Q}/\mathbb{Z}) \xrightarrow{\pi^*} K_0(P_n \mathbb{C};\mathbb{Q}/\mathbb{Z}) \to 0 \quad (1.9)
\]

\( \pi^*(L^r_p) = 1 \) implies \( \pi^*(e(L^r_p)) = \pi^*(1-L^r_p) = 0 \) and so

\[
\pi^*(k(\omega)) \cap e(L^r_p) = \pi^*(k(\omega) \cap \pi^*(e(L^r_p))) = 0,
\]

where \( k: \text{Ad}_{2n}(P_n \mathbb{C};\mathbb{Q}/\mathbb{Z}) \to K_0(P_n \mathbb{C};\mathbb{Q}/\mathbb{Z}) \) is the canonical map. Thus by exactness of (1.9) there is an element \( z \in K_0(S(L^r_p);\mathbb{Q}/\mathbb{Z}) \) with \( \pi_*(z) = k\pi_*(\omega) \). The exact sequence of the pair \(((B\mathbb{Z}/p^r)^{(2n)},(B\mathbb{Z}/p^r)^{(2n+1)})\) shows that

\[
j: K_0((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to K_0((B\mathbb{Z}/p^r)^{(2n+1)};\mathbb{Q}/\mathbb{Z}) \text{ is isomorphic. By the injectivity of } j_* \text{ and } \pi_* \text{ we have } c(\psi^k - k^n)j_*^{-1}(z) = 0, \text{ because this is true for } k(\omega).
\]

Hence \( j_*^{-1}(z) \in \text{im}(\text{Ad}_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to K_0((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z})) \).

Because \( k \) and \( \pi_*: \text{Ad}_{2n}(B\mathbb{Z}/p^r,\mathbb{Q}/\mathbb{Z}) \to \text{Ad}_{2n}(P_\infty \mathbb{C};\mathbb{Q}/\mathbb{Z}) \) are injective it follows by naturality, that

\[
\omega \in \text{im}(\text{Ad}_{2n}((B\mathbb{Z}/p^r)^{(2n)};\mathbb{Q}/\mathbb{Z}) \to \text{Ad}_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}))
\]

which means \( \omega \in \text{im}(A_2n(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z}) \to \text{Ad}_{2n}(B\mathbb{Z}/p^r;\mathbb{Q}/\mathbb{Z})) \).

Remarks:

1. By (1.7) the conditions \( c_r = 0 \) for \( r > n \) can be expressed explicitly as

\[
\sum_i (a_{i,r}d_i)/p^{ji} \in \mathbb{Z}/p
\]

for \( r > n \).

Example: \( p = 3, x(1,1) = (\rho^3 + 2\rho^5)/(2) \) has \( \pi_*(x(1,1)) = (b_3 - b_6 + b_9)/3 \) in \( A_{9}(P_\infty \mathbb{C}) = \mathbb{Z}/3 \) and gives the first torsion element in \( A_{8}(P_\infty \mathbb{C}) \).
2. Whereas the elements $b_n$ and so $\sum_i x(i,k-i\cdot n)$ have skeleton filtration $2n$, the individual elements $x(i,k)$ are not closely related to filtration. Modulo the summand generated by $y_o = \sum_i x(i,r+v_p(n)-i\cdot n)$ the group $A_{2n-1}(BZ/p^r)$ becomes stationary as a function of $r$ if $r>r_o$, namely

$$A_{2n-1}(BZ/p^r)/<y_o> \cong A_{2n-1}(P_\infty)$$

The Adams operations $\psi^m$ of $K_*(P_\infty;Z/p^\infty)$ with $m \neq o(p)$ induce operations in $Ad_*(P_\infty;Q/Z)$, because they commute with $\psi^k$. To see that the use of these $\psi^m$ cannot improve the upper bound for $\text{im}(e)$ given by $A_*(P_\infty)$, we show

**Lemma 1.10:** Let $m \neq o(p)$, then $c\psi^m_0$ acts on $A_{2n-1}(P_\infty)$ as multiplication by $m^n$.

**Proof:** By surjectivity of $\pi_*: A_{2n-1}(BZ/p^r) \rightarrow A_{2n-1}(P_\infty)$, $r$ large, the above statement follows from the corresponding statement for $A_*(BZ/p^r)$ and because $A_{2n-1}(BZ/p^r)$ is a subgroup of $Ad_{2n-1}(BZ/p^r)$ from that for $Ad_{2n-1}(BZ/p^r)$. The effect of $\psi^m$ on elements of $Ad_{2n-1}(BZ/p^r) \subset K_1(BZ/p^r)$ is calculated in [22].

To determine the number of cyclic summands in $A_{2n-1}(P_\infty)$ we require some preparation. We set $a(n) =$ number of cyclic summands in $A_{2n-1}(P_\infty)$. Multiplication by $p$ on the H-space $P_\infty$ induces a map $m_p : A_{2n-1}(P_\infty) \rightarrow A_{2n-1}(P_\infty)$.

**Lemma 1.11:** For $x_i = \pi_*(x(i,1)) \in Ad_{2n}(P_\infty;Q/Z)$ we have

$$m_{p^*}(x_i) = x_{i-1}(i\cdot 1) \text{ and } m_{p*}(x_0) = 0$$
Proof: The natural projection \( \pi: \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \) induces a map \( Bpr: B\mathbb{Z}/p^k \to B\mathbb{Z}/p^{k-1} \) which fits into a commuting diagram

\[
\begin{array}{ccc}
B\mathbb{Z}/p^k & \xrightarrow{\pi} & P_\infty C \\
\downarrow Bpr & & \downarrow m_p \\
B\mathbb{Z}/p^{k-1} & \xrightarrow{\pi} & P_\infty C
\end{array}
\]

The map \( Bpr_*: K_1(B\mathbb{Z}/p^k) \to K_1(B\mathbb{Z}/p^{k-1}) \) is calculated in [22]. We have \( Bpr_*(\rho^i) = 0 \) if \( i \not\equiv 0(p) \) and \( Bpr_*(\rho^i \mathbb{Z}/p) = \rho^i \), which immediately gives \( Bpr_*(x(i,1)) = x(i-1,1) \). Using the diagram above it follows \( m_p^*(x_i) = x_{i-1} \).

The idea in computing the number of summands is to find the maximal value of \( i \) for which \( \beta x_i \) can be in \( A_{2n-1}(P_\infty C) \).

Lemma 1.12: The number of cyclic summands in \( A_{2n-1}(P_\infty C) \) is

\[
\max \{ i \mid \beta(x_i) \text{ lies in } \text{im}(A_{2n-1}(P_\infty C) \to Ad_{2n-1}(P_\infty C)) \} = a(n)
\]

Proof: Let \( U \) (resp. \( U' \)) be the subgroup of elements of order \( p \) in \( A_{2n-1}(P_\infty C) \) (resp. in \( A_{2n}(P_\infty C; Q/Z) \)). The kernel of \( \beta |_{U'} \) is \( Z/p \cdot x_0 \), so \( a(n) = \dim Z/p \cdot U = \dim Z/p \cdot U'-1 \). We know that every element in \( A_{2n-1}(P_\infty C) \) is a linear combination of the elements \( \beta(\tau_*(x(i,j))) \text{ (i\neq 0)} \), so every element in \( U' \) is a sum of \( x_i \text{ (i\neq 0)} \). Let \( e \) be the maximal index of an \( x_i \) appearing in a linear combination for an element in \( U' \), say \( y = \sum_{m=0}^e a_m x_m \), \( a_e \neq 0 \). We can assume \( a_m = 1 \). Clearly \( a + 1 \leq e + 1 \). By (1.11) we have \( m_p \cdot (y) = x_0 \), so \( x_0 \in U' \), \( m_p \cdot (y) = a_e x_0 + x_{i-1} \), so \( x_{i-1} \in U' \). Inductively it follows that \( x_0, \ldots, x_e \in U' \). The equation \( \sum_{i=0}^e a_i \cdot x_i = 0 \) with \( a_r \neq 0 \) implies \( m_p \cdot (x) = a_m x_0 = 0 \) so the \( x_0, \ldots, x_e \) are linear independent and we have \( e + 1 \geq a + 1 \).
Remark:
In general that is for elements not of order p it may happen that the maximal value of $i$ for which $x(i,j)$ appears in an element $y \in A_{2n-1}(P_\infty \mathbb{C})$ is greater than the number of summands: For example $y = x(1,2)+x(2,1) \in \text{Ad}_{13}(B \mathbb{Z}/9)$ is in $A_{13}(B \mathbb{Z}/9)$ and $\pi_*(y)$ generates $A_{13}(P_\infty \mathbb{C}) \cong \mathbb{Z}/9$ $(p=3)$, but $\pi_*(x(2,1)) \notin A_{13}(P_\infty \mathbb{C})$.

Determining $a(n) = \max \{ i \mid \beta(x_i) \in A_{2n-1}(P_\infty \mathbb{C}) \}$ is by (1.7) equivalent to finding the maximal $i$ satisfying $a_j(i)$ is divisible by $p$ for $j > n$.

**Theorem 1.13:** Let $p$ be an odd prime. The number of cyclic summands in $A_{2n-1}(P_\infty \mathbb{C})$ is given by

$$\left\lfloor \log \frac{n+1}{s+1} \right\rfloor$$

where $n = t(p-1)+s$ and $0 < s < p-1$.

**Proof:** The element $x_r$ is the image of $\sum_{j=1}^{r-1} \sum_{j=p}^{j=p} a_j / p$ under the map $\text{Ad}_{2n}(B \mathbb{Z}/p^{r+1}; \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Ad}_{2n}(P_\infty \mathbb{C}; \mathbb{Q}/\mathbb{Z})$. Let $x_r = \sum_{m} a_m b_m / p$ with $a_m(r)$ defined as in (1.7). The first $m$ with non vanishing $a_m$ is $m = p^{r+1}$. We set $q = p^{r+1}$ then $\pi^*(L-1)^q = \pi^*(L^q-1)$ mod $p$ and $\pi^*(L^q) = 1$ implies $\pi^*(L-1)^m$ is divisible by $p$ for $m > q$, so

$$a_m(r) = \langle \pi^*(L-1)^m, x(r,1) \rangle = 0 \text{ if } m > q$$

and we are left with $a_m(r)$ with $p^r < m < p^{r+1}$.

Let $\sum a_k p^k$ be the $p$-adic expansion of $m$. By the formula for the
mod p value of \( \binom{a}{b} \) we have

\[
a_m(r) = \sum_{j=1}^{p-1} \binom{m}{p-r} j^n(-1)^{j+m} \equiv \sum_{j=1}^{p-1} \binom{a_r}{j} j^n(-1)^{j+m} \mod p
\]

Let \( n = t(p-1)+s \) with \( 0 < s \leq p-1 \), then \( j^n \equiv j^s \mod p \) because \( j^n-j^s = j^s(j^{(p-1)}-1) \) is divisible by \( p \) if \( j \neq o(p) \), so

\[
a_m(r) = (-1)^{ar+m} \sum_{j=1}^{p-1} \binom{ar}{j} j^s(-1)^{s+ar} = (-1)^{ar+m}a_r! \cdot S(s, ar)
\]

where \( S(a,b) \) is a Stirling number of the second kind. We have 

\[
a_m(r) = o(p) \text{ iff } S(s, ar) = o(p).
\]

Now \( S(a,a) = 1 \) and \( S(a,b) = 0 \) for \( b > a \) implies \( a_m(r) = o(p) \) for \( m > (s+1)p^r \). We fix \( s \) with \( 0 < s \leq p-1 \) and consider now only such \( n \) satisfying \( n \equiv s \mod (p-1) \).

The above calculations show \( x_r \in \text{Ad}_{2n}(\mathbb{P}_n,\mathbb{C};\mathbb{Q}/\mathbb{Z}) \) if \( n > (s+1)p^r \) and \( x_r \in \text{Ad}_{2n}(\mathbb{P}_n,\mathbb{C};\mathbb{Q}/\mathbb{Z}) \) for \( n < (s+1)p^r - 1 \). So for all \( n \) (with \( n \equiv s \mod (p-1) \)) satisfying \( (s+1)p^r \leq n + 1 < (s+1)p^{r+1} \) we have

\[
\max \{ i \mid x_i \in \text{Ad}_{2n}(\mathbb{P}_n,\mathbb{C};\mathbb{Q}/\mathbb{Z}) \} = r = a(n).
\]

The function \( \log \) is monotonic, so

\[
(s+1)p^r \leq n + 1 < (s+1)p^{r+1} \iff \log ((s+1)p^r) \leq \log (n+1) < \log ((s+1)p^{r+1})
\]

\[
\iff r \leq \frac{\log(n+1)}{\log(s+1)} < r+1
\]

\[
\iff r = \left[ \frac{\log(n+1)}{\log(s+1)} \right]
\]

where \([x]\) denotes the greatest integer not exceeding \( x \).

Another application of (1.11) is the following lemma which allows us to calculate the order of an element \( x \in A_{2n-1}(\mathbb{P}_n,\mathbb{C}) \) given as
a rational linear combination of the elements $b_1^i$ in $K_0(P_\infty; Q/Z)$ in a simple way.

**Lemma 1.14:** Let $z$ be an element in $Ad_{2n}(P_\infty; Q/Z) \subset K_0(P_\infty; Q/Z)$ with order not exceeding $p^n$. Let $m_p: P_\infty \to P_\infty$ be the map defined by multiplication with $p$ in the $H$-space structure of $P_\infty$. Then the order of $m_p^*(z)$ in $Ad_{2n}(P_\infty; Q/Z)$ is exactly the order of $\beta(z)$ in $A_{2n-1}(P_\infty)$.

**Proof:** The kernel of the Bockstein map consists of multiples of $b_1^n$ so we can write $z$ as a linear combination of elements coming from $Ad_*(BZ/p^r; Q/Z)$, say

$$z = \sum_{r=1}^n c_r x(r, k_r) + a \cdot b_1^n$$

with $p^n a = 0$. Then by (1.11) we have $m_p^*(z) = \sum_{r=1}^n c_r x(r-1, k_r)$ because $m_p^*(b_1^n) = p^n b_1^n$. The order of $m_p^*(z)$ is thus the order of $z - a \cdot b_1^n$, which is the order of $\beta(z)$.

If $x \in Ad_{2n}(P_\infty; Q/Z)$ is given as $x = \sum a_i b_1^i$, then the order of $\beta(x)$ is easy to calculate because $m_p^*(b_1^i) = p^i b_1^i$ and there are usually very few terms left.

**Example:** $p=3$. The element $z = 4 \cdot b_1^1/27 + b_1^3/9$ is in $\ker(\phi^{k-7})$.

By (1.14) $\beta(z)$ has order 9. Direct calculation gives

$$z = \frac{1}{9}(b_3 + 5b_4 + b_5 + 3b_7) \mod b_1^7$$

$$= \pi_{2n}(x(1, 2) + x(2, 1))$$

This also allows one to work out the module structure of $A_*(P_\infty)$.

**Proposition 1.15:** Let $\alpha_t$ denote the generator of $A_{2t}(P_\infty)$.
then the order of $b_1^n \cdot \alpha_t$ in $A_2 t(p-1) + 2n-1(P_\infty \mathcal{C})$

is $\max \{ 1 + \nu_p(t) - n, 0 \}$.

**Proof:** The order of $\alpha_t$ is $p^{1 + \nu_p(t)}$ and the order of

$\beta(b_1^n/p^{1 + \nu(t)})$ follows from (1.14).
The number of cyclic summands in $J(\mathbb{P}^n)$

There is a close connection between the groups $\text{Ad}^*(X)$ and $J(X)$, the group of stable vector bundles on $X$ under the relation of stable fibre homotopy equivalence.

Adams [1] has defined a group $J''(X)$ which serves as an upper bound for $J(X)$. By the solution of the Adams conjecture $J''(X)$ is actually $J(X)$. Stated only for the $p$-primary part ($p$ odd) $J''(X)(p)$ is the quotient of $\tilde{K}^0(X)$ by the subgroup generated by the elements $(\psi^k-1)(x)$ for $k \not\equiv 0(p)$ and $x \in \tilde{K}^0(X)$. By definition of $\text{Ad}^1(X)$ we get surjective map from $\text{im}(j: \tilde{K}^0(X) \to \text{Ad}^1(X))$ onto $J''(X)(p)$

**Lemma 1.15:** For an odd prime $p$ $\tilde{\text{Ad}}^1(\mathbb{P}^n) \cong J(\mathbb{P}^n)(p)$

**Proof:** The exact sequence (1.1) shows

$$v_p(|\text{Ad}^1(\mathbb{P}^n)|) = \sum_{i=1}^{m} 1 + v_p(i)$$

whereas the results of [5] (Lemma 5.2) imply

$$v_p(|J(\mathbb{P}^n)|) = \sum_{i=1}^{m} 1 + v_p(i) \text{ with } m = \left\lfloor \frac{n}{p-1} \right\rfloor$$

As preparation we need.

**Lemma 1.16:** Let $n = t(p-1)+s$ with $0 \leq s \leq p-2$, then

$$\text{Ad}^1(\mathbb{P}^n) \cong \text{Ad}^1(\mathbb{P}^t(p-1))$$

**Proof:** The map induced by inclusion is surjective. Because the determinant of $(\psi^k-1)$ does not change if $n$ is altered to $n-s$, $\text{Ad}^1(\mathbb{P}^n)$ and $\text{Ad}^1(\mathbb{P}^t(p-1))$ must have the same order.
Theorem 1.17: The number of cyclic summands in \( J(P_n \mathbb{C})(p) \) is given by
\[
\left\lfloor \frac{\log (n+1)}{\log p} \right\rfloor
\]

Proof: The theory of elementary divisors shows that
\[
\text{Ad}^{2i+1}(P_n \mathbb{C}) \otimes \mathbb{Z}/p \text{ is the cokernel of } K^0(P_n \mathbb{C}) \otimes \mathbb{Z}/p \xrightarrow{\psi} K^0(P_n \mathbb{C}) \otimes \mathbb{Z}/p
\]
Therefore \( \text{Ad}^{2i+1}(P_n \mathbb{C}) \otimes \mathbb{Z}/p \cong \text{Ad}^{2j+1}(P_n \mathbb{C}) \otimes \mathbb{Z}/p \) if \( k_i \equiv k_j \mod p \)
which can be seen as a remnant of Bott periodicity. Let \( n = t(p-1) \), then \( k^n-1 \equiv o(p) \), so \( \text{Ad}^1(P_n \mathbb{C}) \otimes \mathbb{Z}/p \cong \text{Ad}^{1+2n}(P_n \mathbb{C}) \otimes \mathbb{Z}/p \). The Kronecker product \( \langle,\rangle : K^0(P_n \mathbb{C}) \otimes K_0(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \) induces a pairing
\[
1: \text{Ad}^{2m+1}(P_n \mathbb{C}) \times \text{Ad}^{2m}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}
\]
by \( 1(x,y) = \langle \overline{x}, k(y) \rangle \) where \( j(x) = x \) and \( j: K^{2m}(P_n \mathbb{C}) \to \text{Ad}^{2m+1}(P_n \mathbb{C}) \)
and \( k: \text{Ad}^{2m}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to K_2m(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) are the natural maps. The pairing 1 induces a map \( \tilde{1}: \text{Ad}^{2m}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\text{Ad}^{2m+1}(P_n \mathbb{C}); \mathbb{Q}/\mathbb{Z}) \).
Because \( \text{Hom}(-, \mathbb{Q}/\mathbb{Z}) \) is exact and the map
\[
K^0(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(K^0(P_n \mathbb{C}), \mathbb{Q}/\mathbb{Z})
\]
corresponding to \( \tilde{1} \) is an isomorphism, the 5-Lemma implies that \( \tilde{1} \)
is bijective. So the number of summands in \( \text{Ad}^1(P_n \mathbb{C}) \) and
\( \text{Ad}^{2n-1}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) are the same. By (1.13) it follows that there are \( \left\lfloor \frac{\log (t(p-1)+1)}{\log p} \right\rfloor \) summands in \( A_{2t(p-1)-1}(P_n \mathbb{C}) \).
In \( \text{Ad}^{2n-1}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \) there is one additional summand \( \ker \beta \), so the number of summands in \( \text{Ad}^1(P_n \mathbb{C}) \) is \( \left\lfloor \frac{\log(t(p-1)+1)}{\log p} \right\rfloor \). It is easy to see, that for \( 0 < s < p-2 \)
\[
\left\lfloor \frac{\log(t(p-1)+1+s)}{\log p} \right\rfloor = \left\lfloor \frac{\log(t(p-1)+1)}{\log p} \right\rfloor + 1
\]
so the theorem is proved.

Remark:
In [37] an upper bound is given for the number of cyclic summands in \( J(P_{2n} \mathbb{C})(p) \) to be \( \left\lfloor \frac{\log n}{\log p} \right\rfloor + 1 \) and for \( p=2 \) it is proved, that the number of summands in \( J(P_{2n} \mathbb{C})(p) \) is \( \left\lfloor \frac{\log(2n)}{\log 2} \right\rfloor \)
§ 2 Transfer maps

In this chapter we discuss the transfer maps

$$t^k: \pi^S_\infty(\mathbb{C}^k\tilde{H}) \rightarrow \pi^S_\infty(S^0)$$

and establish some of their properties. These transfer maps can be induced by stable maps $\tau^k \in \pi^{-1}_\infty(\mathbb{C}^k\tilde{H})$. We show that $\tau^0$ is in the image of the $J$-homomorphism $J:K^{-2}(\mathbb{C}^\infty) \rightarrow \pi^{-1}_\infty(\mathbb{C}^\infty^+)$ and describe the connection between transfer maps and generalized $J$-homomorphisms. We identify the cofibre spectrum of $\tau^k$ with a Thom spectrum and give a formula relating the images of the various $t^k$. We close with a description of $t^{-1}$ in terms of $t^0$.
The $S^1$-transfer establishes a close connection between
\[ \pi^S_\ast(P_\infty \mathbb{C}^+) \text{ and } \pi^S_\ast(S^0). \]
In particular, the part of \( \pi^S_\ast(P_\infty \mathbb{C}) \)
detected by the $e$-invariant is related to elements in \( \pi^S_\ast(S^0) \)
coming from the $\text{Ext}_{2^r}^{2^r}$-term of the BP-Adams spectral sequence.

The simplest definition of the $S^1$-transfer

\[ t^0: \Omega^\text{fr}_n(P_\infty \mathbb{C}) \longrightarrow \Omega^\text{fr}_{n+1} \]
is by identifying \( \Omega^\text{fr}_n(P_\infty \mathbb{C}) \) with the bordism group of equivariantly framed free \( S^1 \)-manifolds \( \Omega^\text{fr}_{n+1}(S^1; \text{free}) \) and then defining \( t^0 \) as the forgetful map to framed bordism. Or more explicitly:

Given an element \([M, \phi, f]\) in \( \Omega^\text{fr}_n(P_\infty \mathbb{C}) \), by pulling the universal \( S^1 \)-bundle back via \( f \) we get an induced \( S^1 \)-bundle \( (\tilde{M}, \pi, M) \) over \( M \).

The tangent bundle of \( \tilde{M} \) splits as a direct sum of \( \pi^*TM \) and \( T_\mathbb{P} \),
the tangent bundle along the fibres. Let \( H \) denote the Hopf bundle over \( P_\infty \mathbb{C} \), then \( T_\mathbb{P} \oplus 1 \) is isomorphic to \( f^*\pi^*H \) and thus canonically trivialized. We can frame \( T\tilde{M} \) by putting the framings of \( \pi^*\phi \) and \( T_\mathbb{P} \) together to get a framed manifold \( (\tilde{M}, \tilde{\phi}) \). Different splittings of \( T\tilde{M} \) do not alter the bordism class of \( (\tilde{M}, \tilde{\phi}) \) and we define \( t^0[M, \phi, f] = [\tilde{M}, \tilde{\phi}] \).

We first recall what is known about \( t^0 \) (see [26],[35],[7]).

Let \( \sigma \in \pi^S_2(P_\infty \mathbb{C}^+) \) denote the element defined by the Hopf bundle
\( S^3 \longrightarrow S^2 \). Then it is well known, that

\[ \pi^S_\ast(P_\infty \mathbb{C})/\text{tor} \cong \mathbb{Z}[\sigma] \]
under the Pontrjagin product. A computation of \( e^tt(\sigma^N) \) can be found in [26],[35]. Using this and the fact that \( t^0 \) raises the degree in the filtration associated to the BP-Adams spectral sequence
by 1, it follows that the elements \( \mu_r \) constructed by Adams in \( \pi^S_*(S^0) \) are not in the image of \( t^0 \). Similarly it follows that the generator of \( \text{im}(J) \) in \( \pi^S_{8n-1}(S^0) \) is not in the image of \( t^0 \) (see also [7]). For the odd primes we have
\[
\text{im}(J) \subset \text{im}(t^0).
\]

Denote by \( U \) the infinite unitary group. Then we have an \( S \)-map \( J': U \rightarrow S^0 \) defined as the image of the identity in \([U,U] \cong K^{-1}(U)\) under the \( J \)-homomorphism \( J: K^{-1}(U) \rightarrow \pi^0_s(U) \). The map induced by \( J' \) in stable homotopy is called the bistable \( J \)-homomorphism. Let \( w \) be the complex reflection map
\[
w: S^1 \wedge P_n^+ \rightarrow U(n+1)
\]
defined as follows: To the pair \((\lambda, x) \in S^1 \wedge P_n^+\) we associate the unitary map \( A \) which is the identity on the orthogonal complement of the line \( x \) in \( \mathbb{C}^{n+1} \) defined by \( x \) and maps \( v \) in \( x \) to \( \lambda \cdot v \). The transfer \( t^0 \) can be represented by a \( S \)-map \( \tau: S^1 \wedge P_{\infty}^+ \rightarrow S^0 \) see [23] or the discussion below, then \( t^0 \) is related to \( J' \) as follows:

**Theorem 2.2:** Up to a minus sign \( w \cdot J' \) is homotopic to \( \tau \).

We first state some corollaries of (2.2).

**Corollary 2.3:** The following diagram commutes
\[
\begin{array}{ccc}
\pi^S_n(P_{\infty}^+_
fty) & \xrightarrow{t} & \pi^S_n(S^0) \\
\downarrow w \quad & & \quad \downarrow -J' \\
\pi^S_{n+1}(U) & \xrightarrow{-J'} & \pi^S_{n+1}(U)
\end{array}
\]

This solves a problem posed in [26].

Now \( w: K^{-1}(U) \rightarrow K^{-1}(SP_{\infty}^+) \cong K^{-2}(P_{\infty}^+) \) maps \( \text{id}_U \) onto the
element $\pm \beta(H)$ where $\beta: K^{-2}(X) \cong \tilde{K}(X)$ is Bott periodicity and $H$ the Hopf line bundle over $P_\infty C$. Therefore we have:

**Corollary 2.5:** Under the $J$-homomorphism $J: K^{-2}(P_\infty C) \to \pi^{-1}_s(P_\infty C^+)$ the element corresponding to the Hopf line bundle is mapped onto the element $\tau$ which represents $t^0$ as an $S$-map.

**Remark:**
If $\pi: \tilde{M} \to M$ is a principal $S^1$-bundle and $E$ a generalized multiplicative cohomology theory, we have a transfer map $t_\pi: E_i(\tilde{M}) \to E_{i-1}(M)$ with transfer index $t_\pi(1) \in E_{i-1}(M)$. It is then clear how (2.5) can be used to determine $t_\pi(1)$.

All maps in the diagram (2.4) can be interpreted geometrically. The description of $J'$ is as follows: An element in $\pi^s_n(U) = \tilde{n}_n^{fr}(U)$ is a triple $(M, \phi, f)$ with $[M, \phi] = 0$ and $f: M \to U$ a map. Then $J'([M, \phi, f]) = [M, \phi^f]$ where $\phi^f$ is the new framing of $M$ got by twisting the framing $\phi$ by $f$ (that is, we compose $\phi$ by the automorphism of the trivial bundle over $M$ defined by $f$). Starting with an element $[M, \phi, f] \in \tilde{n}_n^{fr}(P_\infty C)$ we get under $t^0$ the manifold $\tilde{M}$, which is the total space of the $S^1$-bundle defined by $f$. The other way around we arrive at the manifold $S^1 \times M$. Now the twist is not in the manifold but in the framing.

My original proof for (2.2) was by looking directly at the construction of an $S$-map for $\tau$, observing that $\tau$ is homotopic to a map which is in the image of the $J$-homomorphism $J: K^{-2}(P_\infty C) \to \pi^{-1}_s(P_\infty C^+)$ and then comparing $e$-invariants. It seems nicer to see an explicit framed bordism between $\tilde{M}$ and $M \times S^1$. This is done in [41], where also a simpler proof of (2.2) is given. For
a proof of (2.2) we therefore refer to [4]. A simple proof of (2.2) can also be deduced from the results of [10].

