Bergische Universität Wuppertal

# Jordan Decomposition for the Alperin–McKay Conjecture

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*vorgelegt von Lucas Ruhstorfer*

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# Introduction

## The Global-Local Conjectures

The representation theory of finite groups is a field of mathematics which was introduced to study finite groups by means of linear algebra. This theory is concerned with the study of group homomorphisms

$$\mathcal{X} : G \to \mathrm{GL}_n(K)$$

from a finite group $G$ to a general linear group $\mathrm{GL}_n(K)$ over a field $K$. From the very beginning of this topic, it became apparent that many properties of the group itself are encapsulated in its representations and there are theorems in group theory which can only be proved by using representation-theoretic methods.

Motivated by the work of Brauer, the so-called *local-global conjectures* became an important area of research. These conjectures predict that for a prime number $\ell$ the information about a finite group $G$ (*global information*) should relate to properties of $\ell$-local subgroups of $G$, that is, normalizers or centralizers of non-trivial $\ell$-subgroups of $G$ (*local information*).

One of the most simple yet extremely difficult conjectures is the so-called McKay conjecture, see [Mal17, Section 2]. Let $\ell$ be a prime and $K$ a finite field extension of $\mathbb{Q}_\ell$ large enough for all finite groups considered. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible $K$-representations and by $\mathrm{Irr}_{\ell'}(G)$ the subset corresponding to irreducible representations $\mathcal{X} : G \to \mathrm{GL}_n(K)$ with $\ell \nmid n$.

**Conjecture** (McKay). *Let $G$ be a finite group and $P$ a Sylow $\ell$-subgroup of $G$. Then $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(\mathrm{N}_G(P))|$.*

Later, Alperin [Alp76] refined this conjecture by taking into account the representation theory over a field of positive characteristic $\ell$.

Denote by $\mathcal{O}$ the ring of integers of $K$ over $\mathbb{Z}_\ell$ and by $k$ its residue field. For an $\ell$-block $B$ of $kG$ let $\mathrm{Irr}_0(G, B)$ be the set of isomorphism classes of height zero representations of $kB$, i.e., the set of irreducible representations $\mathcal{X} : G \to \mathrm{GL}_n(K)$ associated to the block $B$ such that $n_\ell = |G : D|_\ell$, where $D$ denotes a defect group of $B$.

**Conjecture** (Alperin–McKay)**.** *Let $G$ be a finite group and $b$ an $\ell$-block of $G$ and $B$ its Brauer correspondent in $\mathrm{N}_G(D)$. Then*

$$|\operatorname{Irr}_0(G, b)| = |\operatorname{Irr}_0(\mathrm{N}_G(D), B)|.$$

Both of these conjectures have been reduced (by Isaacs–Malle–Navarro, respectively Späth, see [IMN07, Theorem B] and [Spä13, Theorem C]) to the verification of certain stronger versions of the same conjecture for finite quasi-simple groups. These stronger versions are usually referred to as the inductive conditions.

This approach has turned out to be very fruitful in recent years. In their landmark paper [MS16], using this approach Malle–Späth were able to prove the McKay conjecture for the prime $\ell = 2$ (proving the original conjecture of McKay).

According to the classification of finite simple groups, many finite simple groups are groups of Lie type. These are finite groups which arise as fixed points $\mathbf{G}^F$ of a simple algebraic group $\mathbf{G}$ under a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. In this thesis, we focus on their representation theory and establish a new approach to the inductive Alperin–McKay condition for those groups.

# Representation theory of groups of Lie type

In characteristic zero Deligne and Lusztig have constructed representations by means of $\ell$-adic cohomology groups of the so-called *Deligne–Lusztig varieties.* Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius $F : \mathbf{G} \to \mathbf{G}$ and let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Then the variety

$$\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} = \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}F(\mathbf{U})\}$$

has a left $\mathbf{G}^F$- and a right $\mathbf{L}^F$-action. Recall that $\mathcal{O}$ denotes the ring of integers over $\mathbb{Z}_\ell$ of a finite field extension $K$ of $\mathbb{Q}_\ell$. The $\ell$-adic cohomology groups $H_c^i(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})$ of the *Deligne–Lusztig variety* $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ provide us with a map

$$R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} : G_0(\mathcal{O}\mathbf{L}^F) \to G_0(\mathcal{O}\mathbf{G}^F),\ [M] \mapsto \sum_i (-1)^i [H_c^i(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O}) \otimes_{\mathcal{O}\mathbf{L}^F} M]$$

on the respective Grothendieck groups, see Section 2.2. Let $\mathbf{G}^*$ be the dual group of $\mathbf{G}$ with dual Frobenius $F^* : \mathbf{G}^* \to \mathbf{G}^*$, see Section 2.6. Deligne–Lusztig constructed a decomposition of the irreducible representations into rational Lusztig series

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_{(s)} \mathcal{E}(\mathbf{G}^F, s),$$

where $(s)$ runs over the set of $(\mathbf{G}^*)^{F^*}$-conjugacy classes of semisimple elements of $(\mathbf{G}^*)^{F^*}$. For a given $s \in (\mathbf{G}^*)^{F^*}$, let $\mathbf{L}^*$ be an $F^*$-stable Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$. Let $\mathbf{L} \subseteq \mathbf{G}$ be in duality with $\mathbf{L}^*$. Then Lusztig showed that the map

$$\mathcal{E}(\mathbf{L}^F, s) \to \mathcal{E}(\mathbf{G}^F, s),\ \psi \mapsto \pm\, \mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\psi),$$

provides a bijection. This is the first important step in establishing the so called *Jordan decomposition of characters*. This bijection has become an indispensable tool to study the representation theory of groups of Lie type since it reduces the question of understanding the representation theory of Lusztig series of general semisimple elements to the question of understanding Lusztig series of quasi-isolated semisimple elements, i.e., semisimple elements whose centralizers are not contained in a proper Levi subgroup.

# Modular representation theory of groups of Lie type

The Deligne–Lusztig theory has been generalized to positive characteristic. Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order. Then we define

$$\mathcal{E}_\ell(\mathbf{G}^F, s) = \coprod_{t} \mathcal{E}(\mathbf{G}^F, t),$$

where $(t)$ runs over the set of $(\mathbf{G}^*)^{F^*}$-conjugacy classes of semisimple elements of $(\mathbf{G}^*)^{F^*}$ whose $\ell'$ part is $(s)$. By the work of Broué–Michel [BM89] this turns out to be a union of $\ell$-blocks of $\mathbf{G}^F$. We denote by $e_s^{\mathbf{G}^F} \in \mathrm{Z}(\mathcal{O}\mathbf{G}^F)$ the central idempotent associated to this sum of blocks. Similar to the characteristic zero case we have a decomposition

$$\mathcal{O}\mathbf{G}^F\text{-mod} \cong \bigoplus_{(s)} \mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}\text{-mod}$$

where $(s)$ runs over the set of $(\mathbf{G}^*)^{F^*}$-conjugacy classes of semisimple elements of $(\mathbf{G}^*)^{F^*}$ of $\ell'$-order.

Establishing a conjecture by Broué, Bonnafé–Rouquier [BR03] and later Bonnafé–Dat–Rouquier [BDR17a] proved a Jordan decomposition for blocks of groups of Lie type. Fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order and suppose that $\mathbf{L}^*$ is the minimal $F^*$-stable Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Denote by $\mathbf{N}^F$ the stabilizer of $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$ in $\mathbf{G}^F$. Then the following was proved in [BDR17a] (see also Theorem 2.35 below):

**Theorem A** (Bonnafé–Dat–Rouquier)**.** *Suppose that $\mathbf{N}^F/\mathbf{L}^F$ is cyclic. Then the bimodule $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to a $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule such that $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ are Morita equivalent. Moreover, $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ are splendid Rickard equivalent.*

Note however that [BDR17a, Theorem 7.7] was announced without the assumption that $\mathbf{N}^F/\mathbf{L}^F$ is cyclic. This assumption is necessary to apply their proof, see Section 3.1 for more details. As a first main result of this thesis we partly remove this technical assumption and therefore extend the results of Theorem A. Assume that $\mathbf{G}$ is a simple algebraic group. In this case, the quotient group $\mathbf{N}^F/\mathbf{L}^F$ embeds into $\mathrm{Z}(\mathbf{G})^F$. Therefore, a non-cyclic quotient can only appear if $\mathbf{G}$ is simply connected and $\mathbf{G}^F$ is of type $D_n$ with even $n \geq 4$. Hence we focus on this situation and prove the following:

**Theorem B** (see Theorem 3.22)**.** *Suppose that $\mathbf{G}$ is a simple algebraic group. If $\ell \nmid (q^2-1)$ or if $\mathbf{N}^F/\mathbf{L}^F$ is cyclic then the bimodule $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to a $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule such that $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ are Morita equivalent. Moreover, $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ are splendid Rickard equivalent.*

Note that Theorem B has also appeared in the author's article [Ruh18].

## Clifford theory and group automorphisms

The Jordan decomposition by Bonnafé–Rouquier has proved to be extremely useful in the representation theory of finite groups of Lie type. For instance, the Bonnafé–Rouquier Morita equivalence was a crucial ingredient in the verification of one direction of Brauer's height zero conjecture by Malle–Kessar [KM13]. Our main objective in this thesis is therefore to provide a reduction of the verification of the inductive Alperin–McKay condition from Späth [Spä13] to blocks associated to quasi-isolated semisimple elements.

Let us therefore from now on assume that $\mathbf{G}$ is a simple algebraic group of simply connected type and for simplicity let us assume in this and the next section that $\mathbf{G}$ is not of type $D_4$. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism such that $\mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F)$ is a finite simple group. We let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding, i.e. an embedding of $\mathbf{G}$ into a group $\tilde{\mathbf{G}}$ with connected center and same derived subgroup as $\mathbf{G}$. As before, consider a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order and suppose now that $\mathbf{L}^*$ is the minimal $F^*$-stable Levi subgroup of $\mathbf{G}^*$ with $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$.