We now come to the generalization of the $S^1$-transfer. Suppose given an $n$-dimensional manifold $M$ and a map $f: M \to P \otimes C$ defining a complex line bundle $\xi$ over $M$. Suppose further that there is given an isomorphism $\phi$ of the stable normal bundle $\nu$ of $M$ with the $k$-fold Whitney sum of $\xi$ ($k \in \mathbb{Z}$). By the Thom-Pontrjagin construction the triple $(M, \phi, f)$ defines an element in $\Omega_n^{fr}(P \circ \xi^k \hat{H})$ and all bordism classes can be so described. Here we write $P \circ \xi^a$ for the Thom space of a bundle $\alpha$ and because of $\chi_{\alpha \otimes \mathbb{R}^n} \cong S^n \chi_{\alpha}$ we are allowed to consider Thom spaces of virtual bundles (e.g. $\hat{H} = H-1$).

As in the case $k=0$ the total space $\tilde{M}$ of the $S^1$-principal bundle of $\xi$ has a canonical framing: We have $T \tilde{M} \cong \pi^* TM \otimes T_F$ and $T_F \cong \pi^*(\xi)$ is canonically framed. The isomorphism $\phi$ allows us to identify $\pi^* TM$ with $-k$ copies of $\pi^*(\xi)$. Putting these isomorphisms together defines a framing $\phi^k$ of $M$. As in the case $k=0$ one easily checks that $[\tilde{M}, \phi^k]$ is well defined up to framed bordism. So we have defined

$$t^k: \Omega_n^{fr}(P \circ \xi^k \hat{H}) \to \Omega_n^{fr} \quad (2.6)$$

Remark:

As in the case $k=0$ one can describe $t^k$ as a forgetful map from equivariant bordism to bordism. Instead of considering equivariantly framed free $S^1$-manifolds, one has to use free $S^1$-manifolds, where the stable normal bundle is equivariantly isomorphic to the representation $k \cdot (\xi, \rho)$, where $(\xi, \rho)$ denotes $\xi$ with the standard $S^1$-action.

The homomorphisms $t^k$ can be induced by stable maps at least on finite skeletons. The way how to do this is essentially in [8].
and now well known. Therefore we only sketch it. Suppose given a map \( f: X \to Y \) between compact \( C^\infty \)-manifolds. Approximate \( f \) by a smooth embedding \( f^1: X \to \mathbb{R}^n \times Y \) with normal bundle \( v \). The Thom-Pontrjagin construction defines then a map

\[
T_\alpha(f^1): Y^\alpha \to X^\nu-m+f^\prime(\alpha)
\]

(2.7)

for every virtual bundle \( \alpha \) over \( Y \). Applied to the projection map \( \pi: S^{2n+1} \to P_n \mathbb{C} \) and \( \alpha = 0 \) this yields an \( S \)-map

\[
\tau: S^1 \wedge (P_n \mathbb{C}^+) \to (S^{2n+1})^+
\]

which upon composing with the projection \( \text{pr}: (S^{2n+1})^+ \to S^0 \) induces the transfer \( t^0 \). Applied to \( \pi \) with \( \alpha = k \cdot \mathbb{H} \) this gives a stable map

\[
\tau^k: S^1 \wedge P_n \mathbb{C}^{k \cdot \mathbb{H}} \to S^0
\]

(2.8)

which represents \( t^k \). The details for a proof are in [8].

The main consequence of this description of the transfer \( t^k \) is the possibility of working out the cofibre of \( \tau^k \) and so embedding \( t^k \) in an exact sequence.

**Theorem 2.9:** The cofibre of \( \tau^k \) is the Thom space \( P_n \mathbb{C}^{(k-1)\mathbb{H}} \), that is

\[
P_{n+1} \mathbb{C}^{(k-1)\mathbb{H}} \to S^2 P_n \mathbb{C}^{k \cdot \mathbb{H}} \to S^1
\]

is a cofibre sequence.

**Proof:** The Thom Pontrjagin construction provides us with the maps

\[
T_\alpha(i): P_{n+1} \mathbb{C}^{(k-1)\mathbb{H}} \to S^2 P_n \mathbb{C}^{k \cdot \mathbb{H}}
\]

\[
\alpha = (k-1)\mathbb{H} \quad \text{and} \quad T_\alpha(\pi): S^2 P_n \mathbb{C}^{k \cdot \mathbb{H}} \to (S^{2n+1})^R
\]

(\( \alpha = k \cdot \mathbb{H} \)) where \( i: P_n \mathbb{C} \to P_{n+1} \mathbb{C} \) is the inclusion. The naturality
properties of the construction $T_\alpha$ (see [8]) imply that the composition $T_\alpha(\pi)\circ T_\alpha(i)$ factorizes over

$$T_\alpha(g): \mathbb{C}^{(k-1)\tilde{H}} \to \mathbb{C}^{2n+2+(k-1)\tilde{H}} = S^{2n+2}$$

where $g: \ast \to \mathbb{C}$ is the inclusion (because $i\circ \pi$ is constant). So

$$pr \circ T_\alpha(\pi)T_\alpha(i) = \tau^k \circ T_\alpha(i)$$

must be null homotopic, where

$$pr: (S^{2n+1}) \mathbb{R} \to S^1$$

denotes the projection.

Let $C_k$ be the cofibre of $\tau^k$. Because $\tau^k \circ T_\alpha(i) \simeq 0$ we get a map

$$f: S^1 \mathbb{P}^{(k-1)\tilde{H}} \to C_k$$

and a commutative diagram

$$\begin{array}{ccc}
\mathbb{P}_n \mathbb{C}^{k\tilde{H}+\mathbb{C}} & \xrightarrow{\tau^k} & S^1 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{P}_n \mathbb{C}^{k\tilde{H}+\mathbb{C}} & \xrightarrow{\tau^k} & S^1 & \xrightarrow{f} & C_k \\
\end{array}$$

The corresponding diagram in homology has exact rows, because $T_\alpha(i)$ is Poincaré-dual to the inclusion. So $f_\ast$ is an isomorphism by the 5-Lemma and $f$ must be an $S$-homotopy-equivalence. The map induced by $T_\alpha(i)$ we denote by $j^!$.

So for example we have embedded $t^\circ$ in the following exact sequence:

$$\pi_n^s(P_\mathbb{C}^{-\tilde{H}}) \xrightarrow{j^!} \pi_n^s(C^+_{\mathbb{P}}) \xrightarrow{t^\circ} \pi_{n-1}^s(S^0) \xrightarrow{i_\ast} \pi_{n-1}^s(P_\mathbb{C}^{-\tilde{H}})$$

(2.10)

Remark:

The map $j^!$ induced by $T_\alpha(i)$ can be interpreted geometrically as follows: Let $(M, f, \phi) \in \mathbb{C}^{fr}_{\mathbb{P}}(P_\mathbb{C}^{k\tilde{H}})$ be given. We can from the submanifold $N \subset M$ dual to $c_1(f^\ast H)$ which then has $(k+1) \cdot f^\ast(H)$ as stable normal bundle and defines $j^![M,f,\phi]$. In bordism theory this is usually called the Smith homomorphism.
The main application of (2.9) is to the computation of $t^k$ on elements in $\pi_\infty^s(P \infty \mathcal{C}^{kH})$ which are detected by the $e$-invariant, see § 3 and § 4.

The generalization of (2.2) and (2.3) to the case of $t^k$ will be discussed in § 7.

As next we discuss the relation of $t^k$ to transfer maps of sphere bundles.

Given a $m$-dimensional vector bundle $V$ and virtual bundles $W,A$ on a space $X$ we have the following cofibre sequence (see e.g. [18]) where $S(V)$ is the sphere bundle of $V$

$$
\rightarrow S(V)^{\pi^*W} \rightarrow X^W \rightarrow X^{V \otimes W} \xrightarrow{\partial} S^1_{\wedge} S(V)^{\pi^*W} \rightarrow
$$

(2.11)

Applying the Thom Pontrjagin construction to the map $S(V) \xrightarrow{\pi} X$ we get the transfer map of the sphere bundle $S(V)$ with coefficients in $A$ ($\bar{V} = V - m)$:

$$
T_A(\pi): X^{A+m} \rightarrow S^1_{\wedge} S(V)^{\pi^*(A-\bar{V})}
$$

Letting $W = A - \bar{V}$ we have the boundary map of the cofibre sequence (2.11)

$$
\partial: X^{A+m} \rightarrow S^1_{\wedge} S(V)^{\pi^*(A-V)}
$$

Lemma 2.12: The stable maps $\partial$ and $T_A(\pi)$ are homotopic up to sign.

Proof: Both constructions can be made fibrewise i.e. in the category of fibre bundles over $X$. One proves $T_A(\pi) = \pm \partial$ in case where $X$ is a point directly, being careful that all constructions are compatible with the structure groups of the fibre bundles involved. The proof is finished by passing to the associated fibre bundles and maps and proceeding as in [46].
The identification of the Thom space $P_r^mH$ with the stunted projective space $P_{r+m}/P_{m-1}$ is induced by the map
\[ D(mH_r) = S^{2r+1} \times S^{m} \rightarrow S^{2r+1+2m}/S^{1} \]
defined by $[x, y] \mapsto [y, (1 - |y|/2)^2, x]$.

If we identify $P_n^rH$ with a stunted projective space, we can write the usual cofibre sequence of $j: P_{n+r}/P_{r-2} \rightarrow P_{n+r}/P_{r-1}$ as follows:
\[ P_{n+1}^rH \xrightarrow{j} P_n^rH \xrightarrow{\partial} S^1(r-1)H \]
\[ \text{as follows:} \]
\[ P_{n+1}^rH \xrightarrow{j} P_n^rH \xrightarrow{\partial} S^1(r-1)H \tag{2.13} \]

The map $M(i): P_n^rH \rightarrow P_n^rH$ induced by the inclusion followed by $j$ is easily seen to be homotopic to the natural map $P_n^rH \rightarrow P_n^rH$ appearing in (2.11) for $X = P_n^rH$. We therefore get a commutative diagram
\[
\begin{array}{ccc}
P_{n+1}^rH & \xrightarrow{j} & P_n^rH \\
& | & | \\
P_n^rH & \xrightarrow{\partial} & S^1(r-1)H \\
\end{array}
\]
\[
\begin{array}{ccc}
P_{n+1}^rH & \xrightarrow{j} & P_n^rH \\
M(i) & | & | \\
P_n^rH & \xrightarrow{\partial} & S^1(r-1)H \\
\end{array}
\]

By (2.12) we can use instead of $\partial^rH$ the transfer map of $S(H) \rightarrow P_n^rH$, so we have proved:

**Proposition 2.14:** For $k > 0$ the transfer map $t^k: S^1P_n^rH \rightarrow S^0$ is stably homotopic to the boundary map in the cofibre sequence of stunted projective spaces in (2.13).

Let $M$ be a multiple of the order of $J(H_n)$ in $J(P_n^rH)$, then we have a relative Thom isomorphism
for each cohomology theory \( h \), which is a module theory over \( \pi_s^* \).

By choosing \( h = \{ P_{n+1} \mathbb{C}^A, - \} \) we see that \( \phi \) can be represented by a stable map

\[
\phi: P_{n+1} \mathbb{C}^{A+M \cdot \tilde{H}} \rightarrow P_{n+1} \mathbb{C}^A \tag{2.15}
\]

By 4.10 of [8] \( \phi \) commutes with the maps \( j^! \) of (2.9) giving a commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
P_{n+1} \mathbb{C}^{(r-1)H} & \xrightarrow{j^!} & P_n \mathbb{C}^{rH} \\
\uparrow \phi & & \uparrow \phi \\
P_n \mathbb{C}^{(r-1)H+M \tilde{H}} & \xrightarrow{j^!} & P_n \mathbb{C}^{rH+M \cdot \tilde{H}}
\end{array}
\]

The commutativity of the square \( B \) shows

**Proposition 2.16:** \( \phi(\tau^r) = \tau^{r+M \tilde{H}} \), that is on \( P_n \mathbb{C} \) the transfer maps are periodic with period \(|J(H_{n+1})|\).

*Remark:*

If \( M \) is only a multiple of \( J(H_n) \) such that \( M \cdot J(H_{n+1}) \neq 0 \), then the square corresponding to \( B \) need not be commutative.

The \( H \)-space structure of \( P_\infty \mathbb{C} \) allows us to relate various transfer maps \( t^k \). Let

\[
\mu_{n,m}: P_n \mathbb{C} \times P_m \mathbb{C} \rightarrow P_{n+m} \mathbb{C}
\]

be the restriction of the \( H \)-space multiplication. This can explicitly defined by interpreting point in \( P_n \mathbb{C} \) as complex polynomials of degree \( n \) (up to a nonzero factor) and then using polynomial multiplication. We apply the Hopf construction to the map.
\[ u_1, n : S^2 \times P_n \mathbb{C} \to P_{n+1} \mathbb{C} \] to get maps \( S^3 \times P_n \mathbb{C} \to S^1 \wedge P_{n+1} \mathbb{C} \). These maps are compatible with the inclusions between projective spaces, so we get induced maps

\[ S^3 \times P_n \mathbb{C}/P_m \mathbb{C} \to S^1 \wedge P_{n+1} \mathbb{C}/P_{m+1} \mathbb{C} \quad (2.17) \]

or written as a stable map in terms of Thom spaces of virtual bundles \((k \geq 0)\):

\[ \omega : P_s \mathbb{C}^{k \tilde{H}} \to P_s \mathbb{C}^{(k+1) \tilde{H}} \quad (2.18) \]

Let \([P_i \mathbb{C}]\) denote the fundamental class of \( P_i \mathbb{C} \) in \( H_* (P_o \mathbb{C}; \mathbb{Z}) \).

Because the Pontrjagin multiplication in homology is given by

\[ [P_i \mathbb{C}] \cdot [P_j \mathbb{C}] = \binom{i+j}{i} [P_i+j \mathbb{C}] \]

we see that the degree of \( \omega \) on the bottom cell is \( k+1 \) and \( k+s+1 \) on the top cell.

Because \( \omega \) commutes with inclusions we have a commutative diagram

\[
\begin{array}{ccc}
P_{r+1} \mathbb{C}^{s \tilde{H}} & \xrightarrow{j} & P_r \mathbb{C}^{(s+1) \tilde{H}} \\
\uparrow \omega & & \uparrow \omega \\
S^2 P_{r+1} \mathbb{C}^{(s-1) \tilde{H}} & \xrightarrow{j} & S^2 P_r \mathbb{C}^{s \tilde{H}}
\end{array}
\]

where the maps \( j \) are defined by natural maps

\[ P_{r+1+s} \mathbb{C}/P_{s-1} \mathbb{C} \to P_{r+1+s} \mathbb{C}/P_s \mathbb{C} \] as in (2.13).

By (2.14) we extend this square to a diagram of cofibre sequences getting:

\[
\begin{array}{ccc}
P_{r+1} \mathbb{C}^{s \tilde{H}} & \xrightarrow{j} & S^2 \wedge P_r \mathbb{C}^{(s+1) \tilde{H}} & \xrightarrow{\tau^{s+1}} & S^1 \\
\uparrow \omega & & \uparrow \omega & & \uparrow f(s) \\
P_{r+1} \mathbb{C}^{(s-1) \tilde{H}} & \xrightarrow{j} & S^2 \wedge P_r \mathbb{C}^{s \tilde{H}} & \xrightarrow{\tau^s} & S^1
\end{array}
\]

(2.19)
where \( f(s) \) denotes a map of degree \( s \).

Thus we have proved for \( s > 0 \):

**Theorem 2.20:** Let \( r \) be fixed. Then there exist stable maps

\[
\omega^1 : P^r\mathcal{C}^{s\mathbb{H}} \rightarrow P^r\mathcal{C}^{(s+1)\mathbb{H}}
\]

for each \( s \in \mathbb{Z} \), such that

\[
t^{s+1}_* \omega^1 = s \cdot t^s
\]

**Proof:** To settle the case \( s < 0 \) we apply periodicity (2.16) of \( r^k \) and use \( \tilde{\omega} \), the map \( \omega \) conjugated with the Thom isomorphisms (2.15). We then get \( t^{s+1}_* \tilde{\omega} = (s+M)t^s \) where \( M \) is a multiple of \( |J(H;r)| \). But if we take \( M \) large enough then \( (s+M)t^s = st^s \) because \( t^s \) is an \( S \)-map of finite order in \( \pi^{-1}_s(P^r\mathcal{C}^{s\mathbb{H}}) \).

**Corollary 2.21:**

\( s \cdot \text{im}(t^s) \subseteq \text{im}(t^{s+1}) \)

For later use we consider the case \( k = -1 \).

We denote \( \mu_{1,n-1} : S^{2}_x P^{n-1} \rightarrow P^n \mathcal{C} \) by \( \mu \).

Stably we have \( S^{2}_x P^{n-1} \mathcal{C} \cong S^2 \wedge (P^n \mathcal{C} \vee S^0) \vee P^n \mathcal{C} \) and we define \( f : S^2 \wedge (P^n \mathcal{C} \vee S^0) \rightarrow (S^{2}_x P^n \mathcal{C})^+ \) by the inclusion.

**Theorem 2.22:** The sequence of maps

\[
\Omega^r_n(\mathcal{C}^{s_{-\mathbb{H}}}) \xrightarrow{\text{fr}} \Omega^r_n(\mathcal{C}^{s_{-\mathbb{H}}}) \xrightarrow{f^r} \Omega^r_n(S^{2}_x P^n \mathcal{C}) \xrightarrow{\mu^s} \Omega^r_n(P^n \mathcal{C}) \xrightarrow{t^0} \Omega^r_n+1(\mathcal{C}^{s_{-\mathbb{H}}})
\]

is up to a minus sign the transfer \( t^{(-1)} \).

This means in terms of framed manifolds: For a given element \( Z = [M,f,\emptyset] \in \Omega^r_n(\mathcal{C}^{s_{-\mathbb{H}}}) \) we can form the submanifold \( N^{n-2} \) dual to \( c_1(f^*H) \). The isomorphisms \( \nu(N,M) = f^*H \) and \( \emptyset \) put a framing \( \emptyset^1 \) on \( N \). We take \( S^{2}_x N \) with the product framing and the sphere bundle of \( H \hat{\cap} f^*(H) \) as an \( S^1 \)-bundle over \( S^{2}_x N \) with
the induced framing. Then \( t^{(-1)}(Z) = [S(H \times f^*(H), \pi'^*\phi^1] \).

**Proof of (2.22):** We start with the commutative diagram of \( S^1 \)-bundles

\[
\begin{array}{ccc}
S(H \times H) & \xrightarrow{\tilde{\mu}} & S(H) \\
\downarrow & & \downarrow \\
S^2 \times P_{n-1} \mathbb{C} & \xrightarrow{\mu} & P_n \mathbb{C}
\end{array}
\]

Because transfer maps are natural for induced bundles we get a commuting square of transfer maps

\[
\begin{array}{ccc}
S(H \times H) & \xrightarrow{\tilde{\mu}} & (S^2 n + 1) \\
\uparrow t_H & & \uparrow t_H \\
(S^2 \times P_{n-1} \mathbb{C}) & \xrightarrow{\mu} & P_n \mathbb{C}^t
\end{array}
\]

Composing with the projections \( pr: X^t \to S^0 \) we get \( pr \cdot t_\phi \simeq t_0^0 \).

Now the sphere bundle \( S(H \times H) \) is the same as \( S(2H) \) and we have the commuting square

\[
\begin{array}{ccc}
S(H \times H) & = & S(2H) \\
\downarrow & & \downarrow \\
S^2 \times P_{n-1} \mathbb{C} & \xrightarrow{q} & P_{n-1} \mathbb{C}
\end{array}
\]

The use of \( S(H \times H) \) instead of \( S(H \times H) \) gives only a change in orientation and so causes a minus sign.

By naturality of the construction \( T_\alpha \) we arrive at a commuting diagram

\[
\begin{array}{ccc}
S(H \times H) & \xrightarrow{t_H} & S(2H) \\
\uparrow t_\phi & & \downarrow t_{2H} \\
(S^2 \times P_{n-1} \mathbb{C})^+ & \xrightarrow{D(q)} & S^2 (P_{n-1} \mathbb{C} \vee S^0)
\end{array}
\]
We see that $D(q)$ is the map $f$ defined by stable inclusion (observe that $2\pi$ on $S(2H)$ is trivial).

The natural map $P_n C^{-2H} \to P_n C^{-2H+2H} = P_n C^+$ appearing in (2.11) can be factored as

$$P_n C^{-2H} \to P_n C^{-2H+H} \to P_n C^+$$

so we have the commutative diagram $A$ below. Because these natural maps together with the corresponding transfers fit into cofibre sequences (2.12) we get a map $h$ relating $t_H$ and $t_{2H}$:

$$
\begin{array}{ccc}
P_n C^{-2H} & \to & P_n C^-H \\
\downarrow \quad & \quad & \downarrow h \\
P_n C^{-2H} & \to & P_n C^+ \\
\end{array}
\begin{array}{ccc}
A \quad & \quad & S^{1} \wedge S(H)^{-2H} \\
\downarrow h & \quad & \downarrow h \\
S^{1} \wedge S(2H)^{-2H} & \quad & \quad \\
\end{array}
$$

The transfers $t_H$ and $t^{-1}$ are related by the commutative diagram $(i:j! = h$ is easily seen by looking at the construction of $j!$):

$$
\begin{array}{ccc}
P_n C^{-H} & \to & P_n C^+ \\
\downarrow \quad & \quad & \downarrow i \\
P_n C^{-H} & \to & P_{n-1} C^+ \\
\end{array}
\begin{array}{ccc}
i & \quad & S^{1} \wedge *^{-H} \\
\uparrow & \quad & \downarrow \\
S^{1} \wedge *^{-H} & \quad & \quad \\
\end{array}
$$

So $h t_H \simeq t_{2H} h \simeq h_{2H} i \cdot j!$. If we compose all transfers with the appropriate projections $X^A \to X^A$ and use the fact that $t_{2H}$ on $P_n C^+$ restricts to $P_{n-1} C^+$ via $i: P_{n-1} C^+ \to P_n C^+$ we have found

$$t^{-1} \simeq pr \cdot t_{2H} \cdot j!$$

If we put together, we get

$$-t^{-1} \simeq t^0 \cdot u \cdot f \cdot j!$$

(2.23)
Remarks:

(2.22) means that the stable map $\tau^{-1}$ lies almost in the $\text{im}(J)$; more precisely by (2.5) we have $\tau^0 \cdot \text{im}(J)$, therefore $(\mu \cdot f)^\ast \tau^0 \cdot \text{im}(J)$ where $(\mu \cdot f)^\ast \tau^0 \in \pi^{-3}_S(P_\infty \mathbb{C}^+)$.

Under the map $j^1 : \pi^{-3}_S(P_\infty \mathbb{C}^+) \to \pi^{-1}_S(P_\infty \mathbb{C}^{-\tilde{H}})$ the element $(\mu \cdot f)^\ast \tau^0$ gives $\tau^{-1}$ by (2.23), but at odd primes $j^1$ is an isomorphism.

Let $p$ be an odd prime and $n \equiv -2 \mod 2(p-1)$.

Then the maps of (2.22) are onto modulo torsion $(\mu \cdot f)_n$ is multiplication by $\sigma \in \Omega^2_{fr}(P_\infty \mathbb{C})$ and for $j^1$ see proof of 7.35). Because $\text{im}(J) \cap \text{im} (\tau^0)$ we have: $\text{im}(\tau^{-1})$ contains elements with maximal e-invariant.
§ 3 The Whitehead conjecture

The J-homomorphism $J: \pi_n(SO) \rightarrow \pi_n^S$ can be factored through the so-called bistable J-homomorphism $J': \pi_n^S(SO) \rightarrow \pi_n^S$. The conjecture of G.~W. Whitehead is that this map is onto in positive dimensions. For the 2-primary component $J'$ is known to be surjective.

The complex analogue of $J'$ is closely related to the transfer map $t^0: \pi_n^S(P_{\infty}C) \rightarrow \pi_{n+1}^S(SO)$; in particular they have the same image.

Using the cofibre sequence of the representing $S$-map $\tau$ for $t^0$ we construct a map $f: S^2P_{\infty}C \rightarrow MU/S^0$ which allows us to compare $t^0$ with the boundary map of the cofibre sequence $S^0 \rightarrow MU \rightarrow MU/S^0 \rightarrow S^1$. By this procedure the problem of computing $t$ from filtration 1 to filtration 2 is lifted to a pure filtration 1 problem, which can be treated by means of the $e$-invariant. This then allows one to find a counterexample to the conjecture of Whitehead for primes larger than three.
The usual stable J-homomorphism \( J: \pi_n(\text{SO}) \rightarrow \pi_n^s(\text{SO}) \) is induced by a map

\[
\text{SO} \rightarrow \Omega^\infty S^n = Q(\text{SO})
\]

Its adjoint \( J' \) is a stable map which induces the bistable J homomorphism

\[
J': \pi_n^s(\text{SO}) \rightarrow \pi_n^s(\text{SO})
\]

G.W. Whitehead conjectured that \( J' \) is surjective for \( n > 0 \). The reflection map \( \text{PallR} \rightarrow 0 \) followed by a fixed reflection defines a map \( \text{PallR} \rightarrow \text{SO} \) and the composition \( J'. i \), is a surjection onto the 2-primary part by the Kahn-Priddy theorem. So \( J' \) is onto for the prime 2. In this chapter we shall prove that \( J' \) is not onto for the odd primes larger than 3.

For the odd primary components we can equally well use the complex analogue of \( J' \)

\[
J'_U: \pi_n^s(U) \rightarrow \pi_n^s(\text{SO})
\]

This is the same as the map mentioned in §2, where its geometric interpretation in terms of twisting framings was also given. By (2.4) we already know \( \text{im}(t^0) = \text{im}(J'_U) \). Besides the relation between \( t^0 \) and \( J' \) given by the complex reflection map \( w \), there is another one: Let \( F_{S^1} \) be the limit of the spaces \( F_{S^1}(S^N) \) of \( S^1 \)-equivariant selfmaps of spheres with free \( S^1 \)-action. In [7] Becker and Schultz constructed a weak homotopy equivalence

\[
\lambda: F_{S^1} \rightarrow Q(S^1 \wedge P_\infty^+ \mathbb{C})
\]  

(3.1)

Now because unitary maps are \( S^1 \)-equivariant we get a forgetful map

\[
\Theta: U \rightarrow F_{S^1} = Q(S^1 \wedge P_\infty^+ \mathbb{C})
\]  

(3.2)
Let $\theta'$ be the adjoint map of $\theta$. Using the results of [7] it is easy to prove that the following diagram commutes up to a minus sign:

$$
\begin{array}{ccc}
\pi_n^s(P_{n+1}\epsilon^+)^+ & \xrightarrow{t^0} & \pi_n^s(S^0) \\
\uparrow\theta' & & \uparrow J'_U \\
\pi_{n+1}^s(U) & & \\
\end{array}
$$

(3.3)

Remark:
Another approach to the map $\theta'$ via relative Euler class and difference class in equivariant stable homotopy is given by M.C. Crabb in [10], where also a proof for the commutativity of (3.3) can be found.

By (3.3) and (2.4) we have

**Proposition 3.4:** $\text{im}(J'_U) = \text{im}(t^0)$

On the stable homotopy groups of a space $X$ there exists a filtration

$$
\pi_n^s(X) = F^0 \supset F^1 \supset F^2 \supset \ldots \supset F^k
$$

associated to the Adams spectral sequence for any reasonable homology theory [3]. Here we shall always take the filtration belonging to the BP- or MU-Adams spectral sequence. If it should be necessary to distinguish between skeleton filtration and Adams spectral sequence filtration, we shall call the latter Adams filtration.

This filtration is a good measure for the complexity of elements in stable homotopy. If the space $X$ has torsion-free homology, the elements of Adams filtration 0 can be detected by rational homology. The same is true for filtration 1 using K-theory (see (3.7) below).
On the known part of $\pi^s_\ast$ the behaviour of Adams filtration is known. Up to some exceptions at the prime 2 the image of the $J$-homomorphism gives all of filtration 1. In the case $p \neq 2$, $F^1/F^2$ is a direct summand and $F^2$ is called cok $(J)$. There exist infinite families of elements in coker $(J)$ e.g. the $\beta_t$ and $\epsilon_t$ families in $F^2 (p>3)$ and the $\gamma_t$-famille in $F^3 (p>5)$. Besides these families only very few indecomposable elements (e.g. $\xi, \lambda', \mu, \phi, \kappa_s$ and $\lambda_s$, $1 \leq s \leq p-3$) are known. Up to now only an upper bound for $F^2/F^3$ is known, namely the $\text{Ext}^{2,s}_{\ast}$-term of the $BP$-Adams spectral sequence [29].