Using the classification of automorphisms of finite simple groups of Lie type (see Section 4.2) we prove the existence of bijective morphisms $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ stabilizing a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ in duality with $\mathbf{L}^*$ and such that the image of $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$, where $\mathcal{A} := \langle \sigma|_{\tilde{\mathbf{G}}^F}, F_0|_{\tilde{\mathbf{G}}^F}\rangle$, generates the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$. Moreover, these bijective morphisms commute with each other and the Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ is an integral power of $F_0$. Using this explicit description of automorphisms we can prove the following:

**Theorem C** (see Theorem 5.16)**.** *Assume that $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ and the order of $\sigma : \mathbf{G}^F \to \mathbf{G}^F$ is coprime to $\ell$. Then $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\mathcal{A})]$-module $M$. Moreover, the bimodule $\tilde{M} := \mathrm{Ind}_{(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\mathcal{A})}^{\tilde{\mathbf{G}}^F\mathcal{A} \times (\tilde{\mathbf{L}}^F\mathcal{A})^{\mathrm{opp}}}(M)$ induces a Morita equivalence between $\mathcal{O}\tilde{\mathbf{L}}^F\mathcal{A}e_s^{\mathbf{L}^F}$ and $\mathcal{O}\tilde{\mathbf{G}}^F\mathcal{A}e_s^{\mathbf{G}^F}$.*

One of the main ingredients in the proof of Theorem A is to show that the Morita bimodule in Theorem A does not depend on the choice of the parabolic subgroup. This yields that the bimodule $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ is $\Delta(\mathcal{A})$-invariant. However, this does not imply that the Morita bimodule extends. To remedy this problem we use a certain idea introduced by Digne [Dig99] in the context of restriction of scalars for Deligne–Lusztig varieties. This allows us to show that the module $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ can be endowed with a natural diagonal action of the automorphism $F_0|_{\tilde{\mathbf{G}}^F}$. From this we can show using the aforementioned independence result that the so-obtained bimodule is still invariant under the automorphism $\sigma$. Once we have proved this, Theorem C is then a consequence of general results on Clifford theory of Morita equivalences. This result gives us the desired compatibility of the Jordan decomposition with group automorphisms:

**Corollary D.** *In the situation of Theorem C we have the following commutative square of Grothendieck groups:*

$$\begin{array}{ccc}
G_0(\mathcal{O}\tilde{\mathbf{L}}^F\mathcal{A}e_s^{\mathbf{L}^F}) & \xrightarrow{[\tilde{M}\otimes -]} & G_0(\mathcal{O}\tilde{\mathbf{G}}^F\mathcal{A}e_s^{\mathbf{G}^F}) \\
\mathrm{Res}^{\tilde{\mathbf{L}}^F\mathcal{A}}_{\mathbf{L}^F}\Big\downarrow & & \Big\downarrow\mathrm{Res}^{\tilde{\mathbf{G}}^F\mathcal{A}}_{\mathbf{G}^F} \\
G_0(\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}) & \xrightarrow{(-1)^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}R_{\mathbf{L}}^{\mathbf{G}}} & G_0(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F})
\end{array}$$

# Local equivalences

The Alperin–McKay conjecture relates the global height zero characters to the local height zero characters. Thus, in order to reduce the verification of the inductive Alperin–McKay condition to a question about quasi-isolated elements we also need a Jordan decomposition as in Theorem C relating the corresponding blocks of normalizer subgroups.

Let $b$ be a block corresponding to the block $c$ under the Morita equivalence induced by $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\mathcal{O})e_s^{\mathbf{L}^F}$. Then the blocks $b$ and $c$ have a common defect group $D$ contained in $\mathbf{L}^F$. We denote by $B_D$ the Brauer correspondent of $b$ and by $C_D$ the Brauer correspondent of $c$. In addition, we let $B_D' = \mathrm{Tr}^{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(D)}_{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(D,B_D)}(B_D)$ and $C_D' = \mathrm{Tr}^{\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D)}_{\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D,C_D)}(C_D)$ be the corresponding central idempotents of $\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(D)$ and $\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D)$.

**Theorem E** (see Theorem 5.24)**.** *Suppose that the assumptions of Theorem C are satisfied. Then the cohomology module* $H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\mathbf{G}}(D)}_{\mathrm{C}_{\mathbf{U}}(D)},\mathcal{O})C_D$ *extends to an* $\mathcal{O}[(\mathrm{N}_{\mathbf{G}^F}(D)\times \mathrm{N}_{\mathbf{L}^F}(D)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D,C_D))]$*-module* $M_D$*. In particular, the bimodule*

$$\mathrm{Ind}^{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(D)\times \mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D)^{\mathrm{opp}}}_{(\mathrm{N}_{\mathbf{G}^F}(D)\times \mathrm{N}_{\mathbf{L}^F}(D)^{\mathrm{opp}})\Delta \mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D,C_D)}(M_D)$$

*induces a Morita equivalence between* $\mathcal{O}\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(D)B_D'$ *and* $\mathcal{O}\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(D)C_D'$.

# Jordan decomposition for the Alperin–McKay conjecture

In the final chapter of this thesis we then use the strong equivariance properties obtained in Theorem C and Theorem E to show that in order to obtain the inductive McKay condition for arbitrary blocks of groups of lie type it is enough to verify it for quasi-isolated blocks. Quasi-isolated semisimple elements for reductive groups have been classified by Bonnafé [Bon05]. In each case there are a small number of possibilities which have been well-described. Moreover, the quasi-isolated blocks of groups of Lie type are better understood by fundamental work of Cabanes–Enguehard and recent work of Enguehard and Kessar–Malle, see [KM13] for a more precise historical account. This gives us reason to hope that our reduction will provide a simplification of the verification of the inductive Alperin–McKay conditions.

Our main theorem is then as follows:

**Theorem F** (see Theorem 6.40 and Remark 6.41). *Assume that every quasi-isolated $\ell$-block of a finite quasi-simple group of Lie type defined over a field of characteristic $p \neq \ell$ satisfies the inductive Alperin–McKay condition (in the sense of Hypothesis 6.30 below). Let $S$ be a simple group of Lie type with non-exceptional Schur multiplier defined over a field of characteristic $p \neq \ell$ and $G$ its universal covering group. If $G$ is not of type $D$ or Assumption 6.26 holds for $G$ then the inductive Alperin–McKay condition holds for every $\ell$-block of $G$.*

Note that Assumption 6.26 was one of the main ingredients for the previous verifications of the inductive McKay condition. It is essential for constructing projective representations for certain classes of groups associated to characters of groups of Lie type and enables us to explicitly compute the factor set of the projective representation. The proof of Assumption 6.26 for groups of type $D$ is addressed in current work of Späth. Hence, Theorem F is expected to yield a complete reduction of the verification of the inductive Alperin–McKay condition to quasi-isolated blocks.

# Chapter 1

# Representation theory

In this chapter we introduce the necessary background material from the representation theory of finite groups. We will give a brief overview on various categorical equivalences of module categories associated to finite groups. We will then discuss in depth the Clifford theory of these equivalences by considering a theorem of Marcus.

## 1.1 Modular representation theory

Let $\ell$ be a prime and $K$ be a finite field extension of $\mathbb{Q}_\ell$. We say that $K$ is *large enough for a finite group $G$* if $K$ contains all roots of unity whose order divides the exponent of the group $G$. In the following, $K$ denotes a field which we assume to be large enough for the finite groups under consideration. We denote by $\mathcal{O}$ the ring of integers of $K$ over $\mathbb{Z}_\ell$ and by $k = \mathcal{O}/J(\mathcal{O})$ its residue field. We will use $\Lambda$ to interchangeably denote $\mathcal{O}$ or $k$.

Let $A$ be a $\Lambda$-algebra, finitely generated and projective as a $\Lambda$-module. We denote by $A^{\mathrm{opp}}$ its opposite algebra. Moreover, we mean by $A$-mod the category of left $A$-modules, that are finitely generated as $\Lambda$-modules.

We denote by $G_0(A)$ the *Grothendieck group* of the category $A$-mod, see also [Ben98, Section 5.1]. This means that $G_0(A)$ is the abelian group on the set $\{[X] \mid X \in A\text{-mod}\}$ of isomorphism classes satisfying the following relation: Whenever $0 \to M \to N \to P \to 0$ is a short exact sequence of $A$-modules then $[N] = [M] + [P]$ in $G_0(A)$. Since every object in $A$-mod has a finite composition series, it follows that $G_0(A)$ is generated by the subset $\mathrm{Irr}(A)$ of isomorphism classes of irreducible $A$-modules.

## 1.2 Module categories

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Comp}^b(\mathcal{A})$ the category of bounded complexes of $\mathcal{A}$ and by $\mathrm{Ho}^b(\mathcal{A})$ its homotopy category. In addition, $\mathrm{D}^b(\mathcal{A})$ denotes the bounded derived category of $\mathcal{A}$.

When $\mathcal{A} = A$-mod we abbreviate $\mathrm{Comp}^b(\mathcal{A})$, $\mathrm{Ho}^b(\mathcal{A})$ and $\mathrm{D}^b(\mathcal{A})$ by $\mathrm{Comp}^b(A)$, $\mathrm{Ho}^b(A)$ and $\mathrm{D}^b(\mathcal{A})$ respectively.

For $C \in \mathrm{Comp}^b(A)$ there exists (see for instance [BDR17a, 2.A.]) a complex $C^{\mathrm{red}}$ with $C \cong C^{\mathrm{red}}$ in $\mathrm{Ho}^b(A)$ such that $C^{\mathrm{red}}$ has no non-zero direct summand which is homotopy equivalent to 0. Moreover, $C \cong C^{\mathrm{red}} \oplus C_0$ with $H^\bullet(C_0) \cong 0$.

Let $A$-proj denote the subcategory of projective $A$-modules. We then denote by $A$-perf the full subcategory of $\mathrm{D}^b(A)$ consisting of complexes quasi-isomorphic to complexes of $\mathrm{Comp}^b(A\text{-proj})$.

Let $B$ and $D$ be two $\Lambda$-algebras. The tensor product

$$- \otimes_A - : \mathrm{Comp}^b(B \otimes_\Lambda A^{\mathrm{opp}}) \times \mathrm{Comp}^b(A \otimes_\Lambda D^{\mathrm{opp}}) \to \mathrm{Comp}^b(B \otimes_\Lambda D^{\mathrm{opp}})$$

of complexes gives rise to the tensor product

$$- \otimes_A^{\mathbb{L}} - : \mathrm{D}^b(B \otimes_\Lambda A^{\mathrm{opp}}) \times \mathrm{D}^b(A \otimes_\Lambda D^{\mathrm{opp}}) \to \mathrm{D}^b(B \otimes_\Lambda D^{\mathrm{opp}})$$

on derived categories.

For two complexes $C \in \mathrm{Comp}^b(A \otimes_\Lambda B^{\mathrm{opp}})$ and $C' \in \mathrm{Comp}^b(A \otimes_\Lambda D^{\mathrm{opp}})$ we denote by $\mathrm{Hom}_A^\bullet(C, C')$ the total Hom-complex with $n$th term $\oplus_{i+j=n} \mathrm{Hom}_A(C^i, C'^j)$. The complex $\mathrm{Hom}_A^\bullet(C, C')$ is a complex of $B \otimes_\Lambda D^{\mathrm{opp}}$-modules and we get a functor

$$\mathrm{Hom}_A^\bullet(-,-) : \mathrm{Comp}^b(A \otimes_\Lambda B^{\mathrm{opp}}) \times \mathrm{Comp}^b(A \otimes_\Lambda D^{\mathrm{opp}}) \to \mathrm{Comp}^b(B \otimes_\Lambda D^{\mathrm{opp}}),$$

the Hom-functor with right derived functor

$$\mathbb{R}\mathrm{Hom}_A^\bullet(-,-) : \mathrm{D}^b(A \otimes_\Lambda B^{\mathrm{opp}}) \times \mathrm{D}^b(A \otimes_\Lambda D^{\mathrm{opp}}) \to \mathrm{D}^b(B \otimes_\Lambda D^{\mathrm{opp}}).$$

If $i$ is an integer and $C = (C^n, d_C^n)$ is a complex of $A$-modules then we define $C[i]$ to be the complex of $A$-modules with terms $C[i]^n = C^{n+i}$ and differential $d_{C[i]}^n = (-1)^i d_C^{n+i}$. If $M$ is an $A$-module we denote by $M[i]$ the complex with all terms equal to 0 except the $-i$th taken to be $M$. Moreover, via the functor $A\text{-mod} \to \mathrm{Comp}^b(A)$, $M \mapsto M[0]$ we identify $A$-modules with complexes of $A$-modules concentrated in degree 0. This identification also yields fully faithful functors from $A$-mod to $\mathrm{Ho}^b(A)$ and $\mathrm{D}^b(A)$.