The bistable $J$-homomorphism and the transfer maps $t^S$ have the property that they raise the degree in the Adams filtration by at least 1. This is a simple consequence of the vanishing of the complex cobordism groups $\overline{MU}^0(U)$ and $\overline{MU}^{-1}(P_\infty C^kH)$ and the multiplicative properties of the filtration (see e.g. [23]). So if $x \in F^2\pi^s_{n+1}(S^0)$ is in the image of $t^0$, it must come from $F^1\pi^s_n(P_\infty C^+)$.

Because $t^0$ is an $\pi^s_\ast(S^0)$-module map the lowest dimensional element in cok $(t^0)$ must be indecomposable.

The knowledge of the cofibre of the transfer map $t^S$ allows us to compute

$$ t^S: F^1\pi^s_n(P_\infty C^SH) \rightarrow F^2\pi^s_{n+1}(S^0) $$

modulo higher filtration.

Let $f^S: P_\infty C \rightarrow BU$ be the classifying map of the vector bundle $sH$ and $f^S: P_\infty C^SH \rightarrow MU$ the induced map of Thom spaces. We map the cofibre sequence of $t^S$ into the cofibre sequence

$$ S^0 \xrightarrow{h} MU \rightarrow \overline{MU} $$

where $h$ is the Hurewicz map and $\overline{MU}$ is the relative Thom spectrum.
MU/S^0 classifying bordism of stably almost complex manifolds with framed boundaries. We get the following commuting diagram:

\[
\begin{array}{ccccccccc}
S^0 & \xrightarrow{i} & P_\infty c(s-1)H & \xrightarrow{j} & S^2P_\infty c^\vee H & \xrightarrow{t^s} & S^1 \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
\]

Here \( f^s \) is the map induced by \( f \) and \( \text{id}_{S^0} \).

Now the stable homotopy of the lower sequence is the well known bordism sequence

\[
\pi^s_n(S^0) \rightarrow \Omega^U_n(\ast) \rightarrow \Omega^U_n,fr(\ast) \rightarrow \pi^s_{n-1}(S^0)
\]

Because \( \Omega^U_{2n-1} = 0 \) the boundary map

\[
\partial: \pi^s_{2n-1}(\overline{MU}) \rightarrow \pi^s_{2n-2}(S^0)
\]

is bijective, so we have described \( t^s \) by \( f^s \). The point is now that \( \partial \) also raises filtration by 1 in the Adams spectral sequence, simply by definition of this filtration. Thus elements in \( \tilde{F}_\ast^2 \pi^s_n(S^0) \) must come from \( F_\ast^1 \pi^s_n(\overline{MU}) \) under \( \partial \) and we have reduced the problem of computing

\[
t^s: F_\ast^1 \pi^s_n(P_\infty c^\vee H) \rightarrow F_\ast^2 \pi^s_n(S^0)
\]

to a problem in filtration 1.

To use the K-theory e-invariant instead of the bordism e-invariant we need the following lemma:
Lemma 3.7: Let $X$ be a space with torsion-free homology then the $K$-theory $e$-invariant is injective on $F^1 \pi_s^s(X)/F^2 \pi_s^s(X)$, where the filtration is associated to the $\text{MU}$-Adams spectral sequence.

Proof: This is a well known consequence of the Hattori-Stong-theorem. Under the assumption of torsion-free homology the bordism spectral sequence collapses for the space $X \wedge M(Q/Z)$, where $M(Q/Z)$ is a Moore space for $Q/Z$. This implies that the Hurewicz map
\[ \mu: \Omega^U_*(X;Q/Z) \longrightarrow K_*(X;\text{MU};Q/Z) \]
is monomorphic. But $\mu$ maps the image of framed bordism into the subgroup $K_*(X;Q/Z)$. So on the subgroup of spherical classes in $\Omega^U_*(X;Q/Z)$ the natural map $\Omega^U_*(X;Q/Z) \longrightarrow K_*(X;Q/Z)$ is injective. This immediately gives the conclusion.

Because the spectra $\tilde{\text{MU}}$ and $\text{P}_\infty \text{CSH}$ have torsion-free homology we can apply (3.7) in these cases. Thus we have proved:

Theorem 3.8: Let $x \in \pi_{2n-1}^s(\text{P}_\infty \text{CSH})(p)$. If $e(f_\infty^s(x))$ in $A_{2n+1}(\text{MU})$ is zero, then $t^s(x)$ is in $\pi_{2+2(p-1)n}^s(S^0)(p)$.

Remark: Because $\beta: \pi^s_*(\tilde{\text{MU}}) \longrightarrow \pi^s_*(S^0)$ factorizes through $\pi^s_*(BP)$ we can use $BP$ instead of $\tilde{\text{MU}}$ in (3.8).

For later use we reformulate this:

Corollary 3.9: Let $x \in F^1 \pi_{2n-1}^s(\text{P}_\infty \text{CSH})(p)$ and $e f_\infty^s(x) \neq 0$ in $A_{2n+1}(BP)$ then $t^s(x) \neq 0$ in $F^2 \pi_{2n}^s(S^0)(p)$.
We apply this now to the case of the element $\beta_{p+1}$ in $F^2\pi^s_{2n}(S^0)(p)$ $(p>3)$.

Set $n = (p-1)\cdot((p+1)^2-1)-1$. From (1.4) we get

$$\nu|\tilde{A}_{2n-1}(P_\infty C)| = p+3$$

and from (1.13) that $A_{2n-1}(P_\infty C)$ has two summands.

An easy calculation with (1.7) shows that the second summand must have order $p$ and is generated by

$$\pi_*(X(2,1)) = \pi_*(\rho^p + 2^n \rho^2 p + \ldots)/p$$

so

$$A_{2n-1}(P_\infty C) = \mathbb{Z}/p^{p+2} \oplus \mathbb{Z}/p$$

(3.10)

The proof of the following two propositions is deferred to § 6.

**Proposition 3.11:** The generator of the $\mathbb{Z}/p^{p+2}$-summand in $A_{2n-1}(P_\infty C)$ $(n = (p-1)\cdot((p+1)^2-1)-1)$ is not in the image of the e-invariant.

**Proposition 3.12:** $f(x)(\text{im}(e)) = 0$ in $A_{2n+1}(BP)$

**Remark:**

We shall actually construct the elements in $\text{im}(e)$ in § 4, so

$\text{im}(e) = \mathbb{Z}/p^{p+1} \oplus \mathbb{Z}/p$ for $p \gg 3$.

As a corollary of (3.12), (3.11), (3.8) and (3.4) we find

**Theorem 3.13:** For $p \gg 5$ the element $\beta_{p+1}$ is not in the image of the bistable J-homomorphism.

**Remarks:**

1. The formulas describing $f^x_*$ are of the same type for all odd primes, so that the behaviour of $t^0$ on $\text{im}(e)$ seems to be
the same for all odd primes. But for $p=3$ it is known that the element $\beta_4$ does not exist.

2. For a counterexample for $p=3$ one needs more information about the existence of the $\beta_i$ elements. It is known that $\beta_1, \beta_2, \beta_3, \beta_5$ and $\beta_6$ exist, but then one leaves the known part of $\pi_*^{s}(S^O)(3)$. In § 4 we shall show that the above $\beta_t$ are in $\text{im}(t^0)$.

3. In the course of the calculations for proving (3.11) and (3.12) we will see that the next $\beta_t$ which has a chance of not being in $\text{im}(t^0)$ is $\beta_{2t+1}$ or $\beta_{2t+2}$.

Using the Gysin sequence of the sphere bundle of the $p^r$-th power of the Hopf line bundle one can relate $\pi_*^{s}(B\mathbb{Z}/p^r)$ and $\pi_*^{s}(P\mathbb{C}^+)$ by transfer maps (for fixed $n$ and $r$ large the bundle $H_{pr}$ on $P_n\mathbb{C}$ becomes orientable for stable homotopy). This allows us also to conclude that the transfer maps

$$t(r): \pi_*^{s}(B\mathbb{Z}/p^r) \to \pi_*^{s}(S^O)(p)$$

cannot be onto for $r>2$, whereas $t(1)$ is onto by the Kahn-Priddy-theorem.
§ 4 Representing stable homotopy by framed manifolds

In this chapter we show how one can use the transfer to construct framed manifolds representing elements in \( \text{cok} (J) \). Applying the principle that the transfer raises Adams filtration by at least 1 we use the transfer twice, that is we construct elements of filtration 0 in \( \pi^s_\infty(P_\infty \mathbb{C} \times P_\infty \mathbb{C}) \) and map then by a double transfer to \( \pi^s_\infty(S^0) \) to get elements of filtration 2. The transfer

\[
t: \pi^s_{2n}(P_\infty \mathbb{C} \times P_\infty \mathbb{C}^+) \to \pi^s_{2n+1}(P_\infty \mathbb{C}^+)
\]

from filtration 0 to filtration 1 can be described by a simple power series and the transfer from \( F^1\pi^s_{2n+1}(P_\infty \mathbb{C}) \) to \( F^2\pi^s_\infty(S^0) \) can be computed using (3.9).
One can use the fact that the transfer raises filtration by 1 not only but several times. There is also a transfer map

\[ t: \pi_n^S(P_\infty C \times P_\infty C^+) \to \pi_{n+1}^S(P_\infty C^+) \]  

(4.1)

One can define \( t \) as induced by the stable map \( \tau \), where \( \tau \) represents \( t^0 \) or as the transfer of the \( S^1 \)-bundle \( \text{pr}^*(ES^2 \to BS^2) \) where \( \text{pr}: P_\infty C \times P_\infty C \to P_\infty C \) is the projection. If one identifies \( \pi_n^S(P_\infty C \times P_\infty C^+) \) with the equivariant bordism group of free equivariantly framed \( T^2 \)-manifolds \( \Omega_{n+2}^{fr}(T^2;\text{free}) \), then \( t \) is given by restricting the \( T^2 \)-action to an \( S^1 \)-action. Similarly one defines

\[ t: \pi_m^S(BT^n) \to \pi_{m+1}^S(BT^{n-1}) \]

If we start in \( \pi_{2n}^S(BT^2) \) with elements of filtration 0 we arrive in \( \pi_{2n+2}^S(S^0) \) by applying the transfer twice in filtration at least 2. How one can compute \( t \) from filtration 1 to filtration 2 is solved by (3.9). The necessary information for computing

\[ t: \pi_{2n}^S(BT^2) \to \pi_{2n+1}^S(P_\infty C) \]

from filtration 0 to filtration 1 is given by the next proposition. To state this we need some more notation.

Let \( y \) be the element in \( K^{-2}(P_\infty C;\mathbb{Q}/\mathbb{Z}) \) defined by reducing the power series \( 1/x - 1/\log(x+1) \) in \( K^{-2}(P_\infty C;\mathbb{Q}) = \mathbb{Q}[x] \) modulo \( \mathbb{Z} \).

Define a map

\[ \hat{t}: K_0(BT^n) \to K_0(BT^{n-1};\mathbb{Q}/\mathbb{Z}) \]  

(4.2)

by the slant product \( \hat{\cdot}: K^{-2}(P_\infty C;\mathbb{Q}/\mathbb{Z}) \otimes K_0(BT^n) \to K_0(BT^{n-1};\mathbb{Q}/\mathbb{Z}) \) with the element \( y \).

**Proposition 4.3:** For \( x \in \pi_{2m}^S(BT^n) \), the value of \( e_x t(x) \) in

\[ K_0(BT^{n-1};\mathbb{Q}/\mathbb{Z})/H_{2m}(BT^{n-1};\mathbb{Q}) \]

is given by \( \hat{t} \circ h(e) \),

where \( h: \pi_{2n}^S(BT^n) \to K_0(BT^n) \) is the Hurewicz map.
The proof of (4.3) can be found in [24].

If we write \( 1/x - 1/\log(x+1) = \sum_{m=0}^{\infty} r_n \times m \), then the coefficients \( c_n \) can be calculated recursively by (see [20]):

\[
\sum_{m=0}^{n} (-1)^m c_m / (n-m) = 0
\]

Elements in \( K_0(BT^2) \) invariant under \( c^k - k^n \) can be written as a rational linear combination of \( b_{1}^{n-i} b_{1}^{i} \). We then have

\[
t(ab_{1}^{n-i} b_{1}^{i}) = -a \cdot \binom{n-i}{i+1} b_{1}^{n-i}
\]

(4.4)

where \( B_i \) is a Bernoulli number (defined by \( z/(e^z-1) = \sum B_i z^i/i! \)).

This is easily seen by well known formulas relating the \( c_i \) to the \( B_i \) [20] or using the Chern character.

To calculate \( e-t \) we need to know the image of the Hurewicz map \( h: \pi_{2n}^{s}(BT^2) \rightarrow K_o(BT^2) \) or equivalently

\( \text{im}(h: \pi_{2n}^{s}(BT^2) \rightarrow H_{2n}(BT^2)) \) or \( \text{im}(h: \pi_{2n}^{s}(BT^2) \rightarrow A_{2n}(BT^2)) \).

But this is not known in general. In a certain range of dimensions \( A_x \) is a good approximation to \( \pi_{s}^{x} \), so one expects \( h: \pi_{2n}^{s}(BT^2) \rightarrow A_{2n}(BT^2) \) to be onto at least for low values of \( n \).

That this is the case can easily be seen by comparing the Atiyah-Hirzebruch spectral sequences for \( \pi_{x}^{s}(X) \) and \( A_x(X) \) or by using the vanishing line for the higher Ext-groups in the Adams spectral sequence:

**Lemma 4.5:** Let \( p \) be an odd prime and \( X \) an \( s \)-connected CW complex with cells only in dimensions which differ by multiples of \( 2(p-1) \). Then \( h: \pi_{2n}^{s}(X) \rightarrow A_{2n}(X) \) is onto as long as \( n < s+2(p-1)^2 \).
Proof: If we compare the Atiyah-Hirzebruch spectral sequences for \( \pi_{2n}^s(X) \) and \( A_{2n}(X) \) using the Hurewicz map we see that \( h \) fails to be onto only if more nonzero differentials start on \( H_{2n}(X; \pi^s) \) than on \( H_{2n}(X; A^s) \) (there are no boundaries). By the assumption on \( X \) only \( d_r \) with \( r = 0 \mod 2(p-1) \) can be nonzero. So for the case in which \( h \) may fail to be onto, there must be an element of \( \text{cok}(J) \) in \( \pi^s_{r}(S^0) \) with \( r+1 = 0 \mod 2(p-1) \). The first time this happens is \( r = \dim \alpha \beta_1^{p-1} = 2(p-1)^2 - 1 \). So as long as \( 2n < 2(p-1)^2 + s \) the Hurewicz map must be onto and \( \text{im}(h) \) can be calculated via K-theory.

We can apply this lemma to the case of \( BT^n \) and \( P_{\infty}^k \) because the \( p \)-localizations of suspensions of those spaces split into wedges of spaces which have cells only in dimensions differing by multiples of \( 2(p-1) \) (for a proof see [30]).

Obviously the sum

\[
Z = \sum_{i=0}^{n} a_i b^{n-i} x^i \quad (4.6)
\]

with \( a_i \in \mathbb{Q} \) in \( K_0(BT^2; \mathbb{Q}) \) is in the eigenspace of \( c \psi^k \) for the eigenvalue \( k^n \) and the only problem is the integrality of \( Z \). This can be decided by expanding \( Z \) in terms of the usual basis \( b_i \times b_j \).

Example: Define \( x_o = (b_i \times b_j - b_j \times b_i)/p \) in \( K_0(BT^2; \mathbb{Q}) \) then it is easy to see that \( x_o \in K_0(BT^2; \mathbb{Z}(p)) \) and so \( x_o \in A_{2(p+1)}(BT^2) \). We denote a preimage of \( x_o \) in \( \pi_{2(p+1)}^s(BT^2) \) simply by \( x_o \) too.

This way of finding the linear combinations \( Z \) as in (4.6) which are in \( K_0(BT^2; \mathbb{Z}(p)) \) by expanding them in terms of \( b_i \times b_j \) is only a reasonable method in lowest dimensions.

We therefore proceed as follows:
Theorem 4.7: Let $z$ be an element in $A_2r(P_\infty \mathbb{C} \times P_\infty \mathbb{C})$. Then

$$Q(z) := \frac{1}{p} (z^p - z \ast (b_1^{r(p-1)} x_1 + x_1 b_1^{r(p-1)} - b_1^{(r-1)(p-1)b_{1}}))$$

is a well defined class in $A_{2rp}(P_\infty \mathbb{C} \times P_\infty \mathbb{C})$ and $z \mapsto Q(z)$ defines an operator

$$Q : A_{2r}(P_\infty \mathbb{C} \times P_\infty \mathbb{C}) \to A_{2pr}(P_\infty \mathbb{C} \times P_\infty \mathbb{C}).$$

Here $z^p$ denotes the $p$-th power of $z$ in the Pontrjagin product on $A_*(P_\infty \mathbb{C} \times P_\infty \mathbb{C})$.

**Proof:** Clearly $Q(z) \in K_0(P_\infty \mathbb{C} \times P_\infty \mathbb{C};\mathbb{Q})$ is in $\ker(c\psi^k - k^rP)$. We must show that $Q(z)$ is in $K_0(P_\infty \mathbb{C} \times P_\infty \mathbb{C};\mathbb{Z}(p))$, that is

$$<x^i \times x^j, Q(z)> \in \mathbb{Z}(p) \text{ for all } i,j \geq 0.$$

This is equivalent to

$$<(1+x)^m \times (1+x)^n, Q(z)> \in \mathbb{Z}(p) \text{ for all } n,m \geq 0.$$

Set $z_{m,n} = \langle (1+x)^m \times (1+x)^n, z \rangle \in \mathbb{Z}(p)$. Because

$$<(1+x)^m \times (1+x)^n, a \cdot b> = <(1+x)^m \times (1+x)^n, a> \cdot <(1+x)^m \times (1+x)^n, b>$$

we have

$$<(1+x)^m \times (1+x)^n, Q(z)> = \frac{1}{p} (z_{m,n}^p - z_{m,n} \cdot (m^{r(p-1)+n(r(p-1)} - m^{(r-1)(p-1)n} - p-1))$$

If $n$ or $m$ is prime to $p$, then $m^{r(p-1)+n(r(p-1)} - m^{(r-1)(p-1)n} - p-1 \equiv 1 \mod p$ and $z_{m,n}^p = z_{m,n} \mod p$ and the right hand side of (4.8) is an integer. If $n$ and $m$ are both divisible by $p$, then on the right hand side there remains $z_{m,n}^p / p$. But in this case we have $z_{m,n} = 0 \mod p$ because $(1+x)^{mp} \times (1+x)^{np} = \psi^p((1+x)^m \times (1+x)^n)$ and $c\psi^p(z) = p^{r} \cdot z$. So $z_{m,n}^p / p \in \mathbb{Z}(p)$. 
Remarks:

1. If $z \in A_2^*(BT^2)$ comes from stable homotopy, then so does $pQ(z)$, because $b_1^ib_1^j$ is the image of $\sigma^i \times \sigma^j$ where $\sigma \in \pi_2^s(P\mathbb{C}^\times)$ is given by the Hopf bundle over $S^2$. So the only problem in establishing $Q$ as an operator in stable homotopy is the divisibility by $p$.

2. One has other choices for the form of the second summand $z^*(1+b_1^{a+1}b_1^{-1}b_1^{a-p+1})$ in the definition of $Q$, e.g. for $a = \dim(z)/2$ even:

$$Q(z) = \frac{1}{p}(z^p-z^*(1+b_1^{a+1}b_1^{a-p+1})x_1-(b_1^xb_1^y)(a(p-1)/2))$$

As one can easily see this does not affect the property of $Q(z)$ of being in the image of $h: \pi_2^s(BT) \to A_2^0(BT^2)$.

3. There are additional operators for $p=2$ using the fact that $v_2(k^t-1)=t$ if $t \neq 0$ (2) and $v_2(k^t-1)=2+v_2(2)$ if $t \equiv 0 \mod 2$.

For example $Q_3(z) = \frac{1}{8}(z^3-z^*(1+b_1^{2a+1}b_1^{2a-p+1}x_1-b_1^{2a-2}))$ and $\dim(z)=a \geq 6$.

4. One can define similar operators on $A_2^*(BT^m)$.

Example: $m=3, n \geq 2$. Set $b:=p-1, a:=nb, c:=(n-1)b, d:=(n-2)b$ and

$$S(n) = \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma$$

Then $z \mapsto (z^p-z^*h(S(n)))/p$ defines an operator on $A_2^*(BT^3)$.

The operator $Q$ on $A_2^*(BT^2)$ has the following properties:

**Proposition 4.8:** a) $Q(x+y) = Q(x)+Q(y) + \frac{1}{p} \sum_{i=1}^{p-1} x^i y^{p-1} (p)$

b) $Q(x \cdot y) = Q(x)y^p + Q(y)x^p \mod p$ ($A_2^*(BT^2)$ is torsion free).
The proof of (4.8) follows directly from the definition and some straightforward calculations.

Property b) shows that Q is most interesting on indecomposable elements of $A_{2k}(BT^2)$. We define inductively

$$x_i = Q_action(x_{i-1})$$

where $x_0 = (b_1^* b_1^P - b_1^* b_1^P)/p$.

Example: $p=3$

$$x_1 = 1/9 \cdot b_1^1 b_1^1 - 10/81 \cdot b_1^3 b_1^3 - 2/27 \cdot b_1^7 b_1^7 + 2/27 \cdot b_1^5 b_1^5 + 10/81 \cdot b_1^3 b_1^9 - 1/9 b_1^1 b_1^1$$

(We have used the symmetric form of Q as in remark 2 above).

By (4.5) it is then clear that $x_0$ and $x_1$ come from stable homotopy. With a little work and some information on $\pi^s_\infty(P_\infty,C)(p)$ (see e.g. § 6) one can prove that $x_2$ is in the image of $h: \pi^s_\infty(BT^2) \to A_\infty(BT^2)$ too. This is most interesting for the prime 3 as we shall see later. Using now the Pontrjagin product on $\pi^s(BT^2)$ we can construct out of the elements $x_0, x_1$ and $\sigma$ a lot of further elements going far beyond the dimension limit where we know $h: \pi^s_{2n}(BT^2) \to A_{2n}(BT^2)$ to be onto.

Suppose given a compact Lie group G with maximal torus $T^n$. In [23] it is shown how $G/T^n$ defines an element $(G/T^n, f, L)$ in $\pi^s_\infty(BT^n)$ and how to calculate the image of this element in $H_*(BT^n)$ using the root system. This can be used to express the elements $x_0$ in terms of homogeneous spaces for low primes. For $p=2$ the element $x_0$ is given by $SU(3)/T^2$ with the canonical $T^2$-bundle on it. It turns out that $x_1 = Q(x_0)$ for $p=2$ is essentially the element given by $G_2/T^2$, the first exceptional Lie group. Similarly
one finds: For \( p=3 \) \( x_0 \) is given by \( \text{Sp}(2)/T^2 \) and for \( p=5 \) by \( G_2/T^2 \).

It is an unsolved problem to give an explicit construction for \( x_0 \) and \( x_1 \) or a description in terms of known manifolds for the other primes. Even more interesting would be a geometric construction describing the operator \( Q \) in stable homotopy (if it exist).

We proceed to calculate \( t(x_i^ix_j^i) \) and \( t \cdot t(x_i^ix_j^i) \) for some \( i,j \).

As already mentioned, the \( p \)-localization of the suspension of \( P_\infty \) splits into \((p-1)\) pieces

\[ \text{Sp}_\infty \mathbb{C}(p) \cong X_1^p X_2^p \cdots \wedge X_{p-1}^p \]  

(4.10)

The \( e \)-invariant calculation of [24] for the stable map \( \tau \) representing \( t^0 \) and (2.5) shows that \( \tau \) restricts to zero on \( X_j \) with \( j \neq (p-2) \). So only the space \( X_{p-2} \) is relevant for the transfer \( t^0 \).

Set
\[ y_i := x_i^{p-i-1} x_0 \quad \text{for} \quad 1 \leq i \leq p-1 \]
\[ y_p := x_1^{p-2} x_0^p \]
\[ y_{p+1} := x_1^{p-1} x_0^{p-2} \]

or in general
\[ y_t := \prod_{i=0}^{r} x_0^{a_i} x_1^{a_i} \cdots x_r^{a_r} \]

where \( t(p-1) = \sum_{i=0}^{r} a_i i \) \((0 \leq a_i < p)\). We then know that \( y_i \) for \( i \leq p+1 \) come from stable homotopy and we denote a inverse image of \( y_i \) by the same symbol. The \( y_i \) are all supported by the component \( X_{p-2}^p \wedge X_{p-2}^p \wedge BT^2 \) \((p)\).

**Theorem 4.11:** Let \( p \) be an odd prime and \( \beta_i \in \pi^S_r(S^0)(p) \) the \( i \)-th element
of the $\beta_t$-family ($r=2(p-1)\cdot(i(p+1)-1)-2$).

Then $t^\ast t(y_i) = \beta_i$ for $i \leq p$
and $t^\ast t(y_{p+1}) = 0$

Remark:
If we had proved that $x_2$ comes from stable homotopy then the same
method would show $t^\ast t(y_i) = \beta_i$ for $2+p \leq i \leq 2p$.

(4.11) will be proved at the end of § 5. As preparation for this
we collect some properties of $t(y_i)$:

**Proposition 4.12:** a) The order of $t(y_i)$ is as follows:

$$|t(y_i)| = p, \quad |t(y_{i+1})| = p^{i+1} \quad \text{for } 1 \leq i < p,$$

$$|t(y_p)| = p^p \quad \text{and} \quad |t(y_{p+1})| = p^{p+1}$$

b) The elements $t(y_i)$ are of the form

$$t(y_i) = \frac{1}{p^i} b_p(i(p-1)-1) + \text{lower terms (i} \neq \text{p)}$$

$$t(y_p) = \frac{1}{p^{p-2}} b_p^3 p^2 - p^p + \text{lower terms}$$

**Proof:** We choose the symmetric form of the operator $Q$ as in remark 2
above. If we interchange the two factors of $BT^2$ then $x_i$ goes over
into $-x_i$. We recall

$$x_0 = \frac{1}{p} (b_1^x b_1^p - b_1^p b_1^y) \quad (4.13)$$

$$x_1 = \frac{1}{p^2} b_1^x b_1^p - p^2 + \frac{1+p}{p^{p-1}} b_1^p x b_1^p + \frac{a_1 b_1^{2p-1}}{p} b_1^p - p^2 - p + \ldots$$

a) We first determine the order of $t(y_i)$ by applying lemma (1.14)
and formula (4.4) for the transfer. We need only consider summands
$a \cdot b_1^i x b_1^j / p^k$ in which $k$ is larger than $i-2$, because the transfer
can introduce only a factor $b / p^2$ when $i < p-1$. 
1. The case of \( y_1 \) is handled separately: \( t(y_1) \) has order \( p \)

2. The element \( x_1^{i} x_o^{p-2-i} \) \((i>0)\) starts with the terms

\[
\pm (1+p^{p-1})(i-j) b_j^{(i-j)p} b_1^j/p^{j+(i-j)p+i} (0 \leq j \leq i)
\]  

(4.14)

where in the notation \( b_1^i x b_1^* \) the star means a number with \( i+* \) = dimension of the element in consideration.

Only for \( j=1 \) do we find \( i+1 \approx o(p) \), so the transfer gives for \( j=1 \) the factor \( 1/p^2 \), whereas for \( j \neq 1 \) we get a factor \( 1/p \). In the sum \( \sum_k c_k b_1^k \) which we get after applying \( m_{p^*} \) we thus have \( |c_1+(i-1)p| = p^{i+2} \) and \( |c_r| = p^s \) with \( s \leq i+2 \) for \( r \neq (i-1)p+1 \). So no cancellation in the contributions of the different terms to \( b_1 \) can take place and \( t(y_1) \) must have order \( p^{i+2} \).

3. In the same way one finds the order of \( t(y_p) \).

4. One readily checks that one has only to consider the same terms in (4.14) for \( x_1^{p-1} x_o^{p-2} \). As in 2. we find the maximal denominator for \( j=1 \). Then \( |t(y_{p+1})| = p^{p+1} \).

b) The next task is to find the maximal filtration of \( t(y_1) \)

1. The skeleton filtration of \( t(y_1) \) is \( 2p(p-2) \), that is to say

\[
t(y_1) = \frac{1}{p} b_p(p-2) + \text{terms of lower filtration}.
\]

2. The case \( x_1^{i} x_o^{p-2-i} \) for \( i \leq p-2 \)

We look at the same terms as in (4.14) but interchange the factors of \( BT^2 \). Putting \( j=0 \) we find a term

\[
\frac{1}{p^{2+i}p} \cdot b_1^{(p-2+i(p-1))} x b_1^{(p-2)+i(p-1)}
\]  

(4.15)
We have \( v_p(p(p-2+i(p-1))!) = p-2+pi \) and \( b^n_1 = n!b^n + \text{lower terms} \). So by applying the transfer we get from (4.15)

\[
\frac{1}{p} b_p(p-2+i(p-1)) + \text{lower terms}
\]

Terms of filtration \( k \) with \( k > p(p-2+i(p-1)) \) can only come from \( b_p^{p-2+(i-r)(p-1)} b^r_1 \) with \( o < r < i \). But the denominator of a product \( a \cdot b^k_1 \cdot b^e_1 \) is always less than or equal to \( p^{\min(k, e)+i} \) by construction of \( x^i_1 \cdot x^{p-2-i}_o \). Thus in those terms the denominator is strictly less than \( p(p-2+pi) \) even after applying the transfer. The maximal filtration is therefore \( p(p-2+i(p-1)) \).