By [CE04, A1.5] and [CE04, A1.11] we have canonical isomorphisms

$$H^i(\mathrm{Hom}_A^\bullet(C, C')) \cong \mathrm{Hom}_{\mathrm{Ho}^b(A)}(C, C'[i])$$

and

$$H^i(\mathbb{R}\mathrm{Hom}^\bullet_A(C, C')) \cong \mathrm{Hom}_{\mathrm{D}^b(A)}(C, C'[i]).$$

We end this section by stating some facts specific to the representations of finite groups.

Let $H$ and $G$ be finite groups and $C$ be a complex of $\Lambda G$-$\Lambda H$-bimodules. Then we write $C^\vee$ for the complex $\mathrm{Hom}_{\Lambda G}(C, \Lambda G)$ viewed as complex of $\Lambda H$-$\Lambda G$-bimodules. If $\Lambda$ denotes the trivial $\Lambda G$-$\Lambda H$-bimodule then we have by [Bro94, 3.A.] an isomorphism $C^\vee \cong \mathrm{Hom}_\Lambda(C, \Lambda)$. Moreover, if $X$ is another complex of $\Lambda G$-$\Lambda H$ modules and $C$ is projective as $\Lambda H$-module then by [Bro94, 3.A.] there is a canonical isomorphism

$$C^\vee \otimes_{\Lambda G} X \cong \mathrm{Hom}_{\Lambda G}(C, X).$$

Let $\sigma : G \to G$ be an automorphism of a finite group $G$ and $H$ a subgroup of $G$. If $M$ is a left (resp. right) $\Lambda H$-module then we denote by ${}^\sigma M$ (resp. $M^\sigma$) the left (resp. right) $\Lambda\sigma(H)$-module which coincides with $M$ as a $\Lambda$-module but with action of $\sigma(H)$ given by $\sigma(h)m := \sigma^{-1}(h)m$ (resp. by $m\sigma(h) := m\sigma^{-1}(h)$).

A basic tool in the representation theory of finite groups is the theory of sources and vertices, see e.g. [Thé95, Chapter 17] for the following. If $H$ is a subgroup of $G$ and $L$ an $\Lambda H$-module then $\mathrm{Ind}_H^G(L)$ denotes $\Lambda G \otimes_{\Lambda H} L$. This defines a functor right adjoint to the restriction functor $\mathrm{Res}_H^G$. Given a $\Lambda G$-module $M$ we say that $M$ is *relatively $H$-projective* if the natural map

$$M \to \mathrm{Ind}_H^G \mathrm{Res}_H^G(M)$$

splits. Assume now that $M$ is an indecomposable $\Lambda G$-module which is free as $\Lambda$-module. If $H$ is a minimal subgroup such that $M$ is relatively $H$-projective then $H$ is necessarily an $\ell$-subgroup of $G$. Moreover, there exists an indecomposable direct summand $L$ of $\mathrm{Res}_H^G(M)$ such that $M$ is a direct summand of $\mathrm{Ind}_H^G(L)$. In this case $H$ is called *a vertex* of $M$ and $L$ is called the *source* of $M$. The pair $(H, L)$ is then unique up to $G$-conjugation.

## 1.3 The Brauer functor

Let $G$ be a finite group and $Q$ an $\ell$-subgroup of $G$. For a $\Lambda G$-module $M$ we let $M^Q$ denote the subset of $Q$-fixed points of $M$. We consider the *Brauer functor*

$$\mathrm{Br}_Q^G : \Lambda G - \mathrm{mod} \to k\,\mathrm{N}_G(Q)/Q - \mathrm{mod}$$

which for a $\Lambda G$-module $M$ is given by

$$\mathrm{Br}_Q^G(M) = k \otimes_\Lambda (M^Q / \sum_{P<Q} \mathrm{Tr}_P^Q(M^P)),$$

where $\mathrm{Tr}_P^Q : M^P \to M^Q$, $m \mapsto \sum_{g\in Q/P} gm$ is the relative trace map on $M$.

Let $f : M_1 \to M_2$ be a morphism of $\Lambda G$-modules. Then $f$ restricts to a morphism $f : M_1^Q \to M_2^Q$ of $\Lambda \mathrm{N}_G(Q)$-modules. One readibly checks that $f$ maps $\sum_{P<Q} \mathrm{Tr}_P^Q(M_1^P)$ to $\sum_{P<Q} \mathrm{Tr}_P^Q(M_2^P)$ and we hence obtain by taking quotients a morphism $\mathrm{Br}_Q(f) : \mathrm{Br}_Q(M_1) \to \mathrm{Br}_Q(M_2)$.

If $H$ is a subgroup of $G$ containing $Q$ then by definition we have

$$\mathrm{Br}_Q^H \circ \mathrm{Res}_H^G = \mathrm{Res}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)} \circ \mathrm{Br}_Q^G .$$

Therefore, we will sometimes omit the upper index and write $\mathrm{Br}_Q^G = \mathrm{Br}_Q$ if the group under consideration is clear from the context. Since $\mathrm{Br}_Q$ is an additive functor it respects homotopy equivalences and therefore extends to a functor

$$\mathrm{Br}_Q^G : \mathrm{Ho}^b(\Lambda G) \to \mathrm{Ho}^b(k\mathrm{N}_G(Q)/Q).$$

However, the functor $\mathrm{Br}_Q$ is neither left or right exact, so it does not extend to a functor on the respective derived categories.

Recall that a $\Lambda G$-module $M$ is called an *$\ell$-permutation module* if for every $\ell$-subgroup $Q$ of $G$ the module $M$ possesses a $Q$-stable $\Lambda$-basis. Equivalently, an $\ell$-permutation module is a direct summand of a permutation module, i.e., a module of the form $\Lambda[\Omega]$, where $\Omega$ is a $G$-set, see [Rou01, 4.1.3]. From the latter description it is not hard to see that the $\ell$-permutation modules are precisely the $\Lambda G$-modules with trivial source module. We let $\Lambda G - \mathrm{perm}$ be the full subcategory of $\Lambda G - \mathrm{mod}$ consisting of all $\ell$-permutation modules of $\Lambda G$.

If $\Lambda[\Omega]$ is a permutation module then the composition

$$\Lambda[\Omega^Q] \hookrightarrow (\Lambda[\Omega])^Q \twoheadrightarrow \mathrm{Br}_Q(\Lambda[\Omega])$$

induces an isomorphism $\mathrm{Br}_Q(\Lambda[\Omega]) \cong k[\Omega^Q]$, see [Rou01, 4.1.2]. From this it follows that the Brauer functor restricts to a functor

$$\mathrm{Br}_Q^G : \Lambda G\text{-}\mathrm{perm} \to k\,\mathrm{N}_G(Q)/Q\text{-}\mathrm{perm}.$$

Note that we will usually identify $k\,\mathrm{N}_G(Q)/Q - \mathrm{perm}$ via inflation with a subcategory of $k\,\mathrm{N}_G(Q) - \mathrm{perm}$.

An important property of the Brauer functor is that an $\ell$-permutation module $M \in \Lambda G - \mathrm{perm}$ has vertex $Q$ if and only if $Q$ is maximal with the property that $\mathrm{Br}_Q(M) \neq 0$, see [Thé95, Corollary 27.7].

Thus, in particular if we consider $\Lambda G$ as $G$-module via $G$-conjugation, then $\mathrm{Br}_Q(\Lambda G) \cong k\,\mathrm{C}_G(Q)$. The canonical surjection

$$\mathrm{br}_Q^G : (\Lambda G)^Q \to k\,\mathrm{C}_G(Q), \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in \mathrm{C}_G(Q)} \lambda_g g,$$

induces an algebra homomorphism, the so called *Brauer morphism*, see [Rou01, Section 4.2].

For two $\ell$-permutation modules $M_1, M_2 \in \Lambda G$-perm we consider $M_1 \otimes_\Lambda M_2$ as $\Lambda G$-module via the diagonal action of $G$. It follows that the natural map

$$\mathrm{Br}_Q(M_1) \otimes_k \mathrm{Br}_Q(M_2) \to \mathrm{Br}_Q(M_1 \otimes_k M_2)$$

is an isomorphism, see [Ric96, Section 4]. In particular, for $M \in \Lambda G$-perm and $e \in \mathrm{Z}(\Lambda G)$ we have $Me \cong M \otimes_\Lambda \Lambda e$ which (see also [Rou01, Section 4.2]) implies

$$\mathrm{Br}_Q(Me) = \mathrm{Br}_Q(M)\,\mathrm{br}_Q(e).$$

## 1.4 Brauer pairs and the Brauer category

A primitive central idempotent $b \in \mathrm{Z}(\Lambda G)$ is called a *block idempotent of* $G$. Its associated indecomposable subalgebra $\Lambda Gb$ of $\Lambda G$ is called a *block of* $G$.

Since the blocks of $\mathcal{O}G$ and $kG$ correspond to each other via lifting of idempotents, see [Thé95, Theorem 3.1], we will identify blocks of $\mathcal{O}G$ and $kG$ if they correspond to each other via reduction modulo $J(\mathcal{O})$.

A *Brauer pair* $(R, e)$ of $G$ consists of an $\ell$-subgroup $R$ of $G$ and a block $e$ of the group algebra $k\,\mathrm{C}_G(R)$.

**Definition 1.1.** Let $(S, f)$ and $(R, e)$ be two Brauer pairs. Then we write $(S, f) \trianglelefteq (R, e)$ if the following conditions are satisfied:

a) $S$ is a normal subgroup of $R$,

b) $f$ is $R$-stable,

c) $e\,\mathrm{br}_R(f) = e$.

We denote by " $\leq$ " the transitive closure of the relation " $\trianglelefteq$ " on the set of Brauer pairs, see [Thé95, Chapter 40]. We recall the following definition.

**Definition 1.2.** Let $b$ be a block of $kG$. A *$b$-Brauer pair* is a Brauer pair $(R, e)$ such that $(1, b) \leq (R, e)$ or equivalently $\mathrm{br}_R(b)e = e$.

An $\ell$-subgroup $D$ of $G$ is called a *defect group* of the block $b$ if there exists a maximal Brauer pair $(D, e)$ such that $(1, b) \leq (D, e)$. Such a Brauer pair is called maximal $b$-Brauer pair. Moreover, the order relation " $\leq$ " coincides with the ordinary subgroup inclusion inside a fixed maximal $b$-Brauer pair, see [Thé95, Corollary 40.9(b)]. We recall the definition of the Brauer category of a block, see [Thé95, § 47].

**Definition 1.3.** Let $b$ be a block of $G$. Denote by $\mathcal{F}(G, b)$ the category whose objects are the $b$-Brauer pairs and with set of morphisms from $(S, f)$ to $(R, e)$ consisting of all homomorphisms $S \to R$ which are given by conjugation with some $g \in G$ such that ${}^g(S, f) \leq (R, e)$. We say that $\mathcal{F}(G, b)$ is the *Brauer category* of the $G$-block $b$.

If $H$ is a subgroup of $G$ and $f \in \mathrm{Z}(\Lambda H)$ then we write

$$\mathrm{N}_G(H, f) := \{x \in \mathrm{N}_G(H) \mid {}^x f = f\}$$

for the set of elements normalizing $H$ and $f$. Moreover, we write $\mathrm{Tr}_H^G(f) = \sum_{x \in G/H} {}^x f \in \mathrm{Z}(\Lambda G)$ for the trace of the element $f$.

**Remark 1.4.** Denote $\mathcal{F} = \mathcal{F}(G, b)$. It is immediate from the definition of the Brauer category that $\mathrm{Hom}_{\mathcal{F}}((R, e), (R, e)) = \mathrm{Aut}_{\mathcal{F}}(R, e) = \mathrm{N}_G(R, e)/\mathrm{C}_G(R)$.

Let $(D, b_D)$ be a maximal $b$-Brauer pair. We denote by $\mathcal{F}(G, D)_{\leq (D, b_D)}$ the full subcategory of $\mathcal{F}(G, b)$ with objects consisting of all $b$-Brauer pairs contained in $(D, b_D)$. A well known theorem asserts that the natural inclusion functor $\mathcal{F}(G, b)_{\leq (D, b_D)} \hookrightarrow \mathcal{F}(G, b)$ induces an equivalence of categories, see e.g. [Thé95, Lemma 47.1] and afterwards.

For a block $e$ of $k\,\mathrm{C}_G(Q)$ we call $[\mathrm{N}_G(Q, e) : \mathrm{C}_G(Q)Q]$ the inertial index of the block $e$. We recall Brauer's first main theorem, see for instance [Thé95, Theorem 40.14]:

**Theorem 1.5** (Brauer's first main theorem)**.** *Let $D$ be an $\ell$-subgroup of $G$. There is a bijection between the set of blocks of $\mathcal{O}G$ with defect group $D$ and the set of all $\mathrm{N}_G(D)$-conjugacy classes of blocks $e$ of $k\,\mathrm{C}_G(D)$ with defect group $\mathrm{Z}(D)$ and inertial index of $\ell'$-order. The bijection maps a block $b$ to the unique $\mathrm{N}_G(D)$-conjugacy class of blocks $e$ such that $(D, e)$ is a maximal $b$-Brauer pair.*

We will be interested in the blocks of normalizers of $\ell$-subgroups. For these the following remark is useful:

**Remark 1.6.** Let $b$ be a block of $G$ and $(Q, b_Q)$ a $b$-Brauer pair. Then the idempotent $b_Q$ is a block of $\mathrm{N}_G(Q, b_Q)$ by [Thé95, Exercise 40.2(b)]. Consequently, $B_Q := \mathrm{Tr}_{\mathrm{N}_G(Q,b_Q)}^{\mathrm{N}_G(Q)}(b_Q)$ is a block of $\mathrm{N}_G(Q)$. By Theorem 1.5 all maximal $b$-Brauer pairs are $G$-conjugate. In particular, if $(D, b_D)$ is a maximal $b$-Brauer pair then it follows that

$$\mathrm{br}_D(b) = \mathrm{Tr}_{\mathrm{N}_G(D,b_D)}^{\mathrm{N}_G(D)}(b_D).$$

For a finite group $G$ and $D$ an $\ell$-subgroup of $G$ we denote by $\mathrm{Bl}(G \mid D)$ the set of blocks of $G$ with defect group $D$. Then Brauer's first main theorem implies that the map

$$\mathrm{br}_D : \mathrm{Bl}(G \mid D) \to \mathrm{Bl}(\mathrm{N}_G(D) \mid D)$$

is a bijection. This bijection is sometimes referred to as the *Brauer correspondence.*

## 1.5 Morita equivalences and splendid Rickard equivalences

In this section we introduce some equivalences between module categories which play an important role in the representation theory of finite groups.

For this, let $G$ and $H$ be finite groups and let $e \in \mathrm{Z}(\Lambda G)$ and $f \in \mathrm{Z}(\Lambda H)$ be central idempotents. In addition, denote $A = \Lambda G e$ and $B = \Lambda H f$.

**Definition 1.7.** Let $C$ be a bounded complex of $A$-$B$-bimodules, finitely generated and projective as $A$-modules resp. $B$-modules. We say that $C$ induces a *Rickard equivalence between $A$ and $B$* if the following holds:

a) The canonical map $A \to \mathrm{End}^\bullet_{B^{\mathrm{opp}}}(C)^{\mathrm{opp}}$ is an isomorphism in $\mathrm{Ho}^b(A \otimes_\Lambda A^{\mathrm{opp}})$ and

b) the canonical map $B \to \mathrm{End}^\bullet_A(C)$ is an isomorphism in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$.

An important special case of the previous definition is the following:

**Definition 1.8.** Let $M$ be an $A$-$B$-bimodule. If the complex $M[0]$ induces a Rickard equivalence between $A$ and $B$, then we say that $M$ induces a *Morita equivalence* between $A$ and $B$.

We will often use the following well-known lemma, which is essentially contained in [Zim14, Lemma 6.7.12] and its proof. To give a more direct proof, we follow a strategy outlined in the proof of [Har99, Theorem 1.6].

**Lemma 1.9.** *Suppose that the complex $C$ of $A$-$B$-bimodules induces a Rickard equivalence between $A$ and $B$. Let $f = c_1 + \cdots + c_r$ be a decomposition of $f$ into blocks of $B$. Then for each $i$ there exists a unique block $b_i$ of $A$ such that $b_iCc_i$ is not homotopy equivalent to $0$ and $b_iCc_i$ induces a Rickard equivalence between $\Lambda Gb_i$ and $\Lambda Hc_i$.*

*Proof.* We fix a block $c$ of $\Lambda Hf$. By definition the natural map $B \to \mathrm{End}^\bullet_A(C)$ is an isomorphism in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$. Therefore, the natural map $Bc \to \mathrm{End}^\bullet_A(Cc)$ is an isomorphism in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$. Furthermore, we have a direct sum decomposition $Cc = \bigoplus_{i=1}^s b_iCc$. Therefore, we have an isomorphism

$$\mathrm{End}_{B\otimes B^{\mathrm{opp}}}(Bc) \cong \mathrm{End}_{\mathrm{Ho}^b(B\otimes B^{\mathrm{opp}})}(\mathrm{End}^\bullet_A(Cc)) \cong \prod_{i=1}^s \mathrm{End}_{\mathrm{Ho}^b(B\otimes B^{\mathrm{opp}})}(\mathrm{End}^\bullet_A(b_iCc))$$

of $\Lambda$-algebras. Note that $\mathrm{Z}(Bc) \cong \mathrm{End}_{B\otimes B^{\mathrm{opp}}}(Bc)$ is a local $\Lambda$-algebra since $Bc$ is an indecomposable $B$-$B$-bimodule. It follows that there exists a unique integer $i$ such that the block $b := b_i$ of $A$ satisfies that $\mathrm{End}^\bullet_A(bCc)$ is not isomorphic to 0 in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$.

For $j \neq i$ we denote $X := b_jCc$ and we claim that $X \cong 0$ in $\mathrm{Ho}^b(A \otimes_\Lambda B^{\mathrm{opp}})$. We have $X^\vee \otimes_A X \cong \mathrm{End}^\bullet_A(X) \cong 0$ in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$. On the other hand, since $X$ is a bi-projective complex of $A$-$B$-bimodules, it follows by the proof of [Ric96, Theorem 2.1] that $X$ is a direct summand of $X \otimes_B (X^\vee \otimes_A X)$ in $\mathrm{Comp}^b(A \otimes_\Lambda B^{\mathrm{opp}})$. From this we deduce that $X \cong 0$ in $\mathrm{Ho}^b(A \otimes_\Lambda B^{\mathrm{opp}})$.

Therefore, $bCc \cong Cc$ in $\mathrm{Ho}^b(A \otimes_\Lambda B^{\mathrm{opp}})$ and it follows that the natural map $Bc \to \mathrm{End}^\bullet_A(bCc)$ is an isomorphism in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$. Similarly, one shows that the natural map $Ab \to \mathrm{End}^\bullet_{B^{\mathrm{opp}}}(bC)^{\mathrm{opp}}$ is an isomorphism in $\mathrm{Ho}^b(A \otimes_\Lambda A^{\mathrm{opp}})$. In other words, the complex $bCc$ induces a Rickard equivalence between $Ab = \Lambda Gb$ and $Bc = \Lambda Hc$. □

If $C$ is a complex inducing a Rickard equivalence between $A$ and $B$, then the functor

$$C \otimes_B - : \mathrm{Ho}^b(B\text{-proj}) \to \mathrm{Ho}^b(A\text{-proj})$$

yields an equivalence of categories. We now define the seemingly weaker notion of derived equivalence:

**Definition 1.10.** We say that a complex $C \in \mathrm{Comp}^b(A \otimes B^{\mathrm{opp}})$ induces a *derived equivalence* between $A$ and $B$ if the functor

$$C \otimes^{\mathbb{L}}_B - : \mathrm{D}^b(B) \to \mathrm{D}^b(A)$$

induces an equivalence of triangulated categories.

**Remark 1.11.** A theorem of Rickard, see [Ric96, Section 2.1], asserts that $A$ and $B$ are Rickard equivalent if and only if they are derived equivalent. More precisely, the proof of said theorem shows, that not every complex $C \in \mathrm{Comp}^b(A \otimes B^{\mathrm{opp}})$ inducing a derived equivalence between $A$ and $B$ gives necessarily rise to a Rickard equivalence between $A$ and $B$. There only exists a complex isomorphic to $C$ in $\mathrm{D}^b(A \otimes B^{\mathrm{opp}})$ which induces a Rickard equivalence between $A$ and $B$.

Assume now that $H$ is a subgroup of $G$. For any subgroup $X$ of $H$ we let $\Delta X := \{(x, x^{-1}) \mid x \in X\}$, a subgroup of $G \times H^{\mathrm{opp}}$.

**Definition 1.12.** A bounded complex $C$ of $A$-$B$-bimodules is called *splendid* if $C^{\mathrm{red}}$ is a complex of $\ell$-permutation modules such that every indecomposable direct summands of a component of $C$ has a vertex contained in $\Delta H$. If $C$ is splendid and induces a Rickard equivalence between $A$ and $B$ we say that $C$ induces a *splendid Rickard equivalence* between $A$ and $B$.