An easy calculation shows that by adding a multiple of \( b^n_1 \) with \( 2n-1 = \dim t(y^i) \) (= kernel of the Bockstein map) we cannot reduce the filtration; but for what follows this is not needed.

3. \( t(y_p) \) has the form \( \frac{1}{p^2} b_p^{p^2-p} + \text{lower terms} \)

4. As in 2, one finds a term \( \frac{1}{p} b_p(p^2-2) \) for \( j=0 \) and all other terms give lower filtration.

As a corollary of (4.11) we can state:

**Corollary 4.16:** Let \( p \) be an odd prime. Then up to dimension \( n = 2(p-1)(p^2+1)-4 \) all elements in the \( p \)-component of \( \pi^s_i(S^0) \) can be constructed from the three manifolds \( \sigma \), \( x^o_0 \) and \( x^1_1 \) using only product and Pontrjagin product.

The reason for stopping with dimension \( 2(p^2+1)(p-1)-4 \) is the element \( \varepsilon^i_1 \). The elements \( \varepsilon^i_1 \) are still in the image of the transfer \( t^0 \). One can show this by the same method as above, but with some
alterations. Let $\bar{\varepsilon}_i$ denote an inverse image of $\varepsilon_i$ in $\pi^S(P_\infty \mathbb{C}^+)$, then $e(\bar{\varepsilon}_i)$ generates the second summand in $A_{2m-1}(P_\infty \mathbb{C})$, that is

$$e(\bar{\varepsilon}_i) = \pi_*(x(2,1))$$

(see § I).

So the appearance of $\varepsilon_i$ goes parallel to the appearance of the second summand of the transfer component in $A_*(P_\infty \mathbb{C})$. Besides the inverse image $t(y_p)$ already constructed, the element $\beta_p = \varepsilon_{p-1}$ has an inverse image of order $p$ in $\pi^S_*(P_\infty \mathbb{C})$. But the elements $\bar{\varepsilon}_i$ ($i < p-1$) are not in the image of

$$t: \pi^S_{2n}(BT^2) \to \pi^S_{2n+1}(P_\infty \mathbb{C})$$

as some elementary calculations show.

To get a lift to filtration $0$ one could try to use the other transfers $t^S$ instead of $t^O$, but I have not done this.

The fact that $\beta_{p+1}$ is not in the image of $t^O$ does not give an insurmountable obstruction in a programme of representing stable homotopy elements by free $T^n$-manifolds. One must only use instead of $t^O$ one of the generalized transfers: For example $\beta_{p+1}$ lies in the image of $t^p(p-1)$ for $p > 3$.

For $p=3$ there exist only parts of the $\beta_t$-family. Using the fact that $x_2$ is in the image of stable homotopy we can construct $\beta_9$ which seemed to be unknown up to now:

**Proposition 4.17:** In the 3-component of $\pi^S_*(S^0)$ we get from $x_0, x_1, x_2$ the elements

$\beta_1, \beta_2, \beta_3, \beta_5, \beta_6, \beta_9$

The method of proof is the same as for (4.11). We therefore omit the calculations.
The elements $x_0, x_1$ can also be used to construct generators for $\pi_*^S(\text{P}_\infty \text{C})$. Up to the dimension $n = 2(p-1)(p^2+1)-3$ nearly all elements of $e(\pi_*^S(\text{P}_\infty \text{C})(p))$ can be constructed from $x_0, x_1$ and $\sigma$ using Pontrjagin products and the transfer; see § 6.

At present, for the odd primes, no element in $\pi_*^S(S^0)(p)$ is known which is in the cokernel of all $t^S$.

For the prime 2, the situation is different; see § 7.

For the construction of the $\mu_t$-family of Adams [2] using the transfer $t^S$, see § 7.
§ 5 Some computations with $A_n(BP)$

This chapter contains the computations which are needed in § 3, § 4 and § 6. First the elements of order $p$ in $A_{2n-1}(BP)$ are worked out and then it is shown that one can compute in $A_{2n-1}(BP)$ modulo elements of certain filtration. This allows us to use the formulas for the map $f: S^{2p} \to BP$ which is needed to compute the transfer, in usual homology instead in $K$-theory.

We then compute the values of $f_*$ mod $p$ and mod $t_1(i+1)$, and close with the proof of (4.11).
In actual computations it seems to be better to use BP-homology instead of complex bordism. We therefore recall some well known facts about BP; for proofs and notation see [3].

The Brown-Peterson spectrum for the fixed prime $p$ is a ring spectrum which fits into the commuting diagram

$$
\begin{array}{ccc}
\text{MU}^*(p) & \xrightarrow{\varepsilon} & \text{MU}^*(p) \\
\downarrow \pi & & \downarrow \text{id} \\
\text{BP} & & \text{BP}
\end{array}
$$

with $\pi \cdot \text{id} = \text{id}_{\text{BP}}$. We have $H^*(\text{BP};\mathbb{Z}(p)) = \mathbb{Z}(p)[q_1, \ldots, q_i]$ and $H_*^{BP}(p) = \mathbb{Z}(p)[v_1, \ldots, v_s]$ where $q_i$ and $v_s$ both have degree $2(p^s-1)$. The Hurewicz map $h: BP_* \rightarrow H_*^p(\text{BP};\mathbb{Z}(p))$ is a monomorphism, so we can identify $BP_*$ with the subring $h(BP_*)$ of $H_*^p(\text{BP};\mathbb{Z}(p))$. The generators $v_s$ are defined inductively

$$
h(v_s) = p \cdot q_s - \sum_{j=1}^{s-1} q_j h(v_{s-j})^{p^j}
$$

(5.1)

In $H_*^p(\text{BP};\mathbb{Z}(p))$ $h(v_s)$ is divisible by $p$ but not by $p^2$. We have $BP_* = \pi_*^{BP}(BP)[t_1, t_2, \ldots]$ with $t_i \in BP_2(p^{i-1})(BP)$. To describe the Hurewicz map $\eta_R: BP \rightarrow BP_* BP$ we identify $BPQ_*(BP)$ with $H_*(BP;Q)[t_1, \ldots, t_n]$. Then

$$
\eta_R(q_k) = \sum_{i+j=k} q_i t_j^{p^i}
$$

(5.2)

We define elements $t_i \in K_*^{BP}(\mathbb{Z}(p))$ by mapping $t_i \in BP_* BP$ into $\text{MU}_*(BP)$ using $i$ and then by the Todd map to $K_*^{BP}(BP)$

**Lemma 5.3:** $K_*^{BP}(\mathbb{Z}(p)) \cong \mathbb{Z}(p)[t_1, \ldots]$
Proof: Clearly \( M = Z(p)[t_1, \ldots, t_i] \subset K_0(BP; Z(p)) \). We know that \( K_0(BP; Z(p)) \) is torsion-free and that rationally \( M \otimes Q = K_0(BP; Q) \).

Let \( G \) denote the multiplicative part in the splitting of \( K(M; Z(p)) \) into \( (p-1) \)-homology theories \([4]\), then \( BP_+BP \to G_*(BP) \) is surjective \([6]\) with image \( Z(p)[t_1, \ldots, t_r, \ldots] \). But \( G_*(X) \) is a direct summand of \( K_*(X; Z(p)) \) so \( M \) must be a direct summand and we are done.

The Hurewicz map \( \eta \) followed by \( i \) and then by the Todd map is the Hurewicz map \( h_K: BP_+ \to K_*(BP; Z(p)) \). Because \( q_n = [p^{n-1}] / p^n \in \pi_*^{s}(MU) \otimes Q \), the Todd map sends \( q_n \) to \( 1/p^n \) in \( K_0(BP; Q) \). Therefore it follows from (5.2):

\[
 h_K(q_k) = \sum_{i+j=k} \frac{1}{p^i} t^j (5.4)
\]

Because \( h \otimes Q \) is the inverse map to the Chern character (5.4) allows us to compute \( \text{ch}(t_j) \).

We have \( A_2n(MU; Q/2) \subset K_0(MU; Q/2) = Q/2[b_1, \ldots, b_k, \ldots] \), set \( \text{deg}(b_i) = i \) and extend this in the obvious way to all elements in \( K_0(MU; Q/2) \). Similarly for \( A_2n(BP; Q/2) \) where we set \( \text{deg}(t_i) = (p^{i-1})/(p-1) \).

Proposition 5.5: Let \( x \in A_2n(MU; Q/2) \) be an element of degree \( r \). If \( n \) is sufficiently large compared with \( r \), then \( \beta(x) \in A_2n-1(MU) \) is zero.

Proof: We have \( A_2n(MU; Q/2) = A_2n+2N(MU(N); Q/2) \) for \( N \) large. Let \( BU(N, q) \) denote the \( q \)-skeleton of \( BU(N) \) and \( MU(N, q) \) the Thom space of the restricted universal bundle \( EU(N) \). Because there exists a \( q \)-dimensional bundle \( E \), stably isomorphic to \( EU(N) \) restricted to \( BU(N, q) \), we have \( MU(N, q) = S^{2N-1} \cdot BU(N, q)^E \). The assumption on \( x \) implies that we can find an element \( \bar{x} \in A_2n+q(BU(N, q)^E; Q/2) \) with
q<r+1 which maps to x under suspension and the map induced by inclusion. Now \( X = BU(N,q)^E \) is a \( 2q \)-dimensional CW-complex, which can therefore be embedded in \( S^{4q+2} \). Let \( D(X) \) be the 4a+2-Spanier-Whitehead dual of X. Then

\[
A_{2n+2q}(X;\mathbb{Q}/2) \cong A^{3q+1-2n}(D(X);\mathbb{Q}/2)
\]

If \( r \) is sufficiently small compared with \( n \), we can choose \( q \) such that \( 3q+1-2n \) is still negative. But in the negative range the \( e \)-invariant

\[
\pi_s^i(X) \rightarrow A^i(X)
\]

is surjective [43]. This means that \( \bar{x} \) is in the image of \( e \) on \( \pi_{2n+q}^s(BU(N,q)^E) \) and so the same is true for \( x \). But if \( x \in A_{2n}(MU;\mathbb{Q}/2) \) comes from stable homotopy, then \( \beta(x) = 0 \) because \( \pi_{2n-1}^s(MU) = 0 \).

**Corollary 5.6:** Let \( x \in A_{2n}(BP;\mathbb{Q}/2) \) be an element of degree \( <r \).

If \( n \) is sufficiently large compared with \( r \), then \( \beta(x) \in A_{2n-1}(BP) \) is zero.

**Proof:** Because \( \pi: A_{2n}(MU;\mathbb{Q}/2) \rightarrow A_{2n}(BP;\mathbb{Q}/2) \) is onto we can find a polynomial in the \( b_i \) which maps to \( x \). In an inverse image of \( t_i \) we only need \( b_j \) with \( j \leq (p-1) \), thus we can find an inverse image \( y \) for \( x \) with degree \( y < r(p-1) \) and can apply (5.5).

Let \( x \in A_{2n}(BP;\mathbb{Q}/2) \subset K_0(BP;\mathbb{Q}/2) \) be an element of order \( p \), then \( x \) also lies in \( A_{2m}(BP;\mathbb{Q}/2) \) with \( m = n + t(p-1) \), \( t > 0 \), because \( (c_k^y - k^n)(x) = 0 \) implies \( (c_k^y - k^m)(x) = 0 \) in this case.

For some large \( t \) (5.6) implies that the element \( x \) is in the kernel of \( \beta \), or what is the same, it is of the form \( y/p \) where \( y \in A_{2n+2t(p-1)}(BP) \). By the Hattori-Stong theorem h: \( \pi_{2m}^s(BP) \rightarrow A_{2m}(BP) \)
is an isomorphism, so \( y \) is a polynomial in the elements \( h(v_i) \).

Example: We have \( \psi^k(t_1) = k(p-1)t_1 + \frac{k(p-1)-1}{p} \equiv t_1 + a \mod p \)

So \( \psi^k(t_1 - t_1) = t_1 - t_1 \mod p \)

Therefore \( x = \frac{(t_1 - t_1)}{p} \) defines an element in \( \mathbb{A}_{2n}(BP; \mathbb{Q}/\mathbb{Z}) \) with \( n = p(p-1) + t(p-1) \). For \( t = 0 \) we have \( \beta(x) \neq 0 \) because \( \ker(\beta) \)

consists of the multiples of \( h(v_1^p) = (pt_1 + 1)^p \). The element \( \beta(x) \)

(\( t = 0 \)) is the first nonzero element in \( \mathbb{A}_i(BP) \) with \( i \) odd. Suppose now \( t \neq 0 \). Because \( n_R(v_2) = v_2 + v_1t_1^p - v_1^p t_1 \mod p \)

we have \( h(v_2) = t_1^p - t_1 \mod p \) in \( K_0(BP; \mathbb{Z}(p)) \). So \( x = h(v_2^\ell - 1)/p \)

and therefore \( \beta(x) = 0 \) in this case. Similarly

\( (c\psi^k - k^n)(t_1^p - t_1)^k \equiv 0 \mod p \) (\( n \neq 0(p-1) \)) and \( (t_1^p - t_1)^k/p \) defines

an element in \( \mathbb{A}_{2n}(BP; \mathbb{Q}/\mathbb{Z}) \) with \( n = kp(p-1) + s(p-1) \) (\( k \geq 1, s > 0 \)). For \( k > p \) it is easy to see, that this element has nonzero Bockstein

as long as \( s < k \). We denote this element by \( h(v_2^k)/p \).

As an immediate corollary of (5.6) we get

Corollary 5.7: The elements of order \( p \) in \( \mathbb{A}_{2n}(p-1)(BP; \mathbb{Q}/\mathbb{Z}) \) are

of the form \( y/p \) where \( y \) is a polynomial in \( h(v_i) \)

and \( \deg(y) \leq n \).

Using the Bockstein sequence

\[
\mathbb{A}_{2n}(BP) \xrightarrow{\partial} \mathbb{A}_{2n}(BP) \xrightarrow{\partial} \mathbb{A}_{2n}(BP; \mathbb{Z}/p) \xrightarrow{\beta} \mathbb{A}_{2n-1}(BP)
\]

one can easily work out which of these polynomials have nonzero

Bockstein.

The order of \( \mathbb{A}_{2n-1}(BP) \) can be computed by a similar method to

that used for \( P_\infty \mathbb{C} \). We know the Hurewicz map \( \mathbb{A}_{2n}(BP) \to H_{2n}(BP) \).
and therefore the order of the differentials in the Atiyah-Hirzebruch spectral sequence on the bottom line $H_\ast(BP;A_0)$. Because there are no other differentials for dimensional reasons, and we know the number of elements in the $E_2$-term, we can compute the number of elements in the $E_\infty$-term.

Lemma 5.8: For $n < p(p^2 - 1)$ the group $A_{2n-1}(BP;\mathbb{Q}/\mathbb{Z})$ is cyclic.

Proof: Because $h(v_3) = t^p_2 - t_2^p +$ terms in $t \mod p$ (see [34]) has degree $p(p+1)$, elements of order $p$ in dimensions less than $2p(p^2 - 1)$ can be expressed using only $h(v_1)$ and $h(v_2)$. One easily sees that the only elements not in ker $(\beta)$ can be given by $h(v_2^k)/p$ for some $k$, that is by the elements discussed above. If there are two values of $k$ such that $h(v_2^k)/p$ is in $A_{2n}(BP;\mathbb{Q}/\mathbb{Z})$, then there is only one with nontrivial image under $\beta$.

For later use we work out $A_{2m}(BP;\mathbb{Z}/p)$ for $m = \deg (v_3v_2) = p(p+1)+p$.

Lemma 5.9: The elements $h(v_3v_2)/p, h(v_2^{p+1})/p$ and $h(v_2^{p+2})/p$ are in $A_{2m}(BP;\mathbb{Z}/p)$ and have nonzero image under $\beta$. The group $A_{2m-1}(BP)$ has 3 cyclic summands.

The proof of (5.9) is an easy calculation which we omit.

Remarks:
1. The cofibre sequence $S^0 \to BP \to BP$ shows

$$A_{2m-1}(BP) \cong A_{2m-1}(BP)$$ (5.10)

2. A cofibre sequence which induces short exact sequences in BP-homology induces a long exact sequence of Ext-groups. In this way one can map

$$\text{Ext}_{BP,BP}^0(BP, BP; (BP; \mathbb{Q}/\mathbb{Z})) \text{ surjectively onto } \text{Ext}_{BP,BP}^2(BP, BP)$$
using the two boundary maps

$$\text{Ext}^{0,\infty}(BP_*, BP_*(\overline{BP}; Q/Z)) \rightarrow \text{Ext}^{1,\infty}(BP_*, BP_*(\overline{BP})) \rightarrow \text{Ext}^{2,\infty}(BP_*, BP_*)$$

But $$\text{Ext}^{0,\infty}(BP_*, BP_*(\overline{BP}; Q/Z))$$ = set of primitive elements in $$BP_*(\overline{BP}; Q/Z)$$ can be mapped into $$A_{2n}(\overline{BP}; Q/Z)$$ and then using the Bockstein map into $$A_{2n-1}(\overline{BP})$$. This defines an injective map

$$\text{Ext}^{2,2n+1}_{BP_*(BP)}(BP_*, BP_*) \rightarrow \text{A}_{2n-1}(BP)$$

The element $$\beta_s$$ of the $$\beta_t$$-family is described in $$\text{Ext}^{0,\infty}(BP_*, BP_*(BP; Q/Z))$$ by $$(\eta_R(v_2)^s - v_2^s)/p v_1$$ and maps to $$h(v_2)^s/p$$ in $$A_{2n-1}(\overline{BP})$$. So we know the e-invariant of an inverse image of $$\beta_s$$ under the boundary map $$\beta: \eta^s_*(BP) \rightarrow \eta^s_*(S^0)$$.

The formulas for $$\eta_R(v_2)^s$$ and $$h(v_2)^s$$ show why $$A_{2n-1}(\overline{BP})$$ is much larger than $$\text{Ext}^{1,2n}_{BP_*(BP)}(BP_*, BP_*(\overline{BP}))$$.

For connected spectra like $$BP$$ there exists an Atiyah-Hirzebruch spectral sequence. There is a filtration of $$BP$$, denoted by $$BP$$ whose associated graded groups are the $$E^\infty$$-terms of the $$E^2$$-terms $$E^2_{s,t} = H_s(BP; A_t(\ast))$$. In the group $$A_{2n}(BP; Q/Z)$$ this filtration is the one given by degree (up to the factor $$2(p-1)$$). The statement of (5.7) means then that nonzero elements in $$A_{2n-1}(BP)$$ must have high filtration. We can use this observation to simplify some calculations. For late use we restrict to the following dimensions $$m = 2(p-1)(p^k+p-1)$$ and $$k \leq p+1$$, although the statement of the following proposition seems to be true in general.

**Proposition 5.11:** The canonical map $$j_*: A_{2m-1}(BP) \rightarrow A_{2m-1}(BP/BP^r)$$

with $$r = 2(p-1)(pk-2)$$ and $$m = (p-1)(pk+k-1)$$

$$k \leq p+1$$ is injective.
(5.11) means that in $A_m(BP; Q/Z)$ we can calculate modulo terms of degree $\varepsilon(pk-2)$ without losing $A_m-1(BP)$. We identify $H_n(BP; Q)$ with its image in $K_0(BP; Q/Z)$. For the proof of (5.11) we require the following lemma:

**Lemma 5.12:** If we have in $K_o(BP; Q)$ the following equation

$$\sum_{i=0}^{p} a(i)q_2q_1^{i}a-i(p+1) + a(p+1)q_3q_1^{(p-1)} \equiv 0 \mod Z(p) \text{ and mod degree } \leq p^2+p-2$$

where

$$a = p^2+2p,$$

then $a(i) \in Z(p)$ and moreover

$$a(i) \equiv 0 \mod p^{p-2}.$$

**Proof:** The statement $a(i) \in Z(p)$ follows from the fact that the terms of highest degree in $q_2^ia_i(p+1)$ are linearly independent.

Then by an easy calculation one sees that the coefficient of $t_2^j t_1^{p^2+p-j(p+1)}$ in $q_2^{k}q_1^{p^2+2p-j(p+1)}$ for $k > j$ has denominator at most $p^2$. The number $a(p+1)$ must be divisible by $p^{(p-1)}$ because of the term $t_3/p^{p-1}$ in degree $p^2+p+1$ of $q_3q_1^{(p-1)}$.

In degree $p^2+p$ the term $t_2^p$ appears in $q_2^{p}q_1^{p^2+2p-(p+1)p} = q_2^{p}q_1^{p}$ with denominator $p^p$ and in $a(p+1)q_3q_1^{(p-1)}$ with denominator $< p$.

So $a(p) \equiv 0 (p^{P-1})$. The term $t_2^{p-1} t_1^{p+1}$ appears in $q_2^{p}q_1^{p}$ and in $q_2^{p-1} q_1^{2p+1}$. Its denominator in $q_2^{p-1} q_1^{2p+1}$ is $p^p$ and in $a(p)q_2^{p}q_1^{p}$ at most $p$. So at least $a(p-1) \equiv o(p^{P-1})$. Proceeding inductively the conclusion follows. One factor $p$ is lost because the denominator of the coefficient of $t_2^t_1^{p^2-1}$ in $q_2^{p^2+p-1}$ is only $p^{p-1}$. 


Proof of (5.11):

1. We first treat the case $k \leq p$. Then $A_{2m-1}(BP)$ is cyclic and the element of order $p$ is given by $\beta(h(v_2^k)/p)$. In $A_{2m}(BP/BP^r; Q/Z)$ we have

$$h(v_2^k)/p = \frac{1}{p} \cdot (t_1^{p-1} - t_1)^k = \frac{1}{p} \cdot t_1^{pk} \quad (5.13)$$

Suppose that $t_1^{pk}/p$ lies in the kernel of $\beta: A_{2m}(BP/BP^r; Q/Z) \to A_{2m-1}(BP/BP^r)$, that is in the image of $A_{2m}(BP/BP^r; Q) \to A_{2m}(BP/BP^r; Q/Z)$; then we can write

$$\frac{1}{p} \cdot t_1^{pk} = \sum_{i=0}^{k-1} a_i \cdot q_1^{(p+1)(k-i)-1} \cdot q_2^i \quad (5.14)$$

But if we look at the coefficients of $t_1^{pk}$ and $t_1^{pk-1}$ in $q_1^{(k-i)(p+1)-1} q_2^i$ we see that they are equal. So an equation like (5.14) cannot exist and $\beta(t_1^{pk}/p) \neq 0$ in $A_{2m-1}(BP/BP^r)$.

Remark: If we try to compare a given element $z \in A_{2m}(BP; Q/Z)$ with $h(v_2^k)/p$ we may not only calculate mod degree $\leq pk-2$ but can also introduce the additional relation $t_2 = 0$. Then $h(v_2^k)/p$ is still nonzero modulo those relations, which then all reduce to $t_1^{pt}/p = t_1^{pt-1}/p$.

2. The case $k = p+1$.

As for $k \leq p$ the coefficients of $t_1^{p_2+p}$ and $t_1^{p_2+p-1}$ are the same in $q_2^2 \cdot q_1^{2p-2i(p+1)}$ with the one exception $i=1$, where we have

- the coefficient of $t_1^{p_2+p}$ in $q_2^2 \cdot q_1^{2p-2i(p+1)}$ is $\frac{(p^2+p-1)}{p-1}$
- the coefficient of $t_1^{p_2+p-1}$ in $q_2^2 \cdot q_1^{2p-2i(p+1)}$ is $\frac{(p^2+p-1)}{p-1} / p^p + \frac{1}{p^2}$

The group $A_{2m-1}(BP)$ is a sum of 3 cyclic groups and the elements of order $p$ are $\beta(h(v_2^{p+2})/p)$, $\beta(h(v_2^{p+1})/p)$ and $\beta(h(v_2 v_3)/p)$.

Now let
\[ a \cdot \frac{h(v_2^{p+2})}{p} + b \cdot \frac{h(v_2^{p+1})}{p} + c \cdot \frac{h(v_2 v_3)}{p} = \frac{p}{i} \sum_{i=0}^{p-1} a(i) \cdot \frac{q_2^2 q_1^{2p-2i(p+1)} + a(p+1) q_3 q_1^{p-1}}{p} \]

mod degree \( \leq p^2 + p - 2 \) in \( A_{2m}(BP/BP; \mathbb{Q}/\mathbb{Z}) \). Then by (5.12) we have
\[ a(i) \equiv 0 \pmod{p^{p-3}}. \]
Looking at the coefficients of \( t_1^{p^2 + p - 1} \) and \( t_1^{p^2 + p} \) mod \( \mathbb{Z} \) we see that they are the same; the additional summand \( t_1^{p^2 + p - 1} \cdot a(1)/p^2 \) vanishes (for \( p > 3 \), for \( p = 3 \) one calculates directly).

We have
\[
\begin{align*}
\frac{h(v_2^{p+2})}{p} &= \frac{1}{p} \cdot t_1^{p^2 + 2p} - \frac{2}{p} \cdot t_1^{p^2 + p + 1} \\
\frac{h(v_2 v_3)}{p} &= \frac{1}{p} \left( t_2^2 t_1^p - t_1^{p^2 + p + 1} - t_1 t_2^p \right) \\
\frac{h(v_2^{p+1})}{p} &= \frac{1}{p} \cdot t_1^{p^2 + p}
\end{align*}
\]

mod degree \( \leq p^2 + p - 2 \).

On the left hand-side of (5.15) the coefficient of \( t_1^{p^2 + p} \) is \( b/p \) and that of \( t_1^{p^2 + p - 1} \) is zero, but on the right-hand side they are equal, so \( b \equiv 0 \pmod{p} \). Because \( a(i) \equiv 0 \pmod{p} \) we find \( a \equiv 0 \) and \( c \equiv 0 \) mod \( p \) by looking at the terms of highest degree.

The usual generators for \( H_*(MU) \) are the elements \( b_i^H = \phi(b_i) \) where \( \phi: H_*(BU) \to H_*(MU) \) is the Thom isomorphism, whereas \( H_*(BP; \mathbb{Z}(p)) = \mathbb{Z}(p)[q_1, \ldots, q_i, \ldots] \). Let \( \pi: MU \to BP \) be the projection. We need some information about \( \pi_*(b_i^H) \) in terms of the \( q_i \). The values of \( \pi_*(m_i^H) \) (notation as in [3]) are known and there are formulas for computing \( b_i^H \) in terms of \( m_i^H \) (7.5 in [3]). However for computing \( \pi_*(b_i^H) \mod p \) we proceed as follows:

**Proposition 5.16:** \( \pi_*(b_j^H) \equiv 0 \pmod{p} \) if \( j \equiv p^i - 1 \)
\[
\pi_*(b_i^H p^{i-1}) = - \sum_{k=1}^{i} q_i \cdot \left( \pi_*(b_i p^{i-k-1}) \right)^p \pmod{p}
\]
Proof: We have a commutative diagram

\[
\begin{array}{ccc}
H_\ast(MU) & \xrightarrow{\pi} & H_\ast(BP) \\
\uparrow T_{MU} & & \uparrow T_{MU} \\
MU_\ast(MU) & \xrightarrow{\pi} & MU_\ast(BP) \quad \xrightarrow{T_{BP}} BP_\ast(BP)
\end{array}
\]

with \( T_{MU}(b_1^{MU}) = b_1^H \). In \( BP_{\ast}BPQ_\ast(P_{\ast}C) \) the following equation is true (see \([3]\) §16):

\[
\sum_f \eta_L(q_f)(x^f)p_f = \sum_f \eta_R(q_f) \left( \sum_{j>0} \pi_\ast(b_j^H)(x^j)^{j+1} \right)p_f \quad (5.17)
\]

Applying \( T_{BP} \) gives

\[
q_0 = \sum_f q_f \cdot \left( \sum_{j>0} \pi_\ast(b_j^H)(x^j)^{j+1} \right)p_f
\]

Calculating mod \( p \) we arrive at

\[
\sum_f \sum_j \pi_\ast(b_j^H)p_f x^{(j+1)}p_f \cdot q_f \equiv q_0 \mod p
\]

Inductively it follows that \( \pi_\ast(b_j^H) \equiv 0 \mod p \) if \( j \neq p^{s-1} \) and for \( i>0 \)

\[
\sum_{k=0}^{i} q_k \pi_\ast(b_{p^{i-k}-1})^p = 0
\]

Remark: A similar argument gives a corresponding formula in K-theory.