Note that our definition of a splendid Rickard equivalence is not symmetric since we assume that $L$ is a subgroup of $G$.

## 1.6 First properties of splendid complexes

We state some important first properties of splendid complexes. The following is a variant of [Ric96, Lemma 4.3].

**Lemma 1.13.** *Let $L$ be a subgroup of a finite group $G$. Let $M$ be a relatively $\Delta L$-projective $\Lambda[G \times L^{\mathrm{opp}}]$-module. If $Q$ is a subgroup of $L$ then all indecomposable direct summands of $\mathrm{Res}^{G\times L^{\mathrm{opp}}}_{G\times Q^{\mathrm{opp}}}(M)$ are relatively $\Delta Q$-projective.*

*Proof.* Since $M$ is a relatively $\Delta L$-projective module we may assume $M = \mathrm{Ind}^{G\times L^{\mathrm{opp}}}_{\Delta L}(N)$ for some $\Lambda[\Delta L]$-module $N$. There exists a set of representatives of the double cosets of $\Delta L \backslash (G \times L^{\mathrm{opp}})/(G \times Q^{\mathrm{opp}})$ contained in $1 \times L^{\mathrm{opp}}$. By Mackey's formula for every indecomposable direct summand of $\mathrm{Res}^{G\times L^{\mathrm{opp}}}_{G\times Q^{\mathrm{opp}}}(M)$ there exists some $l \in L$ such that this summand is relatively projective with respect to the subgroup

$$\Delta L^{(1,l)} \cap (G \times Q^{\mathrm{opp}}).$$

We have

$$^{(l,1)}(\Delta L^{(1,l)} \cap (G \times Q^{\mathrm{opp}})) = {}^{(l,l)}\Delta L \cap {}^{(l,1)}(G \times Q^{\mathrm{opp}}) = \Delta Q$$

which shows that every indecomposable summand is projective relative to a subgroup which is $G \times Q^{\mathrm{opp}}$-conjugate to $\Delta Q$. It follows that every indecomposable summand is relatively $\Delta Q$-projective. □

Let $L$ be a subgroup of a finite group $G$ and $Q$ an $\ell$-subgroup of $L$. Then we can consider the Brauer functor

$$\mathrm{Br}_{\Delta Q} : \Lambda[G \times L^{\mathrm{opp}}] - \mathrm{perm} \to k\,\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q)/\Delta Q - \mathrm{perm}\,.$$

Notice that

$$\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q) = (\mathrm{C}_G(Q) \times \mathrm{C}_L(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_L(Q)).$$

Let $c \in \mathrm{Z}(\Lambda L)$ and $b \in \mathrm{Z}(\Lambda G)$ be two central idempotents and $C$ a bounded complex of $\Lambda Gb$-$\Lambda Lc$-modules. Since

$$\mathrm{C}_{G\times L^{\mathrm{opp}}}(\Delta Q) = \mathrm{C}_G(Q) \times \mathrm{C}_L(Q)^{\mathrm{opp}} \subseteq \mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q)$$

we can consider the image $\mathrm{Br}_{\Delta Q}(C)$ as a complex of $k\,\mathrm{C}_G(Q)\,\mathrm{br}_Q(b)$-$k\,\mathrm{C}_L(Q)\,\mathrm{br}_Q(c)$ bimodules.

In the following lemma we closely follow the proof of [Ric96, Theorem 4.1].

**Lemma 1.14.** *Assume the notation as above and suppose that $C_1$ and $C_2$ are splendid complexes of $\Lambda Gb$-$\Lambda Lc$ bimodules. Then for any $\ell$-subgroup $Q$ of $L$ we have*

$$\mathrm{Br}_{\Delta Q}(C_1^\vee \otimes_{\Lambda G} C_2) \cong \mathrm{Br}_{\Delta Q}(C_1^\vee) \otimes_{k\,\mathrm{C}_G(Q)} \mathrm{Br}_{\Delta Q}(C_2)$$

*in* $\mathrm{Ho}^b(k[\mathrm{C}_L(Q) \times \mathrm{C}_L(Q)^{\mathrm{opp}}])$.

*Proof.* The complex $\mathrm{Hom}_k^\bullet(C_1, C_2)$ viewed as a complex of $k[G\times L^{\mathrm{opp}}]$-modules via the diagonal action is again a complex consisting of relatively $\Delta L$-projective $\ell$-permutation modules, see [Ben98, Corollary 3.3.5].

It follows that $\mathrm{Res}^{G\times L^{\mathrm{opp}}}_{G\times Q^{\mathrm{opp}}}(\mathrm{Hom}_k^\bullet(C_1, C_2))$ is a complex of relatively $\Delta Q$-projective modules, see Lemma 1.13. Therefore, by the proof of [Ric96, Theorem 4.1] we deduce that

$$\mathrm{Br}_{\Delta Q}(\mathrm{Hom}^\bullet_{kG}(C_1, C_2)) \cong \mathrm{Hom}^\bullet_{k\mathrm{C}_G(Q)}(\mathrm{Br}_{\Delta Q}(C_1), \mathrm{Br}_{\Delta Q}(C_2)).$$

By Lemma 1.13 we also see that $C_2$ and $\mathrm{Br}_{\Delta Q}(C_2)$ are complexes of projective left $kG$-modules and $k\,\mathrm{C}_G(Q)$-modules respectively. We obtain $\mathrm{Hom}^\bullet_{kG}(C_1, C_2) \cong C_1^\vee \otimes_{kG} C_2$ and

$$\mathrm{Hom}^\bullet_{k\mathrm{C}_G(Q)}(\mathrm{Br}_{\Delta Q}(C_1), \mathrm{Br}_{\Delta Q}(C_2)) \cong \mathrm{Br}_{\Delta Q}(C_1)^\vee \otimes_{k\,\mathrm{C}_G(Q)} \mathrm{Br}_{\Delta Q}(C_2).$$

By the proof of [Bro85, Lemma 2.4(2)] we have $\mathrm{Br}_{\Delta Q}(C_1)^\vee \cong \mathrm{Br}_{\Delta Q}(C_1^\vee)$, which proves the claim. $\square$

## 1.7 Brauer categories and splendid Rickard equivalences

In this section we recall an important theorem of Puig showing that the Brauer categories of splendid Rickard equivalent blocks are isomorphic. This will be crucial for many of our applications.

**Theorem 1.15.** *Let $L$ be a subgroup of a finite group $G$. Let $b \in \mathrm{Z}(\Lambda G)$ and $c \in \mathrm{Z}(\Lambda L)$ be primitive idempotents. Suppose that there exists a bounded complex $C$ of $\Lambda Gb$-$\Lambda Lc$-modules inducing a splendid Rickard equivalence between $\Lambda Gb$ and $\Lambda Lc$. If $D$ is a defect group of the block $c$ then $D$ is a defect group of $b$.*

*Proof.* Denote $A = \Lambda Gb$ and $B = \Lambda Lc$. Since $C$ induces a splendid Rickard equivalence between $A$ and $B$ it follows by definition that $B \cong \mathrm{End}^\bullet_A(C)$ in $\mathrm{Ho}^b(B \otimes_\Lambda B^{\mathrm{opp}})$. By Lemma 1.14 it follows that

$$\mathrm{Br}_{\Delta D}(\mathrm{End}^\bullet_A(C)) \cong \mathrm{End}^\bullet_{k\mathrm{C}_G(D)}(\mathrm{Br}_{\Delta D}(C)).$$

Since $\mathrm{Br}_{\Delta D}(B) \cong k\mathrm{C}_L(D)\mathrm{br}_D(c)$ we obtain

$$\mathrm{End}^\bullet_{k\mathrm{C}_G(D)}(\mathrm{Br}_{\Delta D}(C)) \cong k\mathrm{C}_L(D)\mathrm{br}_D(c).$$

Taking cohomology yields $\mathrm{End}_{\mathrm{Ho}^b(k\mathrm{C}_G(D))}(\mathrm{Br}_{\Delta D}(C)) \cong k\mathrm{C}_L(D)\mathrm{br}_D(c)$. Since $D$ is a defect group of $c$ it follows that $\mathrm{br}_D(c) \neq 0$. Therefore, the complex $\mathrm{Br}_{\Delta D}(C)$ is not homotopy equivalent to 0 in $\mathrm{Ho}^b(k\mathrm{C}_G(D))$. As $\mathrm{Br}_{\Delta D}(C)$ is a complex of $k\,\mathrm{C}_G(D)\,\mathrm{br}_D(b)$-$k\,\mathrm{C}_L(D)\,\mathrm{br}_D(c)$ bimodules it follows that $\mathrm{br}_D(b) \neq 0$. This shows that $D$ is contained in a defect group of $b$. Since $C$ induces a splendid Rickard equivalence it follows that $C$ induces a basic Rickard equivalence between the blocks $\Lambda Gb$ and $\Lambda Lc$, see beginning of [Pui99, Section 19.2]. Consequently, [Pui99, Theorem 19.7] shows that the defect groups of $b$ and $c$ are isomorphic. Thus, $D$ is also a defect group of $b$. □

**Proposition 1.16.** *Take the notation as in Theorem 1.15 and fix a maximal $c$-Brauer pair $(D, c_D)$. Then there exists a $b$-Brauer pair $(D, b_D)$ such that the following holds: If $(Q, c_Q) \leq (D, c_D)$ is a $c$-Brauer subpair then the $b$-Brauer subpair $(Q, b_Q) \leq (D, b_D)$ is the unique $b$-Brauer pair such that the complex $b_Q\,\mathrm{Br}_{\Delta Q}(C)c_Q$ induces a Rickard equivalence between $k\,\mathrm{C}_G(Q)b_Q$ and $k\,\mathrm{C}_L(Q)c_Q$. For any other $b$-Brauer pair $(Q, b'_Q)$ we have $b'_Q\,\mathrm{Br}_{\Delta Q}(C)c_Q \cong 0$ in $\mathrm{Ho}^b(k[\mathrm{C}_G(Q) \times \mathrm{C}_L(Q)^{\mathrm{opp}}])$.*

*Proof.* The subgroup $D \subseteq L \subseteq G$ is a common defect group of the blocks $b$ and $c$ by Theorem 1.15. Moreover, the complex $C$ is splendid, so the vertices of all indecomposable direct summands of components of $C$ are by definition contained in $\Delta L$. On the other hand, if $P$ is an $\ell$-subgroup of $L$ then $\mathrm{Br}_{\Delta P}(C) \cong \mathrm{br}_P(b)\,\mathrm{Br}_{\Delta P}(C) \cong 0$, unless $P$ is contained in a defect group of the block $b$. It follows that all indecomposable direct summands of components of $C$ are relatively $\Delta D$-projective. Hence, the complex $C$ induces a splendid Rickard equivalence between $kGb$ and $kLc$ in the sense of [Har99]. The statement is therefore precisely [Har99, Theorem 1.6]. □

Let $b$ be a block of a finite group $G$ and $(D, b_D)$ a maximal $b$-Brauer pair. Recall from Definition 1.3 that we denote by $\mathcal{F}(G, b)$ the Brauer category of the $G$-block $b$ and by $\mathcal{F}(G, D)_{\leq (D, b_D)}$ its full subcategory consisting of all $b$-Brauer pairs contained in $(D, b_D)$.