Examples:

\[
\begin{align*}
\pi(b_{p-1}) &= -q_1 \\
\pi(b_{p^2-1}) &= q_1^{p+1} - q_2 \\
\pi(b_{p^3-1}) &= -q_1^{p+2} + q_1^p q_2 + q_1 q_2^2 - q_3 \mod p
\end{align*}
\]
Corollary 5.19: \[ \pi_*(b_{p^i-1}) = (-1)^i q(p^{i-1})/(p-1) \mod p, q_2, q_3. \]

For computing the transfer \( t^S \) in filtration 1 we need to calculate the map \( S^2 P_\infty C^{sH} \to \mu \to BP \). Recall that the map \( f^S \) was defined by the diagram

\[
\begin{array}{ccc}
S^0 & \to & P_\infty C^{(s-1)\overline{H}} \\
\downarrow & & \downarrow j^! \\
S^0 & \to & S^2 P_\infty C^{sH} \\
\downarrow f^S_{s-1} & & \downarrow f^S \\
MU & \to & \overline{MU} \\
\end{array}
\]

If we define the generators \( b^S_i \) of \( H_*(P_\infty C^{sH}) \) using the Thom isomorphism \( \phi: H_*(P_\infty C) \to H_*(P_\infty C^{sH}) \) we can use the commutative diagram

\[
\begin{array}{ccc}
H_*(P_\infty C^{\overline{H}}) & \overset{f^S}{\to} & \tilde{H}_*(MU) \\
\downarrow \phi & & \downarrow \phi \\
H_*(P_\infty C) & \overset{f^S}{\to} & H_*(BU) \\
\end{array}
\]

Note that \( H_*(MU) \) and \( H_*(BU) \) are both rings and the Thom isomorphism is a ring homomorphism. But \( f^S_*(1+b_1+b_2+\ldots) = (1+b_1+b_2+\ldots)^S \) in \( H_*(BU) \), so

\[ f^S_*(b^S_i) = (b^H_o + b^H_1 + \ldots)^S (i) \tag{5.20} \]

where \( (i) \) means term of degree \( i \). Because \( j^!:\tilde{H}_*(P_\infty C^{(s-1)\overline{H}}) \to \tilde{H}_*(P_\infty C^{sH}) \) maps \( b^S_{i-1} \) to \( b^S_i \) we arrive at

\[ f^S_*(b^S_{i-1}) = (b^H_o + b^H_1 + b^H_2 + \ldots)^{S-1} (i) \tag{5.21} \]

The analogous formulas are true in K-theory.

For calculating with the transfer \( t^0 \) we need therefore

\[ (\pi_*(b_o) + \pi_*(b_1) + \pi_*(b_2) + \ldots)^{-1} \]
Modulo $p$ and $q_i$ for $i>1$ this is easy to calculate:

**Proposition 5.22:** The coefficient $c_t$ of $q^t$ in $(\pi_0^H + \pi_1^H + \pi_2^H + \ldots)^{-1}$ has the following mod $p$ value: write $t(p-1) = \sum a_i \cdot p^i$ with $0 \leq a_i < p$, then $c_t \equiv 0$ if $\sum a_i > p-1$ and $c_t \equiv (-1)^t a_0! a_1! a_2! \ldots a_r!$ if $\sum a_i = p-1$.

**Proof:** We calculate $(1+y+y^p+\ldots)^{-1}$ and set $y = x^{p-1}$, then

\[
(1+x^{p-1}+x^{p^2-1}+\ldots)^{-1} = \frac{1}{x} \left( \frac{x}{1+x^{p-1}+x^{p^2-1}+\ldots} \right)
\]

\[
= 1 - \frac{x^{p}+x^{p^2}+x^{p^3}+\ldots}{x+x^{p}+x^{p^2}+\ldots}
\]

\[
= 1 - \frac{(x+x^{p}+x^{p^2}+\ldots)^{p}}{(x+x^{p}+x^{p^2}+\ldots)}
\]

\[
= 1 - (x+x^{p}+x^{p^2}+\ldots)^{p-1}
\]

\[
= 1 - \sum_{a_o,a_1,\ldots,a_r \geq 0} \frac{(p-1)!}{a_o!a_1!a_2!\ldots a_r!} x^{a_o+a_1p+\ldots a_r+p}
\]

The desired identity follows on comparing the coefficients of $x^{t(p-1)}$ on both sides. (5.19) completes the proof.

Similarly, one can deduce formulas for $\pi_x(b_0+b_1+b_2+\ldots)^s \mod p, q_2, \ldots$

**Proof of (4.11):**

a) We want to show $tt(y_i) \neq 0$ for $i \leq p-1$. Because $\pi_x(S^0)(p)$ in this dimension is generated by $\beta_i$ we must have $tt(y_i) = a\beta_i$ with $a \neq 0(p)$.

**Remark:** Because $\text{Ext}_{BP_*BP}^{2,2n+3}(BP_*,BP_*)$ can be mapped monomorphically into $A_{2n+1}(BP)$ one can also compute the value of $a$ and so does not actually need the information that $\pi_x^S(S^0)(p)$ is generated by $\beta_i$. One would then get $tt(y_i) = a\beta_i \mod$ higher filtration.
We set $f = \pi f^0$. By (3.9) we must show that $\beta f_\ast t(y_i)$ is nonzero in $A_\ast (BP)$ and by (5.11) we are allowed to compute mod terms of degree $\leq (p_i-2)$. We know $t(y_i) = (1/p) b^K_{p,((p-1)i-1)} + z$ with $\deg(z) < p((p-1)i-1)$.

So at once $f_\ast (z) = 0$ and we need to know only the term of top degree in $f_\ast (b^K_{p,((p-1)i-1)})$. This of course can be calculated via homology and we can use (5.22) to obtain

$$f_\ast t(y_i) = \frac{1}{p} t^p_{1} \mod t_2 \text{ and degree } \leq (pi-2)$$

As remarked in the proof of (5.11), this suffices to show $\beta f_\ast t(y_i) \neq 0$ because $h(v^i_2)/p = t^p_{1} + p = -t^p_{1}/p \mod q^p_{1} + 1$.

b) The proof for $t t(y_p) = \beta$ runs along similar, but more complicated, lines.

c) To prove $t t(y_{p+1}) = 0$ we must show that $\beta f_\ast t(y_{p+1})$ goes to zero in $A_\ast (BP)$. By the same filtration argument as in a) we need only look at the term in degree $(p+1)p-1$.

The following two statements are easily proved by direct calculation:

1. The coefficient of $\pi a_0 b^H_1 + b^H_2 + \ldots$ in degree $(p+1)p-1$ is $t^p_{1} t^{p-1}_{2} \mod p$ (using (5.18)).

We have already observed in (5.22) that this is zero mod $t_2, t_3$.

2. We have the relations

$$p^p_{1} q^p_{1} = \frac{1}{p} t^p_{2}$$
$$p^p_{1} q^p_{2} = \frac{1}{p} (t^p_{2} + a_1 t^{p-1}_{2} t^p_{1} + a_2 t^{2+p} + a_2 t^{2+p-1})$$
$$p^p_{1} q^p_{1} = \frac{1}{p} (t^{2+p} + t^{2+p-1})$$

mod degree $p^2+p-2$ with $a_1, a_2 \neq 0 (p)$.

Statement 1. proves $f_\ast (t(y_{p+1})) = \frac{1}{p} t^p_{1} t^{p-1}_{2}$ mod degree $\leq p^2+p-2$ and
2. shows that this goes to zero under the Bockstein map.

As an example we include the explicit formula for \( f_\times(t_{p+1}) \) in the case \( p=3 \):

\[
\begin{align*}
  f_\times(t_{p+1}) &= \frac{1}{3} \cdot t_2^3 + \frac{1}{3} \cdot t_1^6 t_2 + \frac{1}{3} \cdot t_2^3 + \frac{1}{3} \cdot t_1^4 + \frac{8}{9} \cdot t_1^3 t_2 + \frac{8}{9} \cdot t_1^2 + \frac{19}{27} \cdot t_1^9 + \\
  &+ \frac{7}{27} \cdot t_2 + \frac{13}{27} \cdot t_6 + \frac{11}{27} \cdot t_1 + \frac{79}{81} \cdot t_3.
\end{align*}
\]
§ 6 On the image of the e-invariant

In this chapter we discuss the problem of computing the values which the e-invariant can take on \( \pi_*^S(P_\infty \mathbb{C}) \). The computation of the algebraic K-groups for \( P_\infty \mathbb{C} \) in § 1 gives an upper bound for the image of the e-invariant. In contrast to the case of stable cohomotopy in negative dimensions, where the existence of the J-homomorphism implies surjectivity of the e-invariant, the e-invariant need not be surjective. Indeed the deviation from surjectivity is connected with other interesting phenomena such as behaviour of the transfer maps \( t^S: \pi_*^S(\mathbb{H}) \rightarrow \pi_*^S(S^0) \) from Adams filtration 1 to filtration 2. If the e-invariant were surjective, then all \( t^S \) would have to map \( F^1 \pi_*^S(\mathbb{H}) \rightarrow \) to higher filtration than \( F^2 \). Every element of \( F^2 \pi_*^S(S^0) \) which is in the image of some \( t^S \) causes an infinite series of elements in coker(e). We shall use this to find elements in the cokernel of \( e: \pi_*^{2n-1}(P_\infty \mathbb{C}) \rightarrow \Lambda_{2n-1}(P_\infty \mathbb{C}) \).

A further upper bound for \( \text{im}(e) \) is given by the groups

\[ \text{Ext}_{BP_*BP}(BP_*BP_*(P_\infty \mathbb{C})) \subset A_{2n-1}(P_\infty \mathbb{C}) \]

It is certainly difficult but not impossible to calculate these groups using the methods of [29]. But even this would not describe \( \text{im}(e) \) completely, because there are nonzero differentials in the Adams spectral sequence on these ext-groups. Via the transfer map those differentials are related to differentials on \( \text{Ext}_{BP_*BP}(BP_*BP_*) \) and so a complete computation of \( \text{im}(e) \) would give information on \( F^2 \pi_*^S(S^0)(p) \).

For lower bounds for \( \text{im}(e) \), the situation is not even as good as
for upper bounds. There are several ways of giving lower bounds for the image of \( e \). First, we can use S-duality and the J-homomorphism to show that elements of low skeletal filtration in \( A_{2n-1}(P_{\infty}C) \) are always in the image of \( e \). Second, we can use the transfer map \( \pi_*^S(BT^2) \to \pi_*^S(P_{\infty}C) \) to produce elements in \( \text{im}(e) \).

The first method only gives a coarse lower bound in all dimensions. The second can only be used effectively in a certain range of dimensions because the image of the Hurewicz map

\[
h: \pi_2^S(BT^2) \to H_{2n}(BT^2)
\]

whose knowledge is necessary for this method is not known in general.

We start with giving lower bounds using S-duality. We then discuss a simple method to find elements in the cokernel of

\[
e: \pi_2^{S}(P_{\infty}C) \to A_{2n-1}(P_{\infty}C),
\]

describe the relation to the transfer maps and use this to calculate \( \text{im}(e) \subset A_{2n-1}(P_{\infty}C) \) for \( n \leq (p-1)p^2 - p \).

This then allows us to compute \( \pi_*^S(P_{\infty}C)_{(p)} \) in the same range of dimensions. We also show how one can describe and construct the elements in \( \pi_*^S(P_{\infty}C) \) in this range by framed manifolds (up to some exceptions). As an application, we compute the image of the Hurewicz map

\[
h: \pi_{2n-1}^S(B\mathbb{Z}/p^2) \to K_1(B\mathbb{Z}/p^2) \text{ for } n \not\equiv 0(p).
\]
We first discuss the way to give lower bounds by use of S-duality and the J-homomorphism.

**Proposition 6.1:** Let \( x \in A_{2n-1}(P_\infty) \) be in the image of the map induced by inclusion \( i: A_{2n-1}(P_k) \to A_{2n-1}(P_\infty) \) and \( 2k+m \leq 2n-1 \) where \( m \) is the geometric dimension of the stable normal bundle of \( P_k \). Then \( x \) is in the image of \( \pi_{2n-1}^S(P_\infty) \to A_{2n-1}(P_\infty) \).

**Proof:** If the S-dual of an element \( x \) is in the image of \( e \), then \( x \in \text{im}(e) \). We have a commutative diagram:

\[
\begin{array}{ccc}
\pi_{2n-1}^S(P_k) & \xrightarrow{D} & \pi_{1-2n+2k+m}(P_k) \\
\downarrow e & & \downarrow e \\
A_{2n-1}(P_k) & \cong & A_{1-2n+2k+m}(P_k)
\end{array}
\]

where \( D \) is S-duality and \( P_k \) the Thom space of the normal bundle of \( P_k \). If \( r = 1-2n+2k+m \leq 0 \) we can look at the J-homomorphism

\[ J: K_{r-1}(P_k) \to \pi^r_s(P_k) \]

The map \( J \) factors as

\[ K_{r-1}(P_k) \to A^r(P_k) \to \pi^r_s(P_k) \]

and for an odd prime the composition \( e \circ J \) is bijective [43].

This shows that \( e: \pi_{2n-1}^S(P_k) \to A_{2n-1}(P_k) \) is surjective if \( 2k+m \leq 2n-1 \).

**Example:** \( p=3 \) The element \( x = (b_3-b_4+b_5)/3 \) in \( A_{2n-1}(P_\infty) \) with \( n \) odd has filtration 6 (because \( x = a \cdot b_1+b_2+c \cdot b_3 \mod b^n_1 \) in \( K_0(P_\infty;\mathbb{Q}/\mathbb{Z}) \)), so if \( 2n-1 \geq 11 \), \( x \) is in the image of \( e \).
The same argument applied to \( B\mathbb{Z}/p^r \) gives

**Proposition 6.2:** Let \( L^{2k+1}(p^r) \) be the \( 2k+1 \)-skeleton of \( B\mathbb{Z}/p^r \) and \( m \) the geometric dimension of the stable normal bundle of \( L^{2k+1}(p^r) \). Then
\[
\text{im} \left( A_i(L^{2k+1}(p^r)) \to A_i(B\mathbb{Z}/p^r) \right) \text{ is contained in } \text{im}(e) \text{ if } 2k+1+m \leq i.
\]

In [38] it is shown that \( m(L^{2k+1}(p^r)) \leq 2 \left\lfloor \frac{k}{2} \right\rfloor + 1 \) if \( p \) is odd. So we always have
\[
\text{im} \left( A_i(L^{2k+1}(p^r)) \to A_i(B\mathbb{Z}/p^r) \right) \subset \text{im}(e) \text{ if } k+1+ \left\lfloor \frac{k}{2} \right\rfloor \leq i.
\]

**Example:** \( p=3 \) \( z = \frac{3^3+2^n \cdot 6^6}{3} \), \( n \) odd is in \( A_{2n-1}(L^{11}(9)) \to A_{2n-1}(B\mathbb{Z}/9) \) so \( z \in \text{im}(h) \) if \( n \geq 9 \).

**Remark:**

By the surjectivity of \( \pi_{2n-1}^{s}(B\mathbb{Z}/p^r) \to \pi_{2n-1}^{s}(P_{\infty}C)(p) \) for large \( r \) (6.2) also gives information for \( P_{\infty}C \), which is sometimes better than that of (6.1). Given \( x \in A_{2i-1}(B\mathbb{Z}/p^r) \), the value of \( m \) with \( x \in \text{im}(A_{2i-1}(L^{2m-1}(p^r)) \to A_{2i-1}(B\mathbb{Z}/p^r)) \) can be calculated as in (1.8).

To find elements in the cokernel of \( e: \pi_{2n-1}^{s}(P_{\infty}C) \to A_{2n-1}(P_{\infty}C) \) we can look at spaces \( X \) where the image of \( e \) in \( A_*(X) \) is already known, and then use maps \( f: P_{\infty}C \to X \) to compare the cokernels of \( e \) in both groups.

If \( x \in A_{2n-1}(P_{\infty}C) \) is mapped by \( f_\ast \) onto a nonzero element in \( A_{2n-1}(X)/\text{im}(e) \), then \( x \) cannot come from stable homotopy.

Spaces or spectra for which the image of the \( e \)-invariant is known are \( MU \) and \( BP \). Because \( \pi_{2n-1}(MU) = \pi_{2n-1}^{U} = 0 \) and \( \pi_{2n-1}(BP) = BP_{2n-1}(\ast) = 0 \) we have \( \text{im}(e) = 0 \) but nevertheless \( A_{2n-1}(MU) \neq 0 \).
Example: p=3 We have \( A_9(P_\infty \mathcal{C}) = \mathbb{Z}/3 \) generated by an element \( z_9 \) with

\[
\begin{align*}
z_9 &= \frac{1}{3}(b_3 - b_4 + b_5) = \frac{2}{9}b_1 - \frac{1}{9}b_1 = \frac{1}{9}b_1 + \frac{1}{3}b_2 + \frac{1}{3}b_3 \mod b_1
\end{align*}
\]

Using (5.17) and (5.4) one finds that \( z_9 \) is mapped under

\[
\pi \cdot f^0 : S^2 P_\infty \mathcal{C} \rightarrow BP
\]

to \( \text{ch}^{-1} \left( \frac{2}{9}q_1 + \frac{4}{3}q_1 \right) = \frac{2}{9}t_1 + \frac{1}{3}t_1^2 \in A_{12}(BP; \mathbb{Q}/\mathbb{Z}) \subset K_0(BP; \mathbb{Q}/\mathbb{Z}). \)

The kernel of the Bockstein map is \( q_1^3 = (t_1 + 1/3)^3 = t_1^3 + t_1^2 + t_1^3 \),

so \( \pi \cdot f^0(z_9) = -t_1^3/3. \)

But \( A_9(P_\infty \mathcal{C}) \cong A_1(X_1) \) where \( SP_\infty \mathcal{C}(3) \cong X_1 \cup X_2 \) and the co-fibre spectrum of \( X_1 \rightarrow BP \) is 15-connected. Because \( A_{11}(BP) \cong \pi_1^{sp}(BP) = \mathbb{Z}/3 \) it then follows that \( z_9 \) is in the image of \( e \) (what we already know, because \( z_9 = e(t(x_0)) \)).

Using this method we shall prove:

**Proposition 3.11:** Let \( x \) be an element of order \( p^{p^2-2} \) in

\[
A_{2m-1}(P_\infty \mathcal{C}) = \mathbb{Z}/p^{b+2} \oplus \mathbb{Z}/p \quad (m = (p-1)((p+1)^2-1)-1)
\]

then \( x \) is not in the image of the \( e \)-invariant.

**Proof:** One can use the formulas of §1 to show that

\[
\bar{x} = \pi_*(x(1, p+2)) \quad \text{in} \quad A_{2m}(P_\infty \mathcal{C}; \mathbb{Q}/\mathbb{Z}) \quad \text{has Bockstein} \quad \beta(\bar{x}) = x \quad \text{of order} \quad p^{p+2}
\]

and then compute the coefficients of the highest terms of \( \bar{x} \). Then by subtracting a suitable multiple of \( b_1^m \) one can lower the filtration of \( x \) so that

\[
\bar{x} = \frac{1}{p} b_1^{m-(p-1)^+} \quad \text{terms of lower degree} \quad (6.3)
\]

in \( A_{2m}(P_\infty \mathcal{C}; \mathbb{Q}/\mathbb{Z}). \)

To avoid such calculations one can argue as follows: We look at the Atiyah-Hirzebruch spectral sequence for \( A_*(P_\infty \mathcal{C}) \). The first differen-
tial is the same as the first differential of the spectral sequence for \( \pi_s^*(P_\infty \mathbb{C}) \) and it is well known and easy to see that

\[
d^1 x = \tilde{c}^1(x) \cdot \alpha_1
\]

(5.4)

where \( \tilde{c}^1 \) is dual to the first mod \( p \) Steenrod power. (We write \( \alpha_1 \in \pi_{2p-3}(S^0)(p) \) for the generators of both \( A_{2p-3}(\mathbb{C}) \) and \( \pi_{2p-3}(S^0)(p) \).)

In the case where \( n+1 = o(p) \) we then have that

\[
d^1: H_{2n}(P_\infty \mathbb{C}; A_0) \longrightarrow H_{2n-2}(p-1)(P_\infty \mathbb{C}; A_{2p-3})
\]

is zero and for dimensional reasons the generator of

\[
H_{2n-2}(p-1)(P_\infty \mathbb{C}; A_{2p-3}) \text{ is a permanent cycle.}
\]

This implies that there is an element \( y \) in \( A_{2m-1}(P_\infty \mathbb{C}) \) which has skeleton-filtration \( 2(m-p+1) \). By comparing the spectral sequences for \( A_*(P_\infty \mathbb{C}) \) and \( A_*(P_\infty \mathbb{C}; Q/\mathbb{Z}) \), it follows that in \( A_{2m}(P_\infty \mathbb{C}; Q/\mathbb{Z}) \) there is an element \( y \) with the same filtration and nonzero Bockstein. It is easy to see that \( z = \beta \pi_\infty(x(2,p)) \) is a generator for the second summand, and from §1 we then know

\[
\bar{z} = \pi_\infty(x(2,p)) = \frac{1}{p} (b^{K_{p^2(p-1)-1}} + b^{K_{p^2(p-1)-2}} + \cdots )
\]

(6.5)

Because we know the filtration of \( px = t(y_{p+1}) \) (see §4) and that of \( z \), \( \bar{y} \) must be \( \bar{x} \) and (6.4) is proved.

We now map \( P_\infty \mathbb{C} \) into \( BP \) as follows:

\[
g: P_\infty \mathbb{C} \longrightarrow P_\infty \mathbb{C}/P_{k-1} \mathbb{C} = P_\infty \mathbb{C}^{kH} \to S^{2k} \text{MU} \to S^{2k} \text{BP}
\]

(6.6)

Here \( pr \) is the natural projection, \( \bar{f}^k \) the map induced by the classifying map of \( k \cdot H \) and \( k = m-(p-1)p = (p-1)(p^2+p)-1 \). In \( A_{2p}(p-1)(BP; Q/\mathbb{Z}) \) we can calculate modulo terms of degree \( < (p-1) \).
and so proceed as in the proof of (4.11): We get

\[ g_\ast \left( \frac{1}{p} \cdot b_{m-(p-1)} + \ldots \right) = \text{term of degree } (p-1) \text{ in } \frac{1}{p} \left( 1 - t_1 + t_1 \cdot t_2 + \ldots \right)^k \]

which is

\[ \pm \left( \frac{k}{p-1} \right) \cdot t_1^{p-1} = \pm \frac{1}{p} \cdot t_1^{p-1}. \]

By (5.11), this is not in the kernel of

\[ g: \Lambda_2^p(p-1)(BP; Q/Z) \longrightarrow \Lambda_2^p(p-1)(BP). \]

We can now prove proposition (3.12) of § 3

**Proposition 3.12:** The image of \( e \) in \( A_{2m-1}(P \infty \mathbb{C}) \), where

\[ m = (p-1)((p+1)^2-1)-1, \]

is mapped to zero in \( A_{2m+1}(BP) \) under \( \pi_\ast f \circ \).

**Proof:** By (4.11) we have proved that \( px=t(y_{p+1}) \) goes to zero under \( \pi_\ast f \circ \). But, as observed in the proof of (3.11), the generator of the second summand \( z \) has filtration \( \leq p^2 \) and in \( A_{2m}(BP; Q/Z) \) we are allowed to compute mod terms of degree \( \leq (p^2+p-2) \) so \( \pi_\ast f \circ (z)=0. \)

We now discuss the relation between the elements in coker \( (e) \) and the behaviour of the transfer-maps on Adams filtration. For simplicity we shall restrict to dimensions \( n \) with \( n \leq \dim \beta^p_1 \).

Define \( C_\ast(X) \) to be the cofibre homology theory of the Hurewicz map \( \pi_\ast \longrightarrow A_\ast \), so that for every space we have an exact sequence

\[ C_\ast(X) \xrightarrow{i} \pi_\ast(X)(p) \xrightarrow{h} \tilde{A}_\ast(X) \xrightarrow{\Delta} C_{\ast-1}(X) \quad (6.7) \]

For an odd prime we see \( C_\ast(S^0) = \text{coker } (J) \subset \pi_\ast(S^0)(p) \). The only way I know of to compute \( C_\ast(X) \) is by means of some spectral sequence. If we use the Atiyah-Hirzebruch spectral sequence, then in the dimension range \( n \leq \dim \beta^p_1 \) and for spaces like \( P \infty \mathbb{C} \) this
is easy. One needs only the knowledge of two differentials $d_2(p-1)$ and $d_2(p-1)^2$. The values of $d_r$ for $C_\cdot$ are deduced from those of the $\pi_\cdot^s$-spectral sequence. There we have

**Lemma 6.8:** $d_{2n,k}^2: H_{2n}(P_\infty C; \pi_k^s) \to H_{2n-2}(P_\infty C; \pi_k^s)$ for $r=2(p-1)$ and $2(p-1)^2$ is given by

$$d_{2(p-1)}^2 ([P_n \pi^s_k] \otimes z) = (n-p+1)[P_{n-p+1} \pi^s_k] \otimes \alpha_1$$

$$d_{2(p-1)^2}^2 ([P_n \pi^s_k] \otimes \alpha_1) = (n-(p-1)^2)[P_{n-(p-1)^2} \pi^s_k] \otimes \beta_1$$

**Proof:** This is known, see [44], [9] and can also be deduced by applying the method for finding elements in $\text{coker}(e)$ discussed above.

Let $X = P_\infty C$ or $P_\infty C/P_k C$.

We then see that the cycles which are left by $d_2(p-1)$ are killed by $d_2(p-1)^2$ with the following exceptions

a) the differentials map into negative dimensions or

b) $d_2(p-1)^2$ starts on an element $[P_\cdot \pi^s_k] \otimes z$ where $z$ is indecomposable in the ring $\pi_\cdot^s$, that is $z \in \beta_1, \beta_2, \ldots, \beta_{p-1}$

For $X = P_\infty C$ there remain for dimensional reasons (case a))

$$[P_j \pi^s_k] \otimes z$$

and $[P_j \pi^s_k] \otimes \alpha_1 \cdot z$ with $1 \leq j \leq (p-1)$

and $z \in \{ \beta_1^k \beta_k \mid k \leq p-1 \text{ and } 0 \leq i+k \leq p-1 \}$ (6.9)

For $X = P_\infty C/P_k C$ there are left, by reason b), the cycles

$$[P_{tp+(p-1)} \pi^s_k] \otimes \beta_1$$

(6.10)

The elements left for dimensional reasons in $C_{2n}(P_\infty C)$ map to nonzero elements in $\pi_{2n}(P_\infty C)$:

Clearly $[P_j \pi^s_k] \otimes z$ $(j < p-1)$ represents $\sigma \cdot z \in \pi_\cdot^s(P_\infty C)$, where

$\sigma \in \pi_2(P_\infty C)$ is the generator.

But we know $t(\sigma^{p-2} \cdot z) = \alpha_1 z \neq 0$ in $\pi_\cdot^s(S^0)$

So $\sigma \cdot z \neq 0$ for $i < p-2$.
Thus elements in the image of $\Lambda: A_{2n-1}(P_{\infty}) \to C_{2n-2}(P_{\infty})$ are represented by cycles $[P_{tp+(p-1)}] \otimes \beta_s$ and we have proved

**Lemma 6.11:** Let $2n-1 \leq \dim \beta^P_1$. Then $\text{coker}(e: \pi^s_{2n-1}(P_{\infty}) \to A_{2n-1}(P_{\infty}))$ has order at most $p$.

**Proposition 6.12:** Let $n \leq \dim \beta^P_1/2$ and $x \in A_{2n-1}(P_{\infty})$ be in $\text{coker}(e)$. Then there exists a number $k$ such that the projection of $x$ into $A_{2n-1}(P_{\infty}/P_k)$ is in the image of $e$ and for an element $\bar{x}$ with $e(\bar{x}) = x$ we have $t^{k+1}((\bar{x}) \neq 0$

**Proof:** $x \in A_{2n-1}(P_{\infty})$ is mapped to an element represented by some $[P_{(p-1)+tp}] \otimes \beta_s$ in $C_{2n-2}(P_{\infty})$. Let $k=tp+1$ and $\text{pr}^k: P_{\infty} \to P_{\infty}/P_k$ be the projection. We look at the exact sequences induced by the transfer $t^{k+1}$:

$$
\begin{array}{cccc}
A_{2n-2k-1}(S^0) & \xrightarrow{k} & A_{2n-1}(P_{\infty}/P_{k-1}) & \xrightarrow{j} & A_{2n-1}(P_{\infty}/P_k) & \xrightarrow{t^{k+1}} & A_s(S^0) \\
\downarrow & & \downarrow & & \downarrow & & \\
C_{2n-2k-2}(S^0) & \to & C_{2n-2}(P_{\infty}/P_{k-1}) & \to & C_{2n-2}(P_{\infty}/P_k) & \to & C_s(S^0)
\end{array}
$$

We have $\Delta \cdot \text{pr}^k_s(x) = 0$, so there exists $\bar{x} \in \pi^s_{2n-1}(P_{\infty}/P_k)$ with $e(\bar{x}) = \text{pr}^k_s(x)$. Then $t^{k+1}(\bar{x}) \neq 0$, because in case $t^{k+1}(\bar{x}) = 0$ we should have $\bar{x} = j^!(\bar{x})$ for some $\bar{x}$ with $e(\bar{x}) = \text{pr}^{k-1}_s(x) + k(z)$ contradicting $\Delta \text{pr}^{k-1}_s(x) = 0$. Clearly $t^{k+1}(x) = a \in \beta_s$ with $a \neq 0$. (p).
Proposition 6.13: If \( x \in \pi_{2n-1}^S(P_\infty \mathbb{C}/P_k) \) is an element with 
\[ t^{k+1}(x) = \beta_s \], then \( e(x) : A_{2n-1}^2(P_\infty \mathbb{C}/P_k) \)
has an inverse image in \( A_{2n-1}^2(P_\infty \mathbb{C}) \) which
is in \( \text{coker} (e) \).

Remark:
Because \( A_{2i}(P_\infty \mathbb{C}) \) is free, \( \text{pr}_s^k : A_{2i+1}(P_\infty \mathbb{C}) \rightarrow A_{2i+1}(P_\infty \mathbb{C}/P_k) \)
is onto.

Proof: Because the transfer map \( t^{k+1} : A_{2n-1}(P_\infty \mathbb{C}/P_k) \rightarrow A_{2n-2k-2}^2(S^0) \)
is zero, \( e(x) \) has inverse image \( y \) in \( A_{2n-1}(P_\infty \mathbb{C}/P_k) \).
This cannot be in the image of \( e \) because the existence of an
inverse image \( \bar{y} \) would imply \( t^{k+1}(x) \neq \beta_s \) \((j!)(\bar{y})-x \) is in Adams-
filtration \( F^{1+2}(p-1) \). Then the lift of \( e(x) \) to \( A_{2n-1}(P_\infty \mathbb{C}) \) can
be mapped onto an element in \( \text{cok}(e) \), so it cannot come from stable
homotopy.

So one can describe the cokernel of \( e \) by computing the transfers
\( t^k \) on filtration 1 and vice versa.

Remark:
The elements in \( \text{cok}(e) \) are always connected with values of the
transfer maps, but they need not always correspond to values of \( t^i \)
in Adams filtration 2. So we cannot determine them simply by com-
puting \( t^k \) via \( t^k \) and \( A_{2n+1}(\overline{BP}) \) and the boundary map
\( \beta : \pi_{2m+1}^S(\overline{BP}) \rightarrow \pi_{2m}^S(S^0) \). The appearance of such a phenomenon -
connected with the relation \( a_{BP} = 0 \) in stable homotopy - in
dimension \( 2p^2(p-1)-3 \) is the reason for our dimension limit.

The method of \$3 \$ allows us to compute \( t^k \) from filtration 1 to
filtration 2 and the results of \$5 \$ reduce this to a computation
of a coefficient in a power series, namely the coefficient of
\( t^{sp-1} \) in \( (1-t_1+t_1^{p+1}-t_1^{p^2+p+1}+\ldots)^{k-1} \).
Theorem 6.14: Let \( s \) be in \( \{1, \ldots, p-1\} \). If the element \( \beta_s \) is in the image of \( t^k \), then \( k \equiv 0 \pmod{p} \).

The element \( \beta_s \) is in the image of \( t^{pt} \) if and only if \( t-1 \pmod{p} \) is in the set \( \{s, \ldots, p\} \).

**Proof:** If \( \beta_s = t^k(x) \), then by (6.13) there exists an element \( y \) in \( A_{2n-1}(P_\infty \mathbb{C}) \) with \( \Delta(y) \neq 0 \) in \( C_{2n-2}(P_\infty \mathbb{C}) \). The element \( \Delta(y) \) has a representing cycle of the form \( [P_{tp+p-1} \mathbb{C}] \times \beta_s \). It follows that \( 2n-2 = 2tp+2(p-1) + \dim \beta_s = \dim \beta + 2k-2 \) and thus \( k \equiv 0 \pmod{p} \).

The proof of the second statement is by the same method as in § 4.

For the existence of the inverse image of \( \beta_s \) one uses the Atiyah-Hirzebruch spectral sequence.

From (6.14) we deduce:

**Corollary 6.15:**

a) Let \( n = s(p^2-1)+tp \) with \( s \in \{1, \ldots, p-1\} \) and \( t \geq 0 \). If \( t \pmod{p} \) is in \( \{s, \ldots, p\} \) then the cokernel of \( e: \pi^s_{2n-1}(P_\infty \mathbb{C}^+) \to A_{2n-1}(P_\infty \mathbb{C}) \) is at least \( \mathbb{Z}/p \).

b) If \( n < (p-1)p^2-p \) then a) describes \( \text{coker}(e) \) completely, that is: if \( n = s(p^2-1)+tp \) with \( s \in \{1, \ldots, p-1\} \) and \( t \geq 0 \), then \( \text{coker}(e) \) is \( \mathbb{Z}/p \) iff \( t \pmod{p} \) is in \( \{s, \ldots, p\} \) and zero otherwise.

Let \( z_i \) be the generator of \( \pi^s_m(P_\infty \mathbb{C})(p) \) with \( m=2p-3+2ip \) and \( 1 \leq i \leq p-2 \) (these are the first torsion elements occurring in the individual components of the splitting of \( SP_\infty \mathbb{C}(p) \) into \( p-1 \) pieces; they can be constructed from \( x_0 \) using the transfer). The cycles \( a \beta_s \) in \( C_{2n-1}(P_\infty \mathbb{C}) \), listed under (6.9), map to nonzero elements in \( \pi^s_{2n-1}(P_\infty \mathbb{C}) \) because \( \pi^s_{2n-2}(P_\infty \mathbb{C}) \to A_{2n-2}(P_\infty \mathbb{C}) \) is onto. Clearly they
represent the elements $\beta_1^j \beta_s^i \cdot z_i$.

Putting the discussion above together gives us a computation of $\pi^S_{i}^{*}(P_{\infty} C)(p)$ for $i \leq \dim \beta^p_1$:

**Corollary 6.16:** Let $p$ be an odd prime and $n \leq \dim \beta^p_1 = 2(p-1)p^2-p$. Then

$$\text{tor} \pi^S_{n}(P_{\infty} C)(p) = \ker(e) \oplus \text{im}(e)$$

where $\text{im}(e)$ is described by (6.15) b). The subgroup $\ker(e)$ is generated by the elements $z_i^i \beta_s^k \beta_s^i (1 \leq i \leq p-2, 0 \leq s \leq p-1, 0 < k+s \leq p-1)$ and elements of order $p$ in dimension $2n-1$ with $n = s(p^2-1)+tp$ ($t > 0$, $s \in \{2, \ldots, p-1\}$ and $t \mod p \in \{1, \ldots, s-1\}$ represented by the cycles $[P_{tp+p-1} C] \otimes \beta_s^i$.

It is of course possible to press the calculations further, but then the results become more complicated to state than to derive.

Almost all elements in $\pi^S_{*}(P_{\infty} C)$ in the dimension range considered in (6.16) can be described by framed manifolds. For elements in $\ker(e)$ this is already in (6.16). It remains to describe the elements in $\text{im}(e)$.

One only needs the three elements $\sigma, x_0$ and $x_1$ and their Pontrjagin products to describe all elements of $\text{im}(e)$ via the transfer $t: \pi^S_{2n}(BT^n) \rightarrow \pi^S_{2n+1}(P_{\infty} C)$ with the following exceptions:

We look at each component of $SP_{\infty} C(p)$ separately. As long as $A_{*}(X_i)$ is cyclic we have $\text{im}(e) = \text{im}(t)$. The generating element in the second summand of $A_{2m}(X_i)$ if it appears for the
first time is not obtained in this way. Then for \( n = s(p^2 - 1) + tp \),
\( s > 2, t > 0 \) and \( t \mod p \in \{2, \ldots, s - 1\} \) a subgroup of order \( p \) is not in \( \text{im}(t) \). The proof is a simple application of (1.14) and (4.4) and will be omitted.

Example: \( p = 5 \).

\[
\text{A}_{113}(\mathbb{P}_\infty \mathbb{C}) = \mathbb{Z}/25 \oplus \mathbb{Z}/5 \quad \text{generated by} \quad t(x_1^4(1 \times \sigma^2)) \quad \text{and} \quad t(x_1^4 x_0^4(1 \times \sigma^2)) \quad (e \text{ is onto}).
\]

It is also possible to describe some of the elements in \( \text{im}(e) \) by \( S^1 \)-bundles over homogeneous spaces using (4.1).

Using (6.15) and arguments as above one can also compute

\[
\pi^S_i(\mathbb{P}_\infty \mathbb{C}/\mathbb{P}_{k-1} \mathbb{C})
\]

for \( i \leq 2((p-1)p^2-p+k) \). The result is similar to (6.16) but slightly more complicated to state. The computation of \( \pi^S_i(\mathbb{P}_\infty \mathbb{C}/\mathbb{P}_{k-1} \mathbb{C}) \) might be more interesting than (6.16) because these groups are closely related to metastable homotopy groups of \( U(n) \), see for example [17] (the dimension range considered there is \( i < 2p^2 - 1 + 2k \)) and §7.
Computation of $h: \pi_{2n-1}^s(\mathbb{Z}/p^2) \to K_1(\mathbb{Z}/p^2)$ for $n \neq 0(p)$

Let $p$ be an odd prime. From (1.6) we know

$$Ad_{2n-1}(\mathbb{Z}/p^2) = \mathbb{Z}/p^{2+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)}$$

so for $n \neq 0(p)$ we have $\text{im}(h) \subset \mathbb{Z}/p^2 \oplus \mathbb{Z}/p$. The generator of the first summand is always in the image of $h$ because this element is the image of $\sigma^{n-1} \in \pi_{2n-2}^s(\mathcal{O}_{\infty}\mathcal{C})$ under the transfer map [21]. The second summand is generated by

$$(p^p + 2n_1p^2 + \ldots + (p-1)p(p-1)p)/p$$

Separating the congruence classes of $n$ mod $(p-1)$ we get $(p-1)$ different classes $z_s$ $(1 \leq s \leq p-1)$ in $K_1(\mathbb{Z}/p^2)$. By (1.8) $z_s$ is in $A_{2n-1}(\mathbb{Z}/p^2)$ iff $\pi_*(z_s)$ is in $A_{2n-1}(\mathcal{O}_{\infty}\mathcal{C})$. Using (1.7) we find that the lowest value of $n$ such that $z_s \in A_{2n-1}(\mathcal{O}_{\infty}\mathcal{C})$ is $n = (p-1)+sp$. So for $n \neq 0(p)$, $n \equiv s \pmod{(p-1)}$ and $n \equiv (p-1)+sp$ we have $A_{2n-1}(\mathbb{Z}/p^2) = \mathbb{Z}/p^2 \oplus \mathbb{Z}/p$. We first show that $z_s$ is in the image of $h$. For $\pi_*(z_s)$ this is true by (6.15). By [25] we have $J(\mathcal{H}p^2) = 0$ in $J(\mathcal{P}_m\mathcal{C})(p)$ as long as $m < (p-1)p^2$. So $\mathcal{H}p^2 \to \mathcal{P}_{(p-1)p^2-1}\mathcal{C}$ is orientable for stable homotopy and we get an exact Gysin-sequence:

$$\pi_{2n-1}(\mathbb{Z}/p^2^+) \overset{\pi_*}{\longrightarrow} \pi_{2n-1}(\mathcal{O}_{\infty}\mathcal{C}^+) \overset{\cap e}{\longrightarrow} \pi_{2n-3}(\mathcal{O}_{\infty}\mathcal{C}^+)/(F) \longrightarrow \ldots$$

as long as $n < (p-1)p^2$.

For $n = (p-1)+sp$, $s=1, \ldots, p-2$, the map $h: \pi_{2n-3}(\mathcal{O}_{\infty}\mathcal{C}^+) \to A_{2n-3}(\mathcal{O}_{\infty}\mathcal{C})$ is injective, so $h^{-1}(z_s)$ must come from $\pi_{2n-1}(\mathbb{Z}/p^2^+)$ because $\pi_*(z_s)$ comes from $A_{2n-1}(\mathbb{Z}/p^2)$. For $s=p-1$ we have $\alpha_1 \beta_1 \gamma_1 \in \pi_{2n-3}(\mathcal{O}_{\infty}\mathcal{C}^+)$.
coming from the base point and we must check that $h^{-1}(\pi_1 \mathbb{Z}_p) \cdot e$ is not a multiple of $\alpha_1 \beta_1 \gamma$. We compose $t : \pi^S_{n}(\mathbb{C}_\infty^+) \to \pi^S_{n}(\mathbb{BZ}/p^2^+)$ with the transfer $t_2 : \pi^S_{n}(\mathbb{BZ}/p^2^+) \to \pi^S_{n}(\mathbb{BZ}/p^2^+)$ and by an easy spectral sequence argument we see $t_2 t(\alpha_1 \beta_1 \gamma) \neq 0$, thus $\alpha_1 \beta_1 \gamma$ cannot be in the image of $-\cdot e$.

So for $s=1, \ldots, p-1$ we have found an element $z_s = \pi^S_{2n-1}(\mathbb{BZ}/p^2^+)$ ($n=sp+p-1$) with $h(z_s) = z_s$ in $K_1(\mathbb{BZ}/p^2^+)$. Because $z_s$ is of order $p$ the Toda bracket $<z_s, p, \alpha_1>$ is defined. We have $h(<z_s, p, \alpha_1>) = h(z_s)$.

Proceeding in this way we come into the range where we can apply (6.1). Thus we have proved:

**Theorem 6.18:** Let $n \neq 0 (p)$. Then the image of the Hurewicz map

$$h : \pi^S_{2n-1}(\mathbb{BZ}/p^2^+) \to K_1(\mathbb{BZ}/p^2^+)$$

is $\mathbb{Z}/p^2^+ \oplus \mathbb{Z}/p$ if $n=t(p-1)+s$ with $s \in \{1, \ldots, p-1\}$ and $t > s$ and $\mathbb{Z}/p^2^+$ if $t < s$.

**Remark:**

Using (6.17) and the computation of $\pi^S_{n}(\mathbb{C}_\infty^+)(p)$ for $n \leq \dim \beta_1^P$ it is possible to work out $\pi^S_{n}(\mathbb{BZ}/p^k)$ for $k \geq 2$ in the same range. The case $\pi^S_{n}(\mathbb{BZ}/p)$ can be handled by using instead of (6.17) the cofibre sequence of the transfer map $S_1, \mathbb{C}_\infty^+ \to \mathbb{BZ}/p^+$. 
§ 7 Some James numbers.

In this chapter we are concerned with the computation of some James numbers $\tilde{U}(n,k)$, which describe the minimal degree of a composition $\pi \circ f$, where $\pi: W_{n,k} \to S^{2n-1}$ is the projection of the Stiefel manifold to the sphere and $f: S^{2n-1} \to W_{n,k}$ a map. We consider stable James numbers $U(n,k)$ which describe the corresponding stable problem for stunted projective spaces. The numbers $U(n,k)$ are the same as the $\tilde{U}(n,k)$ in the stable range. There is a conjecture that the numbers $U(n,k)$ are determined by K-theory, that is $U(n,k)(p) = U_A(n,k)$ where $U_A(n,k)$ is defined in the same way as $U(n,k)$ using algebraic K-theory $A_\infty$ instead of stable homotopy. Using properties of transfer maps, we verify this conjecture for $n=1,2,k-2,k-1,k+2$ and in the case where $n$ is divisible by a large power of $p$. By periodicity this determines some of the numbers $U(n,k)$.

The numbers $U(n,k)$ are closely related to the values of the transfer maps $t^k$ on $F^0_\pi \Omega^s(P_\infty \mathcal{H})$. This allows us to show that the elements $u_r$ in the 2-component of $\pi^s(S^0)$ are in the image of $t^s$ for $s=2$ and $s=-2$.

This chapter should be viewed only as a first attempt at the problem of computing $U(n,k)$. This study is continued in [12], where it is shown that $U(n,k)$ for $\alpha n \leq k$ is determined by K-theory.
Let $W_{n,k}$ denote the complex Stiefel manifold $U(n)/U(n-k)$. We then have a fibration

$$W_{n-1,k-1} \rightarrow W_{n,k} \xrightarrow{p} W_{n,1} = S^{2n-1}$$

(7.1)

inducing an exact sequence in homotopy

$$\pi_{2n-1}(W_{n-1,k-1}) \rightarrow \pi_{2n-1}(W_{n,k}) \rightarrow \pi_{2n-1}(S^{2n-1})$$

The James number $\bar{U}(n,k)$ is defined to be the index of

$$\pi_{2n-1}(W_{n,k}) \text{ in } \pi_{2n-1}(S^{2n-1}) = \mathbb{Z}.$$ 

The numbers $\bar{U}(n,k)$ have the following properties (see [18] or [19] where also a table of $\bar{U}(n,3)$ is given):

1. $\bar{U}(n,k)$ is a multiple of $\bar{U}(n,m)$ if $m \leq k$

(7.2)

2. $\bar{U}(n,k) \cdot \bar{U}(m,k)$ is a multiple of $\bar{U}(n+m,k)$

3. If $\bar{U}(n,k) = 1$ and $m > 2k - 1$ then $\bar{U}(m,k) = \bar{U}(n+m,k)$

4. $\bar{U}(n,k) = 1$ iff $n$ is a multiple of $M_k$

where $M_k$ is the order of $J(H) \text{ in } J(P_{k-1}\mathbb{C})$

Because $p_*: H_{2n-1}(W_{n,k}) \rightarrow H_{2n-1}(S^{2n-1})$ is onto, the commutative diagram

$$\begin{array}{ccc}
\pi_{2n-1}(W_{n,k}) & \xrightarrow{p_*} & \pi_{2n-1}(S^{2n-1}) \\
\downarrow & & \\
H_{2n}(W_{n,k}) & \xrightarrow{p_*} & H_{2n-1}(S^{2n-1})
\end{array}$$

(7.3)

allows us to interpret $\bar{U}(n,k)$ as a property of the Hurewicz map. The numbers $\bar{U}(n,k)$ are related to several interesting geometric problems.
Using the reflection map \( w: \text{SP}_{n-1} \rightarrow U(n) \), in the stable range the fibre sequence (7.1) can be compared with the cofibre sequence of stunted projective spaces

\[
P_{n-2}C/P_{n-k-1}C \rightarrow P_{n-1}C/P_{n-k-1}C \rightarrow S^{2n-2} \quad (7.4)
\]

We therefore define \( U(n,k) \) to be the index of

\[
P_* \pi^{s2n-2}_n(P_{n-1}C/P_{n-k-1}C) \text{ in } \pi^{s2n-2}_n(S^{2n-2}) .
\]

This number is the same as the order of the cokernel of \( h(\pi^{s2n-2}_n(P_{n-1}C/P_{n-k-1}C)) \) in \( H_{2n-2}(P_{n-1}C/P_{n-k-1}C) \). Because the pair \((W_{n,k}, SP_{n-1}C/P_{n-k-1}C)\) is \(4(n-k)+3\) connected (see e.g. [18]) we have

\[
\bar{U}(n,k) = U(n,k) \quad \text{for } n \geq 2k-1 \quad (7.5)
\]

\[
\pi_i(W_{n,k}) \cong \pi_{i-1}(P_{n-1}C/P_{n-k-1}C) \quad \text{for } i \leq 4(n-k) \quad (7.6)
\]

In any case, there is a map \( \Theta \) (see [10], \( \lambda \) in [7])

\[
\Theta: \pi_i(W_{n,k}) \rightarrow \pi_{i-1}(P_{n-1}C/P_{n-k-1}C) \quad (7.7)
\]

so \( \bar{U}(n,k) \) is always a multiple of \( U(n,k) \).

The map \( \Theta \) factorizes through the stable homotopy groups of \( W_{n,k} \) so we get a commutative diagram

\[
\begin{array}{ccc}
\pi^{s2n-1}_n(W_{n,k}) & \xrightarrow{P_*} & \pi^{s2n-1}_n(S^{2n-1}) \\
\downarrow \Theta' & & \downarrow \Theta \\
\pi^{s2n-2}_n(P_{n-1}C/P_{n-k-1}C) & \xrightarrow{P_*} & \pi^{s2n-2}_n(S^{2n-2})
\end{array}
\]

(7.8)

Thus we can interpret \( U(n,k) \) as a sort of stable James number.
Remark:

If we work at a fixed prime $p$ then theorem 3.2 of [19] allows us to extend the range where (7.6) is true to $i \leq 2p(n-k+1)-4$ and

$$U(n,k)(p) = U(n,k)(p) \quad \text{for } n \geq k-1+k/(p-1) \quad (7.9)$$

The numbers $U(n,k)$ depend only on the $S$-type of $P_{n-1}/P_{n-k-1}$ so we can use periodicity of stunted projective spaces to define $U(n,k)$ for all $n \in \mathbb{Z}$ and $k \geq 0$.

The numbers $U(n,k)$ have several other interpretations; an important one is the following: By $S$-duality $U(n,k)$ is the order of the cokernel of

$$h: \pi_s^0(P_{k-1}C^{-nH}) \rightarrow \tilde{H}^0(P_{k-1}C^{-nH};\mathbb{Z}) \quad (7.10)$$

Let $i: S^0 \rightarrow P_{k-1}C^{-nH}$ denote the inclusion of the fibre, then the commutative diagram

$$
\begin{array}{ccc}
\pi_s^0(P_{k-1}C^{-nH}) & \xrightarrow{i^*} & \pi_s^0(S^0) \\
\downarrow h & & \\
\tilde{H}^0(P_{k-1}C^{-nH}) & \rightarrow & H^0(S^0)
\end{array}
$$

shows, that $U(n,k)$ measures the extent to which $-nH$ fails to be orientable for stable homotopy. More generally, for an orientable vector bundle $\xi$ over a connected space $X$, we define the codegree of $\xi$ $\text{cd}(\xi)$ to be the index of $h_{\pi_s^0(X,\xi)}$ in $H^0(X,\xi;\mathbb{Z}) = \mathbb{Z}$.

Thus $U(n,k) = \text{cd}(-nH_{k-1})$ where $H_{k-1}$ denotes the Hopf bundle over $P_{k-1}C$.

It is well known that $\text{cd}(\xi) = 1 \Leftrightarrow J(\xi) = 0$, even
p-locally $\nu_p(\text{cd}(\xi)) = 0 \Rightarrow \nu_p|J(\xi)| = 0$.

So $\text{cd}(\xi)$ and $|J(\xi)|$ contain the same primes, but in general with different exponents. Whereas $|J(-\xi)| = |J(\xi)|$ we have in general $\text{cd}(\xi) \neq \text{cd}(-\xi)$.

The interpretation of $U(n,k)$ as a codegree allows simple proofs for the properties of the numbers $U(n,k)$ which we collect in the following proposition:

**Proposition 7.11**: a) 1. $U(n,k)$ is a multiple of $U(s,m)$ if $m | k$

2. $U(n,k) \cdot U(m,k)$ is a multiple of $U(n+m,k)$

3. $U(n,k) = U(n+r \cdot M_k,k)$

4. $U(n,k) = 1$ iff $n$ is a multiple of $M_k = |J(H_{k-1})|$

b) The statements under a) are true p-locally, that is one considers only the powers of a fixed prime in $U(i,j)$. In 3. and 4. we can use instead of $M_k$ the p-primary part of $M_k$.

**Proof**:

1. Follows by diagram chasing using the map $P_{m-1}C^{-nH} \to P_{k-1}C^{-nH}$.

2. $U(m,k)$ times the Thom class $U$ of $-mH_{k-1}$ comes from stable cohomotopy, say $h(y) = U(m,k)U$. Then multiplication by $y$ defines a map $\Phi_1$ in the commutative diagram

$$
\begin{array}{ccc}
\pi_0(P_{k-1}C^{-nH}) & \xrightarrow{\phi_1} & \pi_0(P_{k-1}C^{-(n+m)H}) \\
\downarrow \text{cd}(-nH) & & \downarrow \text{cd}(-(n+m)H) \\
\tilde{H}^0(P_{k-1}C^{-nH}) & \xrightarrow{\text{cd}(-mH)} & \tilde{H}^0(P_{k-1}C^{-(n+m)H})
\end{array}
$$

which proves 2.
3. Because $r^* M_k \cdot H_{k-1}$ is $J$ trivial and so $\pi^s_{-}$-orientable we have a Thom isomorphism

$$
\pi^0_s(P_{k-1}C^{-nH}) \xrightarrow{\sim} \pi^0_s(P_{k-1}C^{-(n+r \cdot M_k)} \cdot H)
$$

this proves 3.

4. follows from the equivalence of the notions of $\pi^s_{-}$-orientability and $J$-triviality for a vector bundle

The $p$-primary versions of the proofs of 1. - 4. prove b)

The $p$-localization of $SP_n \mathbb{C}/P_{n-k} \mathbb{C}$ splits up into a wedge of $(p-1)$ spaces $X_i$, where each $X_i$ has cells only in dimensions differing by $2(p-1)$ (for a proof see [30]). This immediately implies

**Proposition 7.12:** $U(n,k)(p) = U(n,k+1)(p)$ if $k \neq 0(p-1)$

or what is the same $U(n,1+r+t(p-1))(p)$ is constant and equals $U(n,1+t(p-1))(p)$ as long as $0 \leq r < p-1$.

**Remark:**

By the splitting of $W_{n,k}$ [30] the same is true for the James numbers $\tilde{U}(n,k)(p)$.

The number $U(n,k)$ is the index of $h \pi_{2n-2}^s(P_{n-1} \mathbb{C}/P_{n-1-k} \mathbb{C})$ in $H_{2n-2}(P_{n-j} \mathbb{C}/P_{n-1-k} \mathbb{C})$. We define a number $U_A(n,k)$ by using $A_{2n-2}(P_{n-1} \mathbb{C}/P_{n-1-k} \mathbb{C})$ instead of $\pi^s_{-}(P_{n-1} \mathbb{C}/P_{n-1-k} \mathbb{C})$. Clearly $U(n,k)(p)$ is a multiple of $U_A(n,k)$. It is a conjecture that $U(n,k)(p)$ is actually the same as $U_A(n,k)$ (using the right theory for $p=2$). This can also be stated in the form that $K$-theory detects the attaching maps for $P_n \mathbb{C}/P_k \mathbb{C}$ (stably).
Equivalent to the conjecture \( U(n,k)(p) = U_A(n,k) \) are the statements that the K-theory Hurewicz map

\[
h_k: \pi_{2n-2}(P_{n-1}/P_{n-1-k}) \to K_0(P_{n-1}/P_{n-1-k})(p)
\]

maps \( \pi_{2n-2}(P_{n-1}/P_{n-1-k}) \) onto a direct summand, or that

\[
h_A: \pi_{2n-2}(P_{n-1}/P_{n-1-k}) \to A_{2n-2}(P_{n-1}/P_{n-1-k})
\]
is onto.

By looking at the diagram of exact sequences

\[
\begin{align*}
\pi_{2n-2}(P_{n-1}/P_{n-1-k}) & \to \pi_{2n-2}(S^{2n-2}) & \to & \pi_{2n-3}(P_{n-2}/P_{n-1-k}) \\
\downarrow h_A & & \downarrow \partial_\pi & \downarrow e \\
A_{2n-2}(P_{n-1}/P_{n-1-k}) & \to A_{2n-2}(S^{2n-2}) & \to & A_{2n-3}(P_{n-2}/P_{n-1-k})
\end{align*}
\]
we see the connection between the statements "\( h_A \) is onto" and

\[
|\partial_\pi(1)| = |e\partial_\pi(1)|.
\]

We have \( U_A(n,k) = |\partial_A(1)| \) and either \( \text{coker} \ h_A: A_{2n}(P_n/C/P_k) \to H_{2n}(P_n/C/P_k) \) or \( \partial_A(1) \) can be calculated using K-theory (see below). These numbers express the K-theory restrictions for the existence of a stable \( q \)-section \( f \) of \( P_n/C/P_k \to S^{2n} \), that is for an S-map \( f: S^{2n} \to P_n/C/P_k \) with \( \text{mf} \) of degree \( q \). These restrictions have been found and considered by several authors, e.g. [44], [27], [16].