**Theorem 1.17.** *Suppose that we are in the situation of Proposition 1.16. Then the map $\mathcal{F}(L, c)_{\leq (D, c_D)} \to \mathcal{F}(G, b)_{\leq (D, b_D)}$ given by $(Q, c_Q) \mapsto (Q, b_Q)$ induces an isomorphism of categories. In particular, for any c-Brauer subpairs $(Q, c_Q)$, $(R, c_R)$ contained in $(D, c_D)$ and b-Brauer subpairs $(Q, b_Q)$, $(R, b_R)$ contained in $(D, b_D)$ we have*

$$\mathrm{Hom}_{\mathcal{F}(L,c)}((Q, c_Q), (R, c_R)) = \mathrm{Hom}_{\mathcal{F}(G,b)}((Q, b_Q), (R, b_R)).$$

*Proof.* The paragraph below [Har99, Theorem 1.7] shows that we have an inclusion

$$\mathrm{Hom}_{\mathcal{F}(L,c)}((Q, c_Q), (R, c_R)) \subseteq \mathrm{Hom}_{\mathcal{F}(G,b)}((Q, b_Q), (R, b_R)).$$

By [Pui99, Theorem 19.7] the Brauer categories $\mathcal{F}(L, c)$ and $\mathcal{F}(G, b)$ are equivalent. Consequently, the inclusion above is an equality. □

The following easy corollary will be useful to us.

**Corollary 1.18.** *Suppose that we are in the situation of Proposition 1.16. Then for any subgroup $Q$ of $D$ the inclusion map $\mathrm{N}_L(Q)/\mathrm{C}_L(Q) \hookrightarrow \mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ induces an isomorphism between $\mathrm{N}_L(Q, c_Q)/\mathrm{C}_L(Q)$ and $\mathrm{N}_G(Q, b_Q)/\mathrm{C}_G(Q)$.*

*Proof.* Theorem 1.17 shows that we have an equality

$$\mathrm{Aut}_{\mathcal{F}(L,c)}(Q, c_Q) = \mathrm{Aut}_{\mathcal{F}(G,b)}(Q, b_Q).$$

The corollary follows from this by using Remark 1.4. □

**Remark 1.19.** From Proposition 1.16 it is quite easy to see that we have an injective map $\mathrm{N}_L(Q, c_Q)/\mathrm{C}_L(Q) \hookrightarrow \mathrm{N}_G(Q, b_Q)/\mathrm{C}_G(Q)$. Indeed, $b_Q\,\mathrm{Br}_{\Delta Q}(C) c_Q$

is a complex of $k[(\mathrm{C}_G(Q) \times \mathrm{C}_L(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_L(Q))]$-modules. Thus, for $x \in \mathrm{N}_L(Q)$ the complex

$$^{x}(b_Q \operatorname{Br}_{\Delta Q}(C)c_Q)^{x} \cong {}^{x}b_Q \operatorname{Br}_{\Delta Q}(bCc)\, {}^{x}c_Q$$

of $k\mathrm{C}_G(Q)$-$k\mathrm{C}_L(Q)$-bimodules induces a Rickard equivalence between $k\,\mathrm{C}_G(Q)\,{}^{x}b_Q$ and $k\,\mathrm{C}_L(Q)\,{}^{x}c_Q$. Hence, if $x \in \mathrm{N}_L(Q, c_Q)$ then necessarily ${}^{x}b_Q = b_Q$, since otherwise ${}^{x}b_Q \operatorname{Br}_{\Delta Q}(C)c_Q \cong 0$ by Proposition 1.16.

## 1.8 Properties of splendid Rickard equivalences

In this section we establish some properties of splendid Rickard equivalences. We keep the notation of the previous section. In particular we assume that $L$ is a subgroup of $G$. Furthermore, $b$ and $c$ are block idempotents of $G$ and $L$ respectively.

**Theorem 1.20.** *Suppose that $C_1$ and $C_2$ are two bounded complexes of $\ell$-permutation $\Lambda Gb$-$\Lambda Lc$ bimodules inducing a splendid Rickard equivalence between $\Lambda Gb$ and $\Lambda Lc$. Then the tensor product $C_1^{\vee} \otimes_{\Lambda G} C_2$ induces a splendid Rickard self-equivalence of $\Lambda Lc$.*

*Proof.* It is clear that the tensor product $C_1^{\vee} \otimes_{\Lambda G} C_2$ induces a Rickard equivalence between $\Lambda Lc$ and itself. Therefore, it suffices to prove that the complex $C_1^{\vee} \otimes_{\Lambda G} C_2$ is splendid. This however follows as in the proof of [Rou98, Lemma 10.2.6] by replacing both $H'$ and $D$ by $L$. □

The following lifting theorem from $k$ to $\mathcal{O}$ is crucial and illustrates the strength of the notion of splendidness.

**Theorem 1.21.** *Let $e \in \mathrm{Z}(\Lambda G)$ and $f \in \mathrm{Z}(\Lambda L)$ be central idempotents. Suppose that $C \in \mathrm{Comp}(kGe \otimes_k kLf\text{-perm})$ is a complex inducing a splendid Rickard equivalence between $kGe$ and $kLf$. Then there exists a complex $\tilde{C}$, unique up to isomorphism, inducing a splendid Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$ and satisfying $\tilde{C} \otimes_{\mathcal{O}} k \cong C$.*

*Proof.* This follows from the proof of [Ric96, Theorem 5.2]. □

**Theorem 1.22.** *Let $G$ and $H$ be finite groups and assume that $kGe$ and $kHf$ are blocks.*

*a) Let $C$ be a bounded complex of finitely generated biprojective $kGe$-$kHf$-bimodules. Suppose that $\mathrm{End}^{\bullet}_{kHf}(C) \cong kGe$ in $\mathrm{Ho}^b(kG \otimes_k (kG)^{\mathrm{opp}})$. Then $kGe$ and $kHf$ are Rickard equivalent via $C$.*

*b) Let $M$ be a finitely generated biprojective $kGb$-$kHc$-bimodule. Suppose that $\mathrm{End}_{kHf}(M) \cong kGe$ as $kG$-$kG$-bimodules. Then $kGe$ and $kHf$ are Morita equivalent via $M$.*

*Proof.* The first item is proved in [Ric96, Theorem 2.1]. The second part follows from the first by taking $C := M[0]$. □

We recall the following useful observation, see [Rou98, Section 10.2.3], where it was also explained that the converse statement does not necessarily hold.

**Lemma 1.23.** *Let $G$ and $H$ be finite groups and assume that $e \in \mathrm{Z}(\Lambda G)$ and $f \in \mathrm{Z}(\Lambda H)$ are central idempotents. Suppose that $C$ is a complex of $\Lambda Ge$-$\Lambda Hf$-bimodules inducing a Rickard equivalence between $\Lambda Ge$ and $\Lambda Hf$. If the cohomology of $C$ is concentrated in degree $d$ then $H^d(C)$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Hf$.*

*Proof.* As $C$ induces a Rickard equivalence we have $\mathrm{End}^\bullet_{\Lambda G}(C) \cong \Lambda Hf$ in $\mathrm{Ho}^b(\Lambda Hf \otimes_\Lambda (\Lambda Hf)^{\mathrm{opp}})$ by definition. The cohomology of $C$ is concentrated in degree $d$ and $C$ is a complex of projective $\Lambda Gb$-modules. By [Ben98, Theorem 2.7.1] we therefore obtain

$$H^0(\mathrm{End}^\bullet_{\Lambda G}(C)) \cong \mathrm{End}_{\Lambda G}(H^d(C)) \cong \Lambda Hf$$

as $\Lambda Hf \otimes_\Lambda (\Lambda Hf)^{\mathrm{opp}}$-modules. Similarly, one shows that $\mathrm{End}_{\Lambda H}(H^d(C)) \cong \Lambda Ge$. Hence, the bimodule $H^d(C)$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Hf$. □

## 1.9 Lifting Rickard equivalences

The aim of this section is to introduce a lifting result for Morita equivalences due to Marcus. We first need to introduce some notation. Let $\tilde{L}$ be a subgroup of a finite group $\tilde{G}$. Moreover, let $G$ be a normal subgroup of $\tilde{G}$ and set $L := G \cap \tilde{L}$. In this case, we have an injective map $\tilde{L}/L \hookrightarrow \tilde{G}/G$, which is an isomorphism if and only if $\tilde{L}G = \tilde{G}$.

Let $e \in \mathrm{Z}(\mathcal{O}G)$ and $f \in \mathrm{Z}(\mathcal{O}L)$ be $\tilde{G}$-invariant resp. $\tilde{L}$-invariant central idempotents, such that $e \in \mathrm{Z}(\mathcal{O}\tilde{G})$ and $f \in \mathrm{Z}(\mathcal{O}\tilde{L})$. Consider the subgroup

$$\mathcal{D} := \{(\tilde{g}, \tilde{l}) \in \tilde{G} \times \tilde{L}^{\mathrm{opp}} \mid \tilde{g}G = \tilde{l}^{-1}G\} = (G \times L^{\mathrm{opp}})\Delta(\tilde{L})$$

of $\tilde{G} \times \tilde{L}^{\mathrm{opp}}$.

The following was first proved in [Mar96, Theorem 3.4]. An alternative proof can be found in [Rou98, Lemma 2.8].

**Theorem 1.24** (Marcus)**.** *Suppose that $\tilde{G} = \tilde{L}G$. Let $C$ be a bounded complex of $\Lambda Ge$-$\Lambda Lf$-bimodules inducing a Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$. Suppose that either $C$ is concentrated in one degree or that $\ell \nmid [\tilde{L} : L]$. If $C$ extends to a complex of $\mathcal{D}$-modules $C'$ then $\tilde{C} := \mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')$ induces a Rickard equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$.*

*Proof.* The statement has been proved in the case where $e$ and $f$ are primitive central idempotents in [Rou98, Lemma 2.8]. However, the assumption in the proof of [Rou98, Lemma 2.8] that $e$ and $f$ are primitive is not necessary. $\square$

**Remark 1.25.** As said in [Rou02, Remark 5.4] if we drop the assumption that $[\tilde{L} : L]$ is coprime to $\ell$ in Theorem 1.24 it is still true that $\mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')$ induces a derived equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$.

In the following remark we observe some Clifford-theoretic consequences of Theorem 1.24.

**Remark 1.26.**

a) Suppose that we are in the situation of Theorem 1.24. Let $\varphi : \mathrm{Ho}^b(\Lambda Lf) \to \mathrm{Ho}^b(\Lambda Ge)$ and $\tilde{\varphi} : \mathrm{Ho}^b(\Lambda\tilde{L}f) \to \mathrm{Ho}^b(\Lambda\tilde{G}e)$ be the functors induced by tensoring with $C$ resp. $\tilde{C}$.

   Let $N$ be a complex of $\Lambda\tilde{L}f$-modules. Then by Mackey's formula $\mathrm{Res}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}_{G\times\tilde{L}^{\mathrm{opp}}}(\tilde{C}) \cong \mathrm{Ind}^{G\times\tilde{L}^{\mathrm{opp}}}_{G\times L^{\mathrm{opp}}}(C)$. In particular, we have

   $$\mathrm{Res}^{\tilde{G}}_{G}(\tilde{C}\otimes_{\Lambda\tilde{L}}N) \cong \mathrm{Ind}^{G\times\tilde{L}^{\mathrm{opp}}}_{G\times L^{\mathrm{opp}}}(C)\otimes_{\Lambda\tilde{L}}N \cong (C\otimes_{\Lambda L}\Lambda\tilde{L})\otimes_{\Lambda\tilde{L}}N \cong C\otimes_{\Lambda L}\mathrm{Res}^{\tilde{L}}_{L}(N).$$

   In other words, $\mathrm{Res}^{\tilde{G}}_{G} \circ \tilde{\varphi} \cong \varphi \circ \mathrm{Res}^{\tilde{L}}_{L}$. A similar calculation (or using the fact that Ind and Res are adjoint functors) shows that $\mathrm{Ind}^{\tilde{G}}_{G} \circ \varphi \cong \tilde{\varphi} \circ \mathrm{Ind}^{\tilde{L}}_{L}$.

b) Let $M$ be an $\mathcal{O}Ge$-$\mathcal{O}Lf$ bimodule inducing a Morita equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Lf$. Suppose that $M$ extends to an $\mathcal{O}\mathcal{D}$-module $M'$ and denote $\tilde{M} := \mathrm{Ind}_{\Delta}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(M')$.

   For $R \in \{K, k\}$ the bimodule $M \otimes_{\mathcal{O}} R$ (respectively $\tilde{M} \otimes_{\mathcal{O}} R$) induces a bijection $\varphi : \mathrm{Irr}(RLf) \to \mathrm{Irr}(RGe)$ (respectively $\tilde{\varphi} : \mathrm{Irr}(R\tilde{L}f) \to \mathrm{Irr}(R\tilde{G}e)$) between irreducible modules. Now suppose that $N$ is a simple $R\tilde{L}f$-module. By Clifford's theorem, see [NT89, Theorem 3.3.1], there exists a simple $RL$-module $S$ and an integer $m$ such that

   $$\mathrm{Res}^{\tilde{L}}_{L}(N) \cong (\bigoplus_{\tilde{l}} {}^{\tilde{l}}S)^m,$$

where $\tilde{l} \in \tilde{L}$ runs over a set of representatives of the non-isomorphic $\tilde{L}$-conjugates ${}^{\tilde{l}}S$. Since $M \otimes_{RL} {}^{\tilde{l}}S \cong {}^{\tilde{l}}M^{\tilde{l}} \otimes_{RL} {}^{\tilde{l}}S \cong {}^{\tilde{l}}(M \otimes_{\Lambda L} S)$ we deduce by part (a) that

$$\mathrm{Res}_G^{\tilde{G}}(\tilde{\varphi}(N)) = \varphi(\mathrm{Res}_L^{\tilde{L}}(N)) \cong (\bigoplus_{\tilde{l}} \varphi({}^{\tilde{l}}S))^m \cong (\bigoplus_{\tilde{g}} {}^{\tilde{g}}\varphi(S))^m,$$

where $\tilde{g} \in \tilde{G}$ runs over a set of representatives of the non-isomorphic $\tilde{G}$-conjugates ${}^{\tilde{g}}\varphi(S)$. In particular, we see that the simple $RL$-module $S$ extends to an $R\tilde{L}$-module if and only if $\varphi(S)$ extends to an $R\tilde{G}$-module.

## 1.10 Descent of Rickard equivalences

We keep the assumptions of the previous section. Theorem 1.24 shows that under certain conditions Rickard equivalences can be lifted from normal subgroups. It is therefore natural to ask whether one can also go the other way. For Rickard equivalences we obtain the following converse to Theorem 1.24 which is tailored to our later applications.

**Lemma 1.27.** *Suppose that $\tilde{G} = \tilde{L}G$. Let $C$ be a bounded complex of biprojective $\Lambda Ge$-$\Lambda Lf$-bimodules with cohomology concentrated in degree $d$ such that $H^d(C)$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Lf$. Assume that $C$ extends to a complex of $\Lambda\mathcal{D}$-modules $C'$ such that $\tilde{C} := \mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')$ induces a Rickard equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$. Then also the complex $C$ induces a Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$.*

*Proof.* By the Mackey formula we have

$$\mathrm{Res}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}_{G\times\tilde{L}^{\mathrm{opp}}}(\tilde{C}) \cong C \otimes_{\Lambda L} \Lambda\tilde{L} \text{ and } \mathrm{Res}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}_{\tilde{G}\times L^{\mathrm{opp}}}(\tilde{C}) \cong \Lambda\tilde{G} \otimes_{\Lambda G} C.$$

Since $\tilde{C}$ induces a Rickard equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$ we therefore conclude that

$$\mathrm{Res}^{\tilde{G}\times\tilde{G}^{\mathrm{opp}}}_{G\times\tilde{G}^{\mathrm{opp}}}(\Lambda\tilde{G}e) \cong C \otimes_{\Lambda L} \Lambda\tilde{L} \otimes_{\Lambda\tilde{L}} \tilde{C}^\vee \cong C \otimes_{\Lambda L} C^\vee \otimes_{\Lambda G} \Lambda\tilde{G}.$$

Since $H^d(C)$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Lf$ it follows by the remarks before [Rou98, Lemma 10.2.4] we have an isomorphism

$$C \otimes_{\Lambda L} C^\vee \cong \Lambda Ge \oplus R$$

in $\mathrm{Comp}^b(\Lambda[G \times G^{\mathrm{opp}}])$, where $R$ is a complex of $\Lambda Ge$-$\Lambda Ge$-bimodules such that $H^\bullet(R) \cong 0$ (but not necessarily homotopy equivalent to 0). From this we deduce that

$$\mathrm{Res}^{\tilde{G}\times\tilde{G}^{\mathrm{opp}}}_{G\times\tilde{G}^{\mathrm{opp}}}(\Lambda\tilde{G}e) \cong C \otimes_{\Lambda L} C^\vee \otimes_{\Lambda G} \Lambda\tilde{G} \cong \mathrm{Res}^{\tilde{G}\times\tilde{G}^{\mathrm{opp}}}_{G\times\tilde{G}^{\mathrm{opp}}}(\Lambda\tilde{G}e) \oplus (R \otimes_{\Lambda G} \Lambda\tilde{G})$$

in $\mathrm{Ho}^b(\Lambda[G \times \tilde{G}^{\mathrm{opp}}])$. We conclude that

$$\mathrm{Ind}_{G\times G^{\mathrm{opp}}}^{G\times\tilde{G}^{\mathrm{opp}}}(R) \cong R \otimes_{\Lambda G} \Lambda\tilde{G} \cong 0$$

in $\mathrm{Ho}^b(\Lambda[G\times\tilde{G}^{\mathrm{opp}}])$. Since $R$ is a direct summand of $\mathrm{Res}_{G\times G^{\mathrm{opp}}}^{G\times\tilde{G}^{\mathrm{opp}}}(\mathrm{Ind}_{G\times G^{\mathrm{opp}}}^{G\times\tilde{G}^{\mathrm{opp}}}(R))$ as a complex we thus have $R \cong 0$ in $\mathrm{Ho}^b(\Lambda[G \times G^{\mathrm{opp}}])$. This shows that $C \otimes_{\Lambda L} C^\vee \cong \Lambda Ge$ in $\mathrm{Ho}^b(\Lambda[G \times G^{\mathrm{opp}}])$ and similarly one proves $C^\vee \otimes_{\Lambda G} C \cong \Lambda Lf$ in $\mathrm{Ho}^b(\Lambda[L \times L^{\mathrm{opp}}])$. Consequently, the complex $C$ induces a Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$. □

It would be interesting to know whether the hypothesis that $C$ has cohomology concentrated in degree $d$ such that $H^d(C)$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Lf$ could be weakened or even completely removed. For Morita equivalences the following lemma shows that the situation is much easier:

**Lemma 1.28.** *Suppose that $\tilde{L}G = \tilde{G}$. Let $M$ be a biprojective $\Lambda Ge$-$\Lambda Lf$-bimodule and suppose that $M$ extends to a $\Lambda\mathcal{D}$-module $M'$ such that $\tilde{M} := \mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(M')$ induces a Morita equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$. Then $M$ induces a Morita equivalence between $\Lambda Ge$ and $\Lambda Lf$.*

*Proof.* Since $\tilde{M}$ induces a Morita equivalence between $\Lambda\tilde{L}f$ and $\Lambda\tilde{G}e$ it follows that the natural map $\Lambda\tilde{G}e \to \mathrm{End}_{(\Lambda\tilde{L})^{\mathrm{opp}}}(\tilde{M})^{\mathrm{opp}}$ is an isomorphism. This shows that the natural map

$$\Lambda\tilde{G}e \to \mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(\mathrm{Res}_{\tilde{G}\times L^{\mathrm{opp}}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(\tilde{M}))^{\mathrm{opp}} \cong \mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(\Lambda\tilde{G} \otimes_{\Lambda G} M)^{\mathrm{opp}}$$

is injective. From this it follows that the natural map $\Lambda Ge \to \mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(M)^{\mathrm{opp}}$ is injective as well. Since $\Lambda Ge$ is projective as right $\Lambda G$-module it follows that the map $\Lambda Ge \to \mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(M)^{\mathrm{opp}}$ is a split injection of right $\Lambda G$-modules. Consequently, there exists a right $\Lambda G$-module $R$ such that

$$\mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(M)^{\mathrm{opp}} \cong M \otimes_{\Lambda L} M^\vee \cong \Lambda Ge \oplus R$$

as right $\Lambda G$-modules. We now want to show that $R \cong 0$. According to the proof of Lemma 1.27 we have

$$\mathrm{Res}_{G\times\tilde{G}^{\mathrm{opp}}}^{\tilde{G}\times\tilde{G}^{\mathrm{opp}}}(\Lambda\tilde{G}e) \cong M \otimes_{\Lambda L} M^\vee \otimes_{\Lambda G} \Lambda\tilde{G}.$$

It follows that

$$\Lambda\tilde{G}e \cong \Lambda\tilde{G}e \oplus (R \otimes_{\Lambda G} \Lambda\tilde{G}e)$$

as right $\Lambda\tilde{G}$-modules. We conclude that $R \otimes_{\Lambda G} \Lambda\tilde{G} \cong 0$ which implies that $R \cong 0$. Hence, the natural map $\Lambda Ge \to \mathrm{End}_{(\Lambda L)^{\mathrm{opp}}}(M)^{\mathrm{opp}}$ is an isomorphism. Similarly, one shows that $\Lambda Lf \to \mathrm{End}_{\Lambda G}(M)$ is an isomorphism. □

## 1.11 Morita equivalences and Clifford theory of characters

Let $G$ be a finite group and $e \in \mathrm{Z}(\mathcal{O}G)$ be a central idempotent. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible $KG$-modules and by $\mathrm{Irr}(G, e)$ the subset of isomorphism classes of irreducible $KGe$-modules.