\( U_A(n,k) \) can be calculated as follows \((p+2)\): First we construct a generator of \( A_{2n}(P_n/C/P_k) \subset K_0(P_n/C/P_k) \). The element \( b^n_{11} \in K_0(P_n/C/P_k) \) lies in \( \text{ker}(c^k-k^n) \), therefore \( \text{pr}_x(b^n_{11}) \subset K_0(P_{n+1}/P_k) \) too. But
\[ \text{pr}_r(b^n_j) = \sum_{j=k+1}^{n} \frac{n}{j} S(n,j) j! \cdot b_j \] (7.13)

can be divided by the greatest common divisor \( r \) of the numbers \( S(n,j) j! \), so \( \text{pr}_r(b^n_j)/r \) generates \( A_{2n}(P_n \mathcal{C}/P_k \mathcal{C}) \). Because \( h_A(b^n_j) = n! [P_n \mathcal{C}] \) we find \( \nu_p(n!/a) = \nu_p(U_A(n+1,n-k)) \) or

\[ 2_A(1) = \frac{1}{n!} \sum_{j=k+1}^{n-1} S(n,j) j! \cdot b_j \] (7.14)

in \( A_{2n-1}(P_{n-1} \mathcal{C}/P_k \mathcal{C}) \cong A_{2n}(P_{n-1} \mathcal{C}/P_k \mathcal{C}; \mathbb{Q}/\mathbb{Z}) \subset K_0(P_{n-1} \mathcal{C}/P_k \mathcal{C}; \mathbb{Q}/\mathbb{Z}) \).

It is known that

\[ \sum_{n=0}^{k} \binom{k}{p^{-1}} S(n+k,n) n! = \frac{1}{(n+k)!} \in \mathbb{Z}(p) \] (7.15)

therefore we have

**Proposition 7.16:** \( \nu_p U_A(n,k) \notin \left[ \frac{k-1}{p-1} \right] \) for an odd prime \( p \).

The interpretation of \( U(n,k) \) as a codegree leads to a second possibility of calculating \( U_A(n,k) \):

Let \( \xi \) be a complex vector bundle on a connected finite CW-complex \( X \). we denote by \( U_K \) and \( U_H \) the Thom classes of \( \xi \) in \( K \)-theory or homology, then it is well known that

\[ \text{ch} U_K = T(\xi) \cup U_H \]

where \( T(L) = (e \cdot c_1(L)-1) / c_1(L) \) for a line bundle \( L \). So \( \text{ch}^{-1} U_H = (\text{ch}^{-1} T^{-1}(\xi)) \cup U_K \) in \( K^0(X;\mathbb{Q}) \). If \( m \cdot U_H \) lies in the image of the Hurewicz map \( h \) (or \( h_A \)), then because \( h \) factors over \( K \)-theory via \( \text{ch} \), the class \( m \cdot U_H \) must be in \( \text{ch}(K^0(X)) \). This means the class \( m \cdot \text{ch}^{-1} T^{-1}(\xi) \) must be integral, that is in the image of \( K^0(X) \rightarrow K^0(X;\mathbb{Q}) \).
For a line bundle \( L \) we have
\[
\text{ch}^{-1}T^{-1}(L) = \text{ch}^{-1}(c_1(L)/(e^{c_1(L)}-1)) = \ln(x+1)/x \quad \text{where} \quad x = L-1.
\]
So for \( x = p_{n-1} \xi \) and \( \xi = kH \):
\[
\text{ch}^{-1}T^{-1}(kH_{n-1}) = \left( \frac{\ln(x+1)}{x} \right)^k / x^n = 0
\]
and the denominators of this polynomial must divide
\[
\text{cd}(kH_{n-1}) = U(-k,n), \quad \text{a well known condition on} \quad U(-k,n).
\]
For \( p > 2 \), \( \tilde{A}(P_{n-1} \xi^{kH}) \) is torsion free and so a generator \( z \) of this group gives in \( K^O(P_{n-1} \xi^{kH}) \) an integral class \( \tilde{z} \) which is mapped by the chern character onto \( \pm U_A(-k,n) \cdot U_H \). This implies that the maximal denominator in (7.17) is exactly \( U_A(-k,n) \).

Using this it is easy to prove

**Proposition 7.18:** The \( e_G \)-invariant of the stable map
\[
\tau^r \in \pi^1_s(P_{n-1} \xi^{rH}) \text{ inducing the transfer } t^r
\]
given by
\[
e_G(\tau^r) = \left( \frac{\ln(x+1)}{x} \right)^{r-1} / x^n = 0 \cup \mathcal{U}_K
\]
in \( K^O(P_{n-1} \xi; Q/2) \).

A proof can also be deduced by the method of [24].

This formula also allows to compute the transfer map
\[
\tau^r : \pi^s_{2n}(X^+, P \xi^{rH}) \rightarrow \pi^s_{2n+1}(X^+)
\]
from Adams filtration 0 to 1 by generalizing (4.3) in the obvious way.

The trick used in [47] allows us to find out for which values of \( n \) and \( k \) the numbers \( U_A(n,k) \) take their maximal value: Let \( \xi \) be a \((p-1)\)-th root of \( p \), that is \( \xi^{p-1} = p \), then the denominators of
\[
\ln(x \cdot \xi + 1)/x \cdot \xi = 1 - \frac{x \xi}{2} + \ldots + x^{p-1} - p\xi \frac{x^p}{p+1} + \ldots
\]
do not contain \(p\). Therefore the same is true for \((\ln(x \xi + 1)/x \xi)^n\) for all \(n \in \mathbb{Z}\). If we write

\[
\left(\frac{\ln(x+1)}{x}\right)^n = \sum_t a(n+t,n)x^t \quad (n \geq 0)
\]

\[
\left(\frac{\ln(x+1)}{x}\right)^n = \sum_t b(n+t,n)x^t
\]

we have proved \(p \left[ \frac{t}{p-1} \right] \cdot a(n+t,n) \in \mathbb{Z}(p)\) and \(p \left[ \frac{t}{p-1} \right] \cdot b(n+t,n) \in \mathbb{Z}(p)\). This proves (7.16).

The series \(\ln(x \xi + 1)/x \xi\) reduced mod \(p\) is simply the polynomial of degree \(p-1\):

\[
\ln(x \xi + 1)/x \xi = 1 - \frac{x \xi}{2} + \frac{(x \xi)^2}{3} - + x^{p-1}
\]
The binomial theorem then gives the mod \(p\)-value of \(p^k \cdot a(n+k(p-1),n)\) and \(p^k \cdot b(n+k(p-1),n)\):

\[
p^k \cdot a(n+k(p-1),n) \equiv \binom{n}{k} \mod p \quad (7.19)
\]

\[
(-1)^k \cdot p^k \cdot b(n+k(p-1),n) \equiv \binom{n-1+k}{k} \mod p
\]

From this it follows:

**Proposition 7.20:** \(n \geq 0\)

\[
U_A(n, k(p-1)+1) = p^k \iff \binom{k+n-1}{k} \neq 0(p)
\]

\[
U_A(-n, k(p-1)+1) = p^k \iff \binom{n}{k} \neq 0(p)
\]
Remarks

1. Similarly one can derive formulas for the mod $p^2$ value of $a(n+t,n)$ and so find conditions for $U_A(n,k(p-1)+1)=p^{k-1}$.

2. Another possibility to compute $U_A(n,r)$ in some cases is to compute the mod $p$ value of $S(n,k)$ (see (7.14)). If $S(m,m-t(p-1)) \not\equiv 0(p)$ one gets $U_A(m+1,t(p-1)+1) = ml/(m-t(p-1))!$. In some cases it is possible (see below) to deduce a closed formula for $U_A(n,k)$, but in general this seems to be rather hard.

3. The splitting principle of [46] or calculations with the Chern character imply that for every complex vector bundle $\xi$ on a $2n$-dimensional CW-complex $X$ the number $p^{\left[\frac{n}{p-1}\right]}$ is an upper bound for the index of $h_A: \tilde{A}^0(X^\xi) \longrightarrow \tilde{H}^0(X^\xi;\mathbb{Z})$. At present no example seems to be known, where $\text{cd}(\xi)$ is larger than $p^{\left[\frac{n}{p-1}\right]}$ ($p\neq 2$).

4. Let $f$ be an element in $\mathbb{Q}[x]/(x^n)$ with $f(0) = 1$. We call the minimal positive number $m$ with $m \cdot f \in \mathbb{Z}[x]/(x^k)$ the additive order of $f$ and the minimal number $k$ with $f^k \in \mathbb{Z}[x]/(x^n)$ the multiplicative order of $f$. For $f = (\ln(x+1)/x)^r$, the multiplicative order of $f$ is $|J(x^r)|$ and the additive order is $U_A(-r,n+1)$. Some elementary manipulations with the polynomial $f$ show that in case of $f = (\ln(x+1)/x)^r$ the multiplicative order of $f$ is always a multiple of the additive order. It would be interesting to know under what conditions on $X$ $|J(\xi)|$ is always a multiple of $\text{cd}(\xi)$ (see also (7.25)). At present it is only known that both numbers contain the same set of primes.

5. In cases where $U_A(n,k) = U(n,k)(p)$, (7.16) shows that $|J(nH_{k-1})|$ can be much larger than $\text{cd}(-nH_{k-1})$. 
Before we start with the proofs that \( U_A(n,k) = U(n,k)(p) \) for certain values of \( n \), we discuss shortly the special case \( p=2 \).

Let \( A^*(\_\_) \) be the cohomology theory defined by using real K-theory and \( \psi^3 - 1 \) in the same way as \( A^* \) is defined by complex K-theory; see [36]. Then we can define numbers \( U_{AR}(n,k) \) by

\[
U_{AR}(n,k) = \text{index of } h_{AR}: \widetilde{AR}^0(P_{k-1} C^{-n\tilde{H}}) \to \tilde{H}^0(P_{k-1} C^{-n\tilde{H}})
\]

We still have the corresponding numbers \( U_A(n,k) \), defined by complex K-theory for \( p=2 \). The numbers \( U_A(n,k) \) for \( p=2 \) can be computed in the same way as \( U_A(n,k) \) for \( p \neq 2 \).

**Proposition 7.21:** Let \( r \) be odd, then \( U_{AR}(r,m) = U_A(r,m) \) unless \( m \equiv 3(4) \) and \( U_A(r,m+1) = 2U_A(r,m) \); in this case we have \( U_{AR}(r,m) = 2U_A(r,m) = U_{AR}(r,m+1) \).

If \( r \) is even, then \( U_{AR}(r,m) = U_A(r,m) \) unless \( m \equiv 2(4) \) and \( 8U_A(r,m)/U_A(r,m+1) \neq 0(2) \), where we have \( U_{AR}(r,m) = 2U_A(r,m) \).

**Proof:** We use the commutative diagram

\[
\begin{array}{ccc}
\tilde{A}^0(P_{m-1} C^{-r\tilde{H}}) & \xrightarrow{k_R} & \tilde{K}^0(P_{m-1} C^{-r\tilde{H}}) \\
\downarrow c_A & & \downarrow c \\
\tilde{A}^0(P_{m-1} C^{-r\tilde{H}}) & \xrightarrow{k_C} & \tilde{K}^0(P_{m-1} C^{-r\tilde{H}})
\end{array}
\]

where \( k_R, k_C \) are the maps of the defining long exact sequences for \( AR \) and \( A \), and the vertical sequence is part of the Bott sequence relating real and complex K-theory.

The maps \( k_R \) and \( c \) are inclusions of direct summands as long as
\( \tilde{K}^{O}(P_{m-1} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) \) and \( \tilde{K}^{O-6}(P_{m-1} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) \) are free (modulo torsion in \( \tilde{A}^{R}(P_{m-1} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) \)). This is the case except when A) \( r = 1(2) \) and \( m = 3(4) \) or B) \( r = 0(2) \) and \( m = 2(4) \) (for \( \tilde{K}^{O*}(P_{n} \mathbb{C}) \) see [14]). In the other cases \( c_{A}(\tilde{A}^{R}(P_{m-1} \mathbb{C} \mathbb{H}^{-r \mathbb{H}})) \) must be a direct summand in \( \tilde{A}^{O}(P_{m-1} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) \cong \mathbb{Z} \) or what is the same: \( c_{A} \) must be onto.

Let's first treat A):

In the following commutative diagram

\[
\begin{array}{cccccc}
\tilde{A}^{R}(P_{3+4s} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) & \xrightarrow{j_{AR}} & \tilde{A}^{R}(P_{2+4s} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) & \rightarrow & \tilde{A}^{R}s+5 = 0 \\
\downarrow c_{1} & & \downarrow c_{2} & & \\
\tilde{A}^{O}(P_{3+4s} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) & \xrightarrow{i_{A}} & \tilde{A}^{O}(P_{2+4s} \mathbb{C} \mathbb{H}^{-r \mathbb{H}}) & \rightarrow & \tilde{A}s+5 \cong \mathbb{Z}_{2}
\end{array}
\]

\( j_{AR} \) and \( c_{1} \) must be onto. The map \( i_{A} \) is multiplication by \( \bar{a} = U_{A}(r, 4s+4)/U_{A}(r, 4s+3) \). So the index of \( c_{2} \) must be \( \bar{a} \) too. This completes the case \( r \) odd.

In case B) the problem is the torsion group \( \mathbb{Z}_{2} \) in \( \tilde{K}^{O}(P_{4k+1} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) = \tilde{K}^{O}(P_{4k+1} \mathbb{C}) \). Because this subgroup comes from a sphere, it is in the kernel of \( \psi^{3-1} \), but it may happen that \( \mathbb{Z}_{2} \subseteq \text{im}(\psi^{3-1}) \). In this case it could be that \( U_{AR}(2r, 4k+2) = 2U_{A}(2r, 4k+2) \). By \( i \) we denote the elements in \( \tilde{K}^{O}(P_{j} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \) which correspond under the Thom isomorphism to the generators \( (r \mathbb{H}^{-2}) \) of \( \tilde{K}^{O}(P_{j} \mathbb{C}) \). The torsion group of \( \tilde{K}^{O}(P_{4k+1} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \) is then generated by \( a^{2k+1} \). It is easy to see that \( U_{A}(2r, 4k+2) = U_{A}(2r, 4k+1) = U_{AR}(2r, 4k+1) \). Let \( x \) be the generator of the image of \( k_{R} : \tilde{A}^{R}(P_{4k} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \rightarrow \tilde{K}^{O}(P_{4k} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \).

We consider \( x \) also as an element of \( \tilde{K}^{O}(P_{4k+1} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \) by using the same linear combination of \( a^{i} \) as the one defining \( x \) in \( \tilde{K}^{O}(P_{4k} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \).

If \( (\psi^{3-1}) x = a^{2k+1} \) in \( \tilde{K}^{O}(P_{4k+1} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}}) \) (because \( (\psi^{3-1})a^{2k+1} = 0 \) this does not depend on the choice of the lifting of \( x \) from \( P_{4k} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}} \) to \( P_{4k+1} \mathbb{C} \mathbb{H}^{-2r \mathbb{H}} \) then
Let $\bar{x}$ be the generator of $\text{im} \ (k_{\mathbb{R}}: \tilde{A}^0 \to \tilde{K}^0)$ and $s_1 = U_A(2r, 4k+2)$, $s_2 = U_A(2r, 4k+3)$. Then

$$\bar{x} = (s_2/s_1) \cdot x + b \cdot a^{2k+1}$$

and $(\psi^3 - 1) \bar{x} = 0$ in $\tilde{K}^0$. Because $(\psi^3 - 1)a^{2k+1} = (3^{4k+2} - 1)a^{2k+1}$ we have

$$(\psi^3 - 1)x = 8(s_1/s_2)b' \cdot a^{2k+1}$$

If $v_2(s_2) > v_2(s_1)$ then $b$ and $b'$ are not divisible by 2. By restricting to $\tilde{K}^0$ we see that

$$U_{AR}(2r, 4k+2) = 2U_A(2r, 4k+2) \iff 8s_2/s_1 \neq 0(2).$$

**Remark:**

The examples suggest that $8U_A(2r, 4k+3)/U_A(2r, 4k+2) \neq 0(2)$ never happens.

In [44] G. Walker has shown the equation $|\Theta_\pi(1)| = |\Theta_\pi(1)|$ to be true for $p$ odd and $k \xi(p-1)(p^2-p+1)$ (with some restrictions on $n$ because he worked with $W_n$), and so he has verified the conjecture $U(n, k) = U_A(n, k)$ in this range.

We can easily reprove this: one gets nearly the same range of validity for $U_A(n, k) = U(n, k)(p)$ from (4.5) applied to $P_{n-1} \cdot P_{n-1-k}$. Namely as long as $k \xi(p-1)(p^2-p)$ the Hurewicz map
must be onto, so $U_A(n,k) = U(n,k)(p)$ for $k < (p-1)(p^2-p)$.

By calculating the Atiyah-Hirzebruch spectral sequence for $C_\infty(P_\infty/C/P_\infty)$ (see § 6) one can push the bound on $k$ higher; without difficulty one gets the range $k < (p-1)(2p^2-2p+1)$. For $p=2$ the equation $U_{AR}(n,k) = U(n,k)(2)$ is true for $k < 8$ by the results of [31].

Beyond those values of $k$, the equality $U_A(n,k) = U(n,k)(p)$ is known for $n=k$ and $n=k+1$ because

$$\pi^{s}_{2n}(P_\infty^+) \to A_{2n}(P_\infty)$$ and $$\pi^{s}_{2n}(P_\infty^H) \to A_{2n}(P_\infty^H)$$ are onto

( $U(n,n) = (n-1)!$ and $U(n+1,n) = n!$ ). This indicates that it may be simpler to look first at those values of $n$ for which $(n,k)$ is not in the range where $U(n,k) = \tilde{U}(n,k)$ and then deduce results on $U(n,k)$ using periodicity.

We turn back to the interpretation of $U(n,k)$ as a codegree.

**Theorem 7.22**: If $E$ is an $n$-dimensional vector bundle over a twofold suspension $S^2X$, then the codegree of $E$ is determined by $K$-theory, that is

$$h_A: \pi^n_s(M(E^n)) \to A^n(M(E^n))$$ is onto

**Proof**: We write $\tilde{E}$ for $E-n$. It is well known that for bundles over suspension spaces the inclusion of the fibre $S^0 \to M(E)$ is part of the following cofibre sequence

$$\to S^X \xrightarrow{g} S^0 \xrightarrow{i} M(E) \xrightarrow{j} S^2X$$ \hspace{1cm} (7.23)

where $g$ is the image of $E$ under the $J$-homomorphism.

$J: \tilde{K}_0^{-2}(X) \to \pi^{-1}_s(X)$. We get the following commutative diagram
The $J$-homomorphism $J : KO^{-2}(X) \to \pi_{-1}^{s}(X)$ can be factored through a map $j : A^{-1}(X) \to \pi_{-1}^{s}(X)(p)$ and in [43] it is proved that $e_{j}$ is bijective for odd primes. Because $g^{*}(1) = g$ lies in the image of $J$, $e$ must preserve the order of $g^{*}(1)$, that is

$$|g^{*}(1)| = |e g^{*}(1)|$$

But then $h_{A}$ must be onto.

Remark: For $p=2$ [7.22] remains true with $A_{*}$ replaced by the theory $AR_{*}$ ($AR_{*}(\gamma) = \text{im}(J) + \mu_{r}$ and $\eta \mu_{r}$-elements). We omit the proof.

Corollary 7.25: Let $E$ be a vector bundle over a twofold suspension $S^{2}X$, then the Hurewicz map

$$\pi_{s}^{n}(M(E)) \longrightarrow \tilde{h}_{s}^{n}(M(E);\mathbb{Z}_{p})$$

has cokernel of order $|J(E)|(p)$, where $J(E)$ is the class of $E$ in $J(S^{2}X)(p)$.

As a first application of (7.22) we get

Corollary 7.26: Let $r+l \geq m$ and $s \equiv 0 (M_{r+l})$ then

$$U_{A}(s,r+m+1) = U(s,r+m+1)(p) = |J(sH_{r+m})|(p)$$

Proof: We consider the following cofibre sequence

$$P_{r}C \longrightarrow P_{r+m}C \longrightarrow P_{r+m}C/P_{r}C$$

If $r+l \geq m$ then $P_{r+m}C/P_{r}C$ is a double suspension say $P_{r+m}C/P_{r}C = S^{2}X$. Choose $s$ such that $s \cdot H$ is fibre homotopy trivial.
on $P_r \mathcal{C}$, that is $s=0 \ (M_{r+1})_r(p)$. Because the sequence of $J$-groups

$$0 \to J(P_{r+m} \mathcal{C}/P_r \mathcal{C})(p) \to J(P_{r+m} \mathcal{C})(p) \to J(P_r \mathcal{C})(p) \to 0$$

is exact, we can find a vector bundle $\xi$ on $S^2 X$ with

$$J(p^* \xi) = J(sH)$$

We then have the following cofibre sequence of Thom spaces

$$\longrightarrow M(-sH) \xrightarrow{M(p)} M(-\xi) \longrightarrow S^1 \times P_r \mathcal{C} \longrightarrow S^1 \times M(-sH) \longrightarrow$$

and therefore

$$\tilde{\Omega}^0(M(-\xi)) \longrightarrow \pi_s^0(M(-sH)) \longrightarrow \pi_s^0(P_r \mathcal{C})$$

The map $h_A$ is onto by (7.17) and $M(p)_x$ is onto because $\tilde{\Omega}^0(P_r \mathcal{C}) = 0$, so $\pi_s^0(M(-sH)) \to \tilde{\Omega}^0(M(-sH))$ must be onto.

Therefore $U_A(s,r+1+m) = U(s, r+1+m)(p) = cd(-sH_{r+m})(p)$ and by (7.20) this is $|J(sH_{r+m})(p)|$.

With some more work the result carries over to the case $p=2$.

Remarks:
1. (7.26) is a generalization of a result of James [19], where he considers the situation $m=1$.
2. Because the codegree of bundles induced from a twofold suspension is the same as a $J$-order, it changes in the same manner if we look at multiples of such bundles, that is

$$cd(k \cdot f^*(\xi)) = \frac{cd(f^*(\xi))}{(k, cd(f^*(\xi))}$$

(7.27)
where \((a,b)\) denotes the greatest common divisor of \(a\) and \(b\). By looking at examples of \(U(n,m)\) and \(U(r \cdot n, m)\) one sees that not all (in fact very few) multiples of \(H\) are induced from a suspension.

In using (7.22) one does not really need that the bundle in consideration is induced from a suspension, only that the Thom space is \(S\)-equivalent to the Thom space of such a bundle. This is the case in the next two applications:

**Theorem 7.28:** For all \(n\) we have

\[
U(1,n+1) = U_A(1,n+1) = \prod_{p \geq 2} \mathbb{Z}_p^{-\frac{\lfloor \frac{n}{p-1} \rfloor}{a_n}}
\]

where \(a_n = 2\) if \(n \equiv 2 \pmod{4}\) and \(a_n = 1\) if \(n \not\equiv 2 \pmod{4}\).

**Proof:** We look at the cofibre sequence of the \(S^1\)-transfer map \(\tau\) (§2):

\[
S^0 \longrightarrow \mathbb{C}^\mathbb{Z} \longrightarrow S^2 \mathbb{P}_{n-1} \mathbb{C}^+ \longrightarrow S^1 \longrightarrow (7.29)
\]

By (2.5) we know \(\tau = J(\beta(-H))\) where \(\beta : K^0(P_{n-1} \mathbb{C}^+) \to K^0(S^2 P_{n-1} \mathbb{C}^+)\) is Bott periodicity. Therefore we have a second cofibre sequence inducing \(\tau\):

\[
S^0 \longrightarrow M(\beta(-H)) \longrightarrow S^2 \mathbb{P}_{n-1} \mathbb{C}^+ \longrightarrow S^1 \longrightarrow (7.30)
\]

We map (7.29) into (7.30) and get some map \(P_n \mathbb{C}^\mathbb{Z} \to M(\beta(-H))\) which is a stable homotopy equivalence because the induced map in homology is an isomorphism.

So \(U(1,n+1) = |\tau| = |e(\tau)| = U_A(1,n+1)\) where \(\tau \in \pi^{-1}_s(P_{n-1} \mathbb{C}^+)\). The \(e\)-invariant of \(\tau\) is computed in [24] and is given by

\[
e(\tau) = \frac{1}{x} - \frac{1}{\log(x+1)}
\]

in \(K^{-2}(P_n \mathbb{C}; \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{x}]\).
Up to a dimension shift the denominators in $e(\tau)$ are the same as the denominators in $x/\log(x+1)$. Because $\binom{k}{k} \neq 0(p)$ for all $k$ it follows from (7.19) that the coefficient of $x^{t}(p-1)$ in $x/\log(x+1)$ has denominator exactly $p^{t}$. The extra factor for $p=2$ follows from (7.21).

**Theorem 7.32:** For all $n$ we have $U(2,n)(p) = U_{A}(2,n)$.

**Proof:** The stable map $\gamma$ representing the transfer $t^{-1}$ is an element in $\pi_{-1}^{s}(P_{n} \mathcal{C}^{-\text{H}})$. We look at the exact sequence induced by the cofibre sequence of $t^{0}$:

$$
\begin{align*}
\pi_{-1}^{s}(S^{0}) & \xrightarrow{i^{*}} \pi_{-1}^{s}(P_{n} \mathcal{C}^{-\text{H}}) \xrightarrow{j!} \pi_{-3}^{s}(P_{n-1} \mathcal{E}^{+}) \xrightarrow{t^{0}} \pi_{-2}^{s}(S^{0})
\end{align*}
$$

By (2.22) we know that there is an element of $\pi_{-3}^{s}(P_{n-1} \mathcal{E}^{+})$ lying in the image of the $J$-homomorphism which maps to $\gamma$ under $j!$. But the
behaviour of $t^0$ on $\pi^{-2}_s(S^0)$ can be determined using AR'-theory so $|\gamma| = |e(\gamma)|$ even at the prime 2. Then the cofibre sequence of $\gamma$ in cohomotopy shows $\pi^0_s(P_n \mathbb{C}^{-2\check{H}}) \to \check{A}^0(P_n \mathbb{C}^{-2\check{H}})$ to be onto (for $p=2$ we use $\check{A}^0(P_n \mathbb{C}^{-2\check{H}})$).

There is also a close relation between the transfer maps $t^r$ und the numbers $U(n,k)$: The transfer $t^r$ induces a cofibre sequence

$$\pi^s_{2n}(P_\infty \mathbb{C}^{-U(r)}) \xrightarrow{j^!} \pi^s_{2n-2}(P_\infty \mathbb{C}^{-U(r+1)}) \xrightarrow{t^r} \pi^s_{2n-1}(S^0)$$

which is multiplication by the number

$$a = U(n-r,n+1)/U(n-r,n)$$

or what is the same:

$$\frac{U(n-r,n+1)}{U(n-r,n)} = \min \{ |t^{-r}(x)| \mid x \in \pi^s_{2n-2}(P_\infty \mathbb{C}^{-U(r+1)}) \text{ maps to a generator in } \pi^s_{2n-2}(P_\infty \mathbb{C}^{-U(r+1)})/\text{tor} \}$$

If we look at the situation in $K$-theory, we simply have

$$|t^{-r}_A(x)| = U_A(n-r,n+1)/U_A(n-r,n)$$

An application of this is

**Proposition 7.33:** $U(n,n+1) = U_A(n,n+1) = (n-1)!$ denominator of $(B_n)$

**Proof:** We first prove that $t^0(\sigma^n) \in \text{im}(J)$.

The map $S^3 \wedge P_n \mathbb{C} \to S^1 \wedge P_{n+1} \mathbb{C}$ induced by restricting the $H$-space multiplication map to $S^2 \times P_n \mathbb{C}$ and then applying the Hopf construction (see §2) maps $\sigma^n$ to $\sigma^{n+1}$ in $\pi^s_s(P_\infty \mathbb{C}^+)$. The same map is used to construct a generator $u_n$ for $\pi_{2n+1}(U)$, see [43]. This implies
$w_\ast(\sigma^n) = i(u_n)$ in the commutative diagram

\[
\begin{array}{ccccccc}
\pi_{2n+1}(U) & \xrightarrow{i} & \pi_{2n+1}(U) & \xleftarrow{\text{w}_\ast} & \pi_{2n}(P_\infty C^+) \\
J & \downarrow & J' & \xleftarrow{-t^0} & \\
\pi_{2n+1}(S^0) & & & & \\
\end{array}
\]

Therefore $t(\sigma^n) \in \text{im}(J)$.

For later use we remark that the map $\theta': \pi_{2n+1}^s(U) \to \pi_{2n}^s(P_\infty C^+)$ (see (3.2)) has the property $\theta'\circ w_\ast = \text{id}$ (see [10] or [11]); the equation $i(u_n) = w_\ast(\sigma^n)$ implies

\[
\theta' i(u_n) = \sigma^n \quad (7.34)
\]

The proof of (7.33) is finished by observing that, because $t^0(\sigma^n) \in \text{im}(J)$, we have $|e(t^0(\sigma^n))| \sigma^n \in \text{im} j_{\pi}$ and $|e(t^0(\sigma^n))| b^1 \in \text{im} j_{\pi}$, thus $h_A$ must be onto. The maps mentioned are from the diagram

\[
\begin{array}{ccccccc}
\pi_{2n+2}^s(P_\infty C^{-H}) & \xrightarrow{j^!} & \pi_{2n}^s(P_\infty C^+) & \to & \pi_{2n+1}^s(S^0) \\
\downarrow h_A & & \downarrow & & \downarrow e \\
\tilde{A}_{2n+2}^s(P_\infty C^{-H}) & \xrightarrow{j^A} & A_{2n}^s(P_\infty C) & \to & \tilde{A}_{2n+1}^s(S^0) \\
\end{array}
\]

For $p=2$ one has to attend to the base point: $\eta \cdot u_k[k^\wedge]$ as well as $4 \cdot \sigma^{4k+1}$ has image $\eta^2 \cdot u_k$ = element of order 2 in $\text{im}(J)$ under the transfer $t^0$.

**Proposition 7.35:** $U(n-2,n)(p) = U_A(n-2,n)$

\[e.g. \; v_2(U(4k+i-1,4k+i+1)) = \begin{cases} 1+v_2(4k!) & i=0,i=1 \\ 2+v_2(4k!) & i=2,i=3 \end{cases}\]

**Proof:** We first construct elements in $\pi_{2n}^s(P_\infty C^{-H})$ which are mapped
It is well known that one can include the $J$-homomorphism in the following exact sequence

$$
\pi^s_{k+1}(S^0) \xrightarrow{\Omega^a_{k+1}} \pi^a_{k+1} \xrightarrow{\Delta} \pi_k(SO) \xrightarrow{J} \pi^s_k(S^0) \xrightarrow{\text{(7.36)}}
$$

where $\Omega^a_{k+1}$ is the bordism group of closed smooth almost parallelizable manifolds.

In [13] it is shown that one can map this sequence to the cofibera sequence of the bistable $J$-homomorphism to get a commutative diagram, of which we state the complex analogue:

$$
\pi^s_{k+1}(S^0) \xrightarrow{U_{\text{al}}} \pi^a_{k+1} \xrightarrow{\Delta} \pi_k(U) \xrightarrow{J} \pi^s_k(S^0) \xrightarrow{\text{(7.37)}}
$$

Here $\theta'$ is the map of (3.2) and $\bar{o}$ is induced by $\theta'$ and $\text{id}_{SO}$. From this diagram, (7.34) and (2.22) it follows that the set $\tilde{G}_{SO}(U^a_{\text{al}})$ is mapped into $\text{im}(J)$ by $t^{-1}$. The rest of the proof is then as in (7.33). From the known value of $|t^0(\sigma^n)|$ and (2.22) we get the explicit formula.

As an application of (7.35) we show that the $u_r$-family of Adams [2] (which is not in $\text{im}(t^0)$) is in $\text{im}(t^{-2})$ and can so be constructed via transfer maps. For this we need (see also (7.18)):

Proposition 7.38: Let $z = [M, \phi, f]$ be an element in $\pi^s_{2n}(P_{\infty}C^kH)$ and $x = c_1(f^xH)$ then

$$e_c t^k(z) = \frac{1}{x} \left( \left( \frac{e^x - 1}{x} \right)^{k-1} \right), \quad [M] \mod 2.$$

Proof: The proof is a simple generalization of the proof for $k=0$.

Let $\xi$ be the complex line bundle $f^*(H)$ on $M$. Then the sphere bundle $S(\xi)$ represents $t^k(z)$. To compute $e_\xi t^k(z)$ we use the definition of $e$ via Todd-genus of a bounding manifold. We take as a bounding manifold the disk-bundle $D(\xi)$ and then have to compute the Kronecker product

$$< \text{Todd}(y), [D(\xi), S(\xi)] >$$

where $y$ is the vector bundle on $D(\xi)/S(\xi)$ got from the tangent bundle of $D(\xi)$ and the framing on $S(\xi)$. The map $j^*: \tilde{K}^O(D(\xi), S(\xi)) \to K^O(D(\xi))$ maps $y$ to the class of the tangent bundle of $D(\xi)$. Thus using the difference construction in $K$-theory we can write

$$y = d((1-k)\xi, C^0; \phi)$$

But $\phi$ is constructed universally, so we can compute this on $P_\infty \mathbb{C}$ where $j^*$ is injective. There the class with $j^*U = (1-k)\tilde{H}$ is $(1-k)$-times the standard $K$-theory Thom class $U(H)$ so

$$y = (1-k)U(\xi)$$

If $\phi$ is the Thom isomorphism of $\xi$, then

$$e_\xi t^k(z) = < \text{Todd}(y), [D(\xi), S(\xi)] >$$

$$= < \phi^{-1}(\text{Todd}(U(\xi)))^{1-k-1}, [M] >$$

$$= < \frac{1}{x} \left( \left( e^x - 1 \right)^{k-1} - 1 \right), [M] >$$

**Theorem 7.39:** Let $z \in \pi^{s}_{8k}(P_\infty \mathbb{C}^{2\tilde{H}}(2))$ be an element which maps to a generator in $\pi^{s}_{8k}(P_\infty \mathbb{C}^{2\tilde{H}})/\text{tor}$, then

$$e_\xi t^{-2}(z) = \frac{1}{2} \mod \mathbb{Z}(2)$$

that is $t^{-2}(z) = \mu_k \in \pi^{s}_{8k+1}(S^O(2)) \mod F^3$.

**Proof:** Denote by $\phi$ the Thom isomorphism of $-2\tilde{H}$, then by (7.35) an element $z$, which maps to a generator in $\pi^{s}_{8k}(P_\infty \mathbb{C}^{2\tilde{H}})/\text{tor}$ has Hurewicz
image

\[ h(z) = (4k)!2 \cdot [P_{4k}] \quad \text{(mod odd primes)} \]

We have to calculate \( x = c_1(H) \)

\[ r = \text{et}^{-2}(z) = 4k!2 \cdot \frac{x^2}{(e^x - 1)^3}, \quad [P_{4k}] > \mod \mathcal{Z}(2) \]

If we write \( x/(e^x - 1) = (1-x/2+a) \), then \( a \) contains only elements of even degree and we get

\[ r = 4k!2 \cdot \text{term of degree (4k+1) in } (1-x/2+a)^3 \]
\[ = 4k!2 \cdot 3 \cdot (a-a^2/2)(4k) \]
\[ = 4k!a^2(4k) \mod \mathcal{Z}(2) \]

The proof is finished by using the following lemma:

**Lemma 7.40:**

\[ r = \sum_{i=1}^{2k-1} \binom{4k}{2i} \cdot B_{2i} \cdot B_{4k-2i} = \frac{1}{2} \mod \mathcal{Z}(2) \]

**Proof:**

\[ r = 2 \cdot \frac{1}{2} \sum_{i=1}^{k-1} \binom{4k}{2i} \cdot B_{2i} \cdot B_{4k-2i} + \binom{4k}{2k} \cdot B_{2k}^2 \cdot B_{2k} \cdot B_{2k} \cdot B_{2k}. \] Because \( B_{2i} = \frac{1}{2} \)

\[ \mod \mathcal{Z}(2) , \text{ we get } r = \frac{1}{2} \sum_{i=1}^{k-1} \binom{4k}{2i} + \frac{1}{2} \binom{4k}{2k-1}. \] But

\[ 4r = \sum_{i=1}^{4k-1} \binom{4k}{i} = \sum_{i=0}^{4k} \binom{4k}{i} - 2 = 2^{4k-2} \mod 4, \]

so \( r = \frac{1}{2} \mod \mathcal{Z}(2) \).

**Remarks:**

1. An element \( z \) as above can be characterized as follows: It suffices to know \( h(z) \) is divisible by \( 4k!2 \) and not by any number with larger 2 primary factor, or to be more explicit, there exists a manifold \( M^{2k} \) with a complex line bundle \( \xi \) such that the stable tangent bundle of \( M \) is isomorphic to \( 2\xi \) and such that

\[ \langle c_1(\xi)^{4k}, [M] \rangle = a \cdot 2 \cdot 4k! \]

with \( a \neq 0 \mod 2 \). For all such manifolds we have: \( S(\xi) \) with the
induced framing represents $\mu_k$ mod Adams filtration 3.

2. One can choose an $z$ such that $t^{-2}(z)$ is determined up to $\eta^*\pi_{8k}^{S}(S^0) + \nu^*\pi_{8k-2}^{S}(S^0)$. It should be possible to define an element $z$ such that $t^{-2}(z) = \mu_k$.

3. In course of the calculation of $e \in t^{-2}(z)$ we have seen that the only odd term in $x/(e^x-1)$ is closely related to the $\mu_r$-family.

4. One can show that $\mu_r \notin \text{im}(t^k)$ with $k$ odd.

In (7.39) we have seen that one needs to know the numbers $U(n,s)$ in order to compute the transfer $t^k$ from filtration 0 to filtration 1.

One can now, of course, ask if every element $x \in \pi_n^S(S^0)$ lies in $\text{im}(t^k)$ for some $k$ (depending upon $k$). For the odd primes no counterexample seems to be known. For $p=2$ we have already:

For no $k$ is $\sigma \in \pi_n^S(S^0)(2)$ in $\text{im}(t^k)$.

This is proved by computing the relevant $U(n,m)$. By periodicity these are only finitely many cases to check (see table 1).

We turn now to $U(n+i,n)$ with $i \geq 2$.

**Proposition 7.41:**

$$
U(n+2,n)(p) = U_A(n+2,n) \quad (p \text{ odd})
$$

$$
U(n+2,n)(2) = U_{AR}(n+2,n) \quad (n \neq 0 \quad (4))
$$

e.g. $v_p(U(n+2,n)) = \begin{cases} 
v_p((n+1)!)-1 & \text{if } n \equiv 0 \quad (p-1) \\
0 & \text{if } n \equiv 0 \quad (p-1) 
\end{cases}$

**Proof:** Using the transfer sequence

$$
\tilde{A}_{2n}(P_\infty \mathcal{C}^\mathcal{H}) \rightarrow \tilde{A}_{2n-2}(P_\infty \mathcal{C}^{2\mathcal{H}}) \rightarrow t^2 \tilde{A}_{2n-1}(S^0) \rightarrow i^*_x \tilde{A}_{2n-1}(P_\infty \mathcal{C}^\mathcal{H})
$$

we see $\ker(i_x) \cong \mathbb{Z}/p$ iff $n \equiv 0 \quad (p-1)$, because the order of
i_*(b_1/p^{l+v_p}(n)) in \tilde{A}_{2n-1}(P_\infty \mathbb{C}^\mathbb{H}) \simeq \tilde{A}_{2n+1}(P_\infty \mathbb{C}) is \ p^{v_p}(n) by (1.14).

This gives \ U_A(n+2,n) = U_A(n+2,n)/p if \ n \equiv 0 \ (p-1) \ and \ U_A(n+2,n) = U_A(n+2,n) if \ n \neq 0 \ (p-1). \ The \ following \ diagram \ shows \ that \ it \ suffices \ to \ prove \ that \ \alpha_t \in \pi_{2t(p-1)-1}(S^0)(p) \ is \ in \ \text{im}(t^2).

\[
\begin{array}{cccc}
\pi_{2n}(P_\infty \mathbb{C}^\mathbb{H}) & \longrightarrow & \pi_{2n-2}(P_\infty \mathbb{C}^2\mathbb{H}) & \longrightarrow & \pi_{2n-1}(S^0) \\
\, & \downarrow & \, & \downarrow & \, \\
\tilde{A}_{2n}(P_\infty \mathbb{C}^\mathbb{H}) & \longrightarrow & \tilde{A}_{2n-2}(P_\infty \mathbb{C}^2\mathbb{H}) & \longrightarrow & \tilde{A}_{2n-1}(S^0) \\
\end{array}
\]

For this we need a generalization of (3.3):

The generalized J-homomorphism (see e.g. [40])

\[ J_{n,k} : \pi_i(W_{n,k}) \longrightarrow \pi_{i+2k}(S^{2n}) \]

is defined as follows: we look at \ W_{n,k} as the space of norm-preserving maps from \ \mathbb{C}^k into \ \mathbb{C}^n, so we get an evaluation map

\[ S^{2k-1} \times W_{n,k} \longrightarrow S^{2n-1} \]

to which the Hopf construction is applied. As in the case \ n=k we get \ J_{n,k} and stable versions e.g.

\[ J'_{n,k} : \pi_i(W_{n,k}) \longrightarrow \pi_{i+2k-2n}(S^0) \]

The results of [10], [7] show that the following diagram commutes

\[ (d=n-k): \]

\[ \pi_m(W_{n,k}) \xrightarrow{J_{n,k}} \pi_{m+2k}(S^{2n}) \]

\[ \downarrow \theta \]

\[ \pi_{m-2d-1}(P_{k-1}\mathbb{C}^d\mathbb{H}) \xrightarrow{t^d} \pi_{m-2d}(S^0) \]

There is a commutative diagram due to James which allows one to interpret \ J_{n,k} via boundary maps (see [33], [18]):
\[ \pi_m(W_{n,k}) \xrightarrow{J_{n,k}} \pi_{m+2k}(S^{2n}) \]
\[ \downarrow q_* \]
\[ \pi_m(U(n)/U(k) \times U(d)) \xrightarrow{\vartheta} \pi_{m-1}(S^{2d-1}) \]

where \( q \) is the projection of \( W_{n,k} \) to the Grassmann manifold \( U(n)/U(k) \times U(d) \). We fix \( d=n-k \) and let \( k \) tend to \( \infty \) to get

\[ \pi_m(U/U(d)) \xrightarrow{J_d} \pi_{m-2d}(S^0) \]
\[ \downarrow q_* \]
\[ \pi_m(BU(d)) \xrightarrow{\vartheta} \pi_{m-1}(S^{2d-1}) \]  

(7.43)

Because \( BU(1) \cong K(\mathbb{Z},2) \) we see that for \( d=2 \) \( \vartheta \) is onto. Clearly \( q_* \) is onto. It is known [15] that \( \alpha_t \) has \( S^3 \) as sphere of origin, so \( \alpha_t \) lifts to \( \pi_m(U/U(2)) \) and so to an element in \( \pi_m^s(P \mathbb{CP}^2) \) which maps to \( \alpha_t \) under \( t^2 \). This proves (7.41).

For \( p=2 \) the same argument applies using [28], except when \( n=0 \) (4) because the sphere of origin for the element of order 2 in \( \text{im}(J) \subset \pi_{8k-1}^s(S^0)(2) \) is not \( S^3 \).

Because \( S^3 \) is the sphere of origin for the \( \mu_r \)-family we get:

Corollary 7.44: \( \mu_k \in \text{im}(t^2) \) for all \( k \).

Using the splitting of the \( p \)-localizations of Lie groups [30], one can use the same sort of argument to prove

\[ U(n+i,n)(p) = U_A(n+i,n) \quad 0 \leq i \leq p \]  

(7.45)

It is even possible to use (7.42) and (7.43) to deduce some results about instable homotopy groups in a simple manner:

Because of (7.42) and (7.43) \( \text{im}(t^2) \) is an upper bound for the image
of the suspension map \( \pi_{m-1}(S^3) \to \pi_{m-4}(S^0) \).

**Example:** If \( \alpha_1 \cdot x \neq 0 \) for \( x \in \pi^s_m(S^0) \), then one can show that \( x \notin \text{im}(t^2) \) and therefore

\[
\pi^s_m(S^0) : \alpha_1 \cdot x \neq 0 \implies x \notin \text{im}(\pi_{m-1}(S^3) \to \pi^s_{m-4}(S^0))
\]

In the metastable range, we can put the complex reflection map into the diagrams (7.42) and (7.43) to get

\[
\begin{array}{ccc}
\pi^s_{m-1}(P \mathbb{C}/P_{d-1} \mathbb{C}) & \xrightarrow{t^d} & \pi^s_{m-2d}(S^0) \\
\theta & \downarrow \quad \omega \downarrow & \\
\pi_m(U/U(d)) & \xrightarrow{J \quad d} & \pi^s_{m-2d}(S^0) \\
\downarrow q \downarrow & & \uparrow S \\
\pi_m(BU(d)) & \xrightarrow{\theta} & \pi_{m-1}(S^{2d-1})
\end{array}
\]

We always have \( e_{\mathbb{C}^d} \cdot \theta \subset \text{im}(e_{\mathbb{C}^d} t^d) \), whereas \( e_{\mathbb{C}^d} \cdot \omega \subset \text{im}(e_{\mathbb{C}^d} t^d) \) in the metastable case \((n < 4d)\). So if we know \( U(\mathbb{M}, \mathbb{M}-d) \), \( \mathbb{M} = (m+1)/2 \), we can compute the \( e \)-invariant of the following obstruction class:

Given a \( d \)-dimensional complex vector bundle \( \xi \) on \( S^m \), we can look at the obstruction to the existence of a section, which is an element \( \sigma(\xi) \in H^m(S^m; \pi_{m-1}(S^{2d-1})) \cong \pi_{m-1}(S^{2d-1}) \), then \( e_{\mathbb{C}}(\sigma(\xi)) \in \text{im}(e_{\mathbb{C}^d} t^d) \).

The information \( U_A(i,j) = U(i,j) \) for a fixed pair \((i,j)\) in the metastable range can be used to construct infinite families of elements \( e_{m,d} \) in \( \pi_{2d+2m}(U(d)) \) as follows:

If \( U(d+m+1, m+1) = U_A(d+m+1, m+1) \) we know that

\[
h_A : \pi^s_{2m}(P_\infty \mathbb{C}^{d+\bar{H}}) \to A_{2m}(P_\infty \mathbb{C}^{d+\bar{H}}) \text{ is onto and we can choose an element} \ x_{m,d} \text{ such that } h_A(x_{m,d}) \text{ generates } A_{2m}(P_\infty \mathbb{C}^{d+\bar{H}})/\text{tor.}
\]

Let \( \omega : P_\infty \mathbb{C}^{d+\bar{H}} \to P_\infty \mathbb{C}^{(d+1)\bar{H}} \) be the map of \((2.13)\). From \( x_{m,d} \) we get a family of elements \( x_{m,d+t} = \omega^t(x_{m,d}) \) in \( \pi^s_{2m}(P_\infty \mathbb{C}^{(d+t)\bar{H}}) \) \((t \geq 0)\).
We set

\[ e_{m,r} = \theta(x_{m,r}) \in \pi_{2m+2r}(U(r)) \]

for \( r \leq d \), where \( \theta: \pi_{2m+2r+1}(U/U(r)) \to \pi_{2m+2r}(U(r)) \) is the boundary map.

The exact sequence

\[ A_{2m+2d}(P_\infty \mathbb{C}) \to A_{2m+2d}(P_\infty \mathbb{C}/P_{d-1} \mathbb{C}) \to A_{2m+2d-1}(P_{d-1} \mathbb{C}) \]

shows that the order of \( e(\theta(e_{m,d})) \) is \( \tilde{a} = (m+d)!/U(m+d+1,m+1) \).

The commutative diagram \((r = d+t)\)

\[ A_{2m+2d}(P_\infty \mathbb{C}) \to \tilde{A}_{2m}(P_\infty \mathbb{C}^{d\tilde{H}}) \to A_{2m+2d-1}(P_{d-1} \mathbb{C}) \]

\[ \tilde{A}_{2m+2r}(P_\infty \mathbb{C}) \to \tilde{A}_{2m}(P_\infty \mathbb{C}^{r\tilde{H}}) \to A_{2m+2r-1}(P_{r-1} \mathbb{C}) \]

then gives \( e_\theta(\theta(e_{m,d+t})) = \tilde{a} \).

If \( t \) is large enough, the elements \( e_{m,d+t} \) are independent of the choice of \( x_{m,d} \).

The reason for this is that if \( t \) is large enough \( \omega^t_x \) kills all torsion in \( \pi_{2m}(P_\infty \mathbb{C}^{dH}) \). This can be proved by using the Atiyah-Hirzebruch spectral sequence: The induced map \( \omega^p_x \) in the \( E_2 \)-term is multiplication by at least \( p \). So for every torsion element \( x \), there exist some \( t \) such that \( \omega^t_x(x) = 0 \) in the \( E_2 \)-term. This means that \( \omega^t_x(x) \) has a skeleton filtration which is strictly less than the one of \( x \). By repeating this, \( \omega^s_x(x) = 0 \) follows.

Because the order of the torsion groups in \( \pi_{2m}(P_\infty \mathbb{C}^{dH})(t \in \mathbb{Z}) \) is bounded for \( m \) fixed, it follows that \( j_*(\sigma^{m+t}) \) in \( \pi_{2m}(P_\infty \mathbb{C}^{dH}) \) is divisible in an unique way by some number \( c \) which increases with \( t \) (\( \sigma^k \) is the generator of \( \pi_{2k}(P_\infty \mathbb{C})/\text{tor} \) and \( j: P_\infty \mathbb{C} \to P_\infty \mathbb{C}/P_{k-1} \mathbb{C} \) the natural map). This implies, that for \( t \) large enough, the order of \( e_{m,d+t} \) must be exactly \( \tilde{a} = (m+d)!/U(m+d+1,m+1) \).
We close by mentioning the main results of [12], which show $U_A(i,j) = U(i,j)_{\phi}$ to be true in a large number of cases.

In (7.28) the abstraction class for lifting a generator of $\pi^{m+2}_n(P_{m+n}C^{-H})/\operatorname{tor}$ to $\pi^{m+2}_n(P_{m+n}C^{-H})$ was the transfer image of $\sigma^m$. Because this element was in $\operatorname{im}(J)$, we could conclude that $h: \pi^{m+2}_n(P_{m+n}C^{-H}) \to A^{m+2}_n(P_{m+n}C^{-H})$ is onto. In [12] this is generalized as follows: The transfer of the sphere bundle $r \cdot H$ appearing in the cofibre sequence (2.11)

$$\pi^{m+2}_n(P_{m+n}C^{-rH}) \to \pi^{m+2}_n(P_{m+n}C^+) \xrightarrow{\operatorname{Tr}} \pi^{m+2}_n(S(rH)^{-rH})$$

has the same property, namely $\operatorname{Tr}(\sigma^m)$ is in the image of a $J$-homomorphism. This implies

$$U(r,m+r) = U_A(r,m+r) \text{ for all } m,r \geq 0$$

Closely related to cofibre sequence (7.46) is the following cofibre sequence of projective spaces, $(a,\ldots,n,r \in \mathbb{N})$:

$$P_{m+n+r}C^{aH} \to \pi^{m+n+r}_n(P_{m+n+r}C^{-aH}) \to P_{m+n+r}C^{(a+r)H} \xrightarrow{\operatorname{Tr}} S^1P_{m+n+r}C^{aH}$$

Let $x \in \pi^{m+n+r}_n(P_{m+n+r}C^{(a+r)H})$ be an element with the property

"$e\cdot\pi^{m+n+r}_n(P_{m+n+r}C^{aH})$ in determined by its $e$-invariant" 

(7.47)

that is $|e\cdot x| = |e\cdot \pi^{m+n+r}_n(P_{m+n+r}C^{aH})|$. Then we know that $|e\cdot x| \cdot x$ can be lifted to $\pi^{m+n+r}_n(P_{m+n+r}C^{aH})$. Suppose we have an element $x$ in $\pi^{m+n+r}_n(P_{m+n+r}C^{aH})$ mapping to a generator of the free part of $A_{2n}(P_{n+r}C^{(a+r)H})$ with property (7.47) then we can conclude as above, that $h_A: \pi^{m+n+2r}_n(P_{m+n+r}C^{aH}) \to A_{2n+2r}(P_{m+n+r}C^{aH})$ is onto.

Let $M$ be the $J$-order of the Hopf bundle on $P_wC$ where $w = \max(n,r-1)$. The we have a (in general non-commuting) diagram
defined by the relative Thom isomorphisms of $M \cdot \tilde{H}$. It is then possible to work out the deviation from commutativity for (7.48) and to show that the element describing this deviation is in the image of a $J$-homomorphism. In a large number of cases this implies, that if $\partial_1 \xi$ is determined by its $e$-invariant, the same is true for $\partial_2 \Phi \xi$. This allows us to conclude in case of

$$U_\Lambda( a+n+r+1, n+1 ) = U_\Lambda( a+M \cdot s, n+1 )$$

The element, which describes the deviation from commutativity in (7.48) very often has a small order. This implies that $U(m+M \cdot s, n+r+1)$ is constant as a function of $s$, which explains for example some of the shorter periodicities in table 1.
### Table 1

2-primary part of $U(n,r)$

<table>
<thead>
<tr>
<th>$M_2$ = 2</th>
<th>$U(2,2) = 1$</th>
<th>$M_3$ = 8</th>
<th>$U(3,3) = 2$</th>
<th>$M_4$ = 8</th>
<th>$U(n,4) = U(n,3)$</th>
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<tbody>
<tr>
<td>$U(3,2) = 2$</td>
<td></td>
<td>$U(4,3) = 2$</td>
<td></td>
<td>$U(n,3)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$U(5,3) = 8$</td>
<td></td>
<td>$U(7,3) = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$U(6,3) = 4$</td>
<td></td>
<td>$U(8,3) = 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$U(7,3) = 8$</td>
<td></td>
<td>$U(9,3) = 8$</td>
<td></td>
</tr>
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<td></td>
<td>$U(10,3) = 4$</td>
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</tbody>
</table>

$M_5 = 64$

<table>
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<th>$U(14,5) = 4$</th>
<th>period 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(6,5) = 8$</td>
<td>$U(15,5) = 4$</td>
<td>period 16</td>
</tr>
<tr>
<td>$U(7,5) = 8$</td>
<td>$U(16,5) = 8$</td>
<td>period 16</td>
</tr>
<tr>
<td>$U(n,8,5) = 8/(n,8)$</td>
<td></td>
<td>period 64</td>
</tr>
<tr>
<td>$U(9,5) = 16$</td>
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</tr>
<tr>
<td>$U(10,5) = 16$</td>
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<td>$U(11,5) = 16$</td>
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<td>$U(12,5) = 16$</td>
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</table>

$M_6 = 64$

<table>
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<th>$U(6,6) = 8$</th>
<th>$U(14,6) = 4$</th>
<th>period 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(7,6) = 16$</td>
<td>$U(15,6) = 8$</td>
<td>period 16</td>
</tr>
<tr>
<td>$U(8,6) = 8/(n,8)$</td>
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<td>period 64</td>
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<tr>
<td>$U(13,6) = 16$</td>
<td>$U(21,6) = 8$</td>
<td>period 16</td>
</tr>
</tbody>
</table>

$M_7 = 128$

<table>
<thead>
<tr>
<th>$U(7,7) = 16$</th>
<th>$U(15,7) = 8$</th>
<th>period 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(n,8,7) = 16/(n,16)$</td>
<td></td>
<td>period 128</td>
</tr>
<tr>
<td>$U(9,7) = 128$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U(10,7) = 64$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U(11,7) = 64$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U(12,7) = 16$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U(13,7) = 16$</td>
<td>$U(5,7) = 64$</td>
<td>period 32</td>
</tr>
<tr>
<td>$U(14,7) = 4$</td>
<td>$U(22,7) = 32$</td>
<td>period 32</td>
</tr>
<tr>
<td>$U(46,7) = 8$</td>
<td>$U(54,7) = 32$</td>
<td>period 64</td>
</tr>
<tr>
<td>$U(21,7) = 64$</td>
<td>$U(30,7) = 16$</td>
<td></td>
</tr>
<tr>
<td>$U(38,7) = 32$</td>
<td>$U(70,7) = 32$</td>
<td></td>
</tr>
</tbody>
</table>

$M_8 = 128$

| $U(n,8) = U(n,7)$ | | |

$M_9 = 2048$

<table>
<thead>
<tr>
<th>$U(i,9) = 2^8$</th>
<th>$i = 17 + k \cdot 16, \ldots, 24 + k \cdot 16$</th>
<th>$(k &gt; 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(i,9) = 2^6$</td>
<td>$i = 9 + k \cdot 32, \ldots, 16 + k \cdot 32; 25 + k \cdot 32$</td>
<td>$(k &gt; 0)$</td>
</tr>
<tr>
<td>$U(i,9) = 2^5$</td>
<td>$i = 26, \ldots, 32; i = 50, 59; i = 90, 96; i = 122, 123$</td>
<td></td>
</tr>
<tr>
<td>$U(i,9) = 2^4$</td>
<td>$i = 61, \ldots, 64; i = 124, 125$</td>
<td></td>
</tr>
<tr>
<td>$U(i,9) = 2^3$</td>
<td>$i = 60, 126$</td>
<td></td>
</tr>
</tbody>
</table>
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