We recall the following theorem due to Broué.

**Theorem 1.29** (Broué)**.** *Let $G$ and $H$ be finite groups and $e \in Z(\mathcal{O}G)$ and $f \in \mathrm{Z}(\mathcal{O}H)$ be central idempotents. Let $M$ be an $\mathcal{O}G$-$\mathcal{O}H$-bimodule which is projective as both $\mathcal{O}G$- and as $\mathcal{O}H$-module. Assume that the functor*

$$M \otimes_{KH} - : KH\text{-mod} \to KG\text{-mod}$$

*induces a bijection between $\mathrm{Irr}(H, f)$ and $\mathrm{Irr}(G, e)$ then the $\mathcal{O}Ge$-$\mathcal{O}Hf$-bimodule $Mf$ induces a Morita equivalence between $\mathcal{O}Hf$ and $\mathcal{O}Ge$.*

*Proof.* See [CE04, Theorem 9.18]. □

Theorem 1.29 is in particular useful if one has already constructed a bimodule as candidate for a Morita equivalence. However, the hard part is usually to find such a bimodule.

We now give an application of this theorem. Let $N$ be a normal subgroup of a finite group $G$. Let $f$ be a central idempotent of $\mathcal{O}N$. Let $H$ be the stabilizer of $f$ in $G$. Then $f$ is a central idempotent of $\mathcal{O}H$. We suppose that $f({}^x f) = 0$ for any $x \in G \setminus H$. This ensures that $F := \mathrm{Tr}_H^G(f) = \sum_{g \in G/H} {}^g f$ is an idempotent of $G$. By definition it is clearly central in $\mathcal{O}G$.

Consider the induction functor

$$\mathrm{Ind}_H^G : \mathcal{O}Hf\text{-mod} \to \mathcal{O}GF\text{-mod},\ X \mapsto \mathcal{O}Gf \otimes_{\mathcal{O}H} X$$

and the restriction functor

$$f\mathrm{Res}_H^G : \mathcal{O}GF\text{-mod} \to \mathcal{O}Hf\text{-mod},\ Y \mapsto f\mathcal{O}G \otimes_{\mathcal{O}G} Y.$$

**Lemma 1.30.** *The $\mathcal{O}GF$-$\mathcal{O}Hf$-bimodule $\mathcal{O}Gf$ induces a Morita equivalence between $\mathcal{O}GF$-mod and $\mathcal{O}Hf$-mod.*

*Proof.* The classical Clifford correspondence shows that $\mathrm{Ind}_H^G : \mathrm{Irr}(H) \to \mathrm{Irr}(G)$ restricts to a bijection between $\mathrm{Irr}(H, f)$ and $\mathrm{Irr}(G, F)$. Its inverse is given by $f\mathrm{Res}_H^G : \mathrm{Irr}(G, F) \to \mathrm{Irr}(H, f)$. The statement follows therefore from Theorem 1.29. □

The following lemma turns out to be quite useful.

**Lemma 1.31.** *Let $M$ be a $\Lambda G$-module and $M'$ be a $\Lambda N$-module. We have*

*a)* $\mathrm{Ind}_H^G(f\mathrm{Res}_H^G(M)) \cong \mathrm{Tr}_H^G(f)M$ *and*

*b)* $\mathrm{Ind}_N^H(fM') \cong f\mathrm{Ind}_N^H(M')$.

*Proof.* We have

$$\mathrm{Res}_H^G(\mathrm{Tr}_H^G(f)M) \cong \bigoplus_{x \in G/H} {}^x(f\mathrm{Res}_H^G(M)),$$

as $f^x f = 0$ for every $x \in G \setminus H$. This implies that $\mathrm{Ind}_H^G(f\mathrm{Res}_H^G(M)) \cong \mathrm{Tr}_H^G(f)M$. Part (b) follows from the definition of induction. □

We will frequently use the following classical extension result.

**Lemma 1.32.** *Let $M$ be a $G$-invariant $\Lambda N$-lattice and $G/N$ be cyclic of $\ell'$-order. Then $M$ extends to a $\Lambda G$-module.*

*Proof.* See [Rou98, Lemma 10.2.13]. □

## 1.12 Rickard equivalences for the normalizer

We continue our discussion on Marcus' theorem. Let $\tilde{L}$ be a subgroup of a finite group $\tilde{G}$. Moreover, let $G$ be a normal subgroup of $\tilde{G}$ and set $L := \tilde{L} \cap G$. Let $e \in \mathrm{Z}(\Lambda G)$ and $f \in \mathrm{Z}(\Lambda L)$ be central idempotents and denote by $L' := \mathrm{N}_{\tilde{L}}(f)$ and $G' := \mathrm{N}_{\tilde{G}}(e)$ their respective stabilizers. In this section we suppose that $G' = GL'$. We denote

$$\mathcal{D}' := (G \times L^{\mathrm{opp}})\Delta(L') \text{ and } \mathcal{D} := (G \times L^{\mathrm{opp}})\Delta(\tilde{L}).$$

In what follows, we assume that $f({}^l f) = 0$ for any $l \in \tilde{L} \setminus L'$ and $e({}^g e) = 0$ for any $g \in \tilde{G} \setminus G'$. This ensures that $F := \mathrm{Tr}_{\mathrm{N}_{\tilde{L}}(f)}^{\tilde{L}}(f)$ is a central idempotent of $\Lambda\tilde{L}$ and $E := \mathrm{Tr}_{\mathrm{N}_{\tilde{G}}(e)}^{\tilde{G}}(e)$ is a central idempotent of $\Lambda\tilde{G}$.

We remark a useful consequence of Lemma 1.30.

**Lemma 1.33.** *Let $C$ be a bounded complex of $\Lambda Ge$-$\Lambda Lf$-bimodules inducing a Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$. Suppose that $C$ is either concentrated in one degree or that $\ell \nmid [\tilde{L} : L]$. If $C$ extends to a complex $C'$ of $\Lambda\mathcal{D}'$-modules then the complex $\mathrm{Ind}_{\mathcal{D}'}^{\tilde{G} \times \tilde{L}^{\mathrm{opp}}}(C')$ induces a Rickard equivalence between $\Lambda\tilde{L}F$ and $\Lambda\tilde{G}E$.*

*Proof.* By Theorem 1.24 the $\Lambda$-algebras $\Lambda L'f$ and $\Lambda G'e$ are Rickard equivalent via the complex $\mathrm{Ind}_{\mathcal{D}'}^{G'\times(L')^{\mathrm{opp}}}(C')$. By Lemma 1.30 $\Lambda L'f$ is Morita equivalent to $\Lambda\tilde{L}F$. The same argument shows that $\Lambda G'e$ and $\Lambda\tilde{G}E$ are Morita equivalent. Thus, the algebras $\Lambda\tilde{L}F$ and $\Lambda\tilde{G}E$ are Rickard equivalent and the Rickard equivalence is given by the complex

$$\Lambda\tilde{G}e\otimes_{\Lambda G'}\mathrm{Ind}_{\mathcal{D}'}^{G'\times(L')^{\mathrm{opp}}}(C')\otimes_{\Lambda L'}f\Lambda\tilde{L}\cong\mathrm{Ind}_{\mathcal{D}'}^{\tilde{G}\times(\tilde{L})^{\mathrm{opp}}}(C'). \quad \square$$

**Remark 1.34.** Suppose that we are in the situation of Lemma 1.33 and let $\varphi:\mathrm{Ho}^b(\Lambda Lf)\to\mathrm{Ho}^b(\Lambda Ge)$ and $\tilde{\varphi}:\mathrm{Ho}^b(\Lambda\tilde{L}F)\to\mathrm{Ho}^b(\Lambda\tilde{G}E)$ be the functors obtained by tensoring with $C$ and $\mathrm{Ind}_{\mathcal{D}'}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')$ respectively. By Remark 1.26 and the construction in the proof of Lemma 1.33 it follows that $e\mathrm{Res}_G^{\tilde{G}}\circ\tilde{\varphi}\cong\varphi\circ f\,\mathrm{Res}_L^{\tilde{L}}$. Moreover, we have $\mathrm{Ind}_G^{\tilde{G}}\circ\varphi\cong\tilde{\varphi}\circ\mathrm{Ind}_L^{\tilde{L}}$.

In most applications we can give a more explicit description of the bimodule inducing the Rickard equivalence in Lemma 1.33.

**Lemma 1.35.** *Let $C$ be a bounded complex of $\Lambda G$-$\Lambda L$-bimodules and assume that $eCf$ induces a Rickard equivalence between $\Lambda Ge$ and $\Lambda Lf$. In addition, suppose that ${}^l eCf\cong 0$ in $\mathrm{Ho}^b(\Lambda[G\times L^{\mathrm{opp}}])$ for all $l\in\tilde{L}\setminus L'$. Suppose that $C$ is either concentrated in one degree or that $\ell\nmid[L':L]$. If $C$ extends to a complex of $\Lambda\mathcal{D}$-modules $C'$ then $\Lambda\tilde{L}F$ and $\Lambda\tilde{G}E$ are Rickard equivalent via the complex*

$$E\,\mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')\,F.$$

*Proof.* The complex $e\mathrm{Res}_{\mathcal{D}'}^{\mathcal{D}}(C')f$ is clearly a $\Lambda\mathcal{D}'$-complex extending $eCf$. By Lemma 1.33, it follows that the complex $\mathrm{Ind}_{\mathcal{D}'}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(e\mathrm{Res}_{\mathcal{D}'}^{\mathcal{D}}(C')f)$ induces a Rickard equivalence between $\Lambda\tilde{L}F$ and $\Lambda\tilde{G}E$.

Recall that $\mathcal{D}'$ is by definition the stabilizer in $\tilde{G}\times\tilde{L}^{\mathrm{opp}}$ of the idempotent $e\otimes f$. Since $\tilde{L}/L'\cong\mathcal{D}/\mathcal{D}'$, we have

$$\mathrm{Tr}_{\mathcal{D}'}^{\mathcal{D}}(e\otimes f)=\sum_{l\in\tilde{L}/L'}{}^{(l,l^{-1})}(e\otimes f).$$

By assumption we have ${}^l eCf\cong 0$ for all $l\in\tilde{L}\setminus L'$. From this it follows that

$$\mathrm{Tr}_{\mathcal{D}'}^{\mathcal{D}}(e\otimes f)C\cong\mathrm{Tr}_{\mathrm{N}_{\tilde{G}}(e)}^{\tilde{G}}(e)C\mathrm{Tr}_{\mathrm{N}_{\tilde{L}}(f)}^{\tilde{L}}(f)=ECF.$$

By Lemma 1.31(a) we have $\mathrm{Ind}_{\mathcal{D}'}^{\mathcal{D}}(e\mathrm{Res}_{\mathcal{D}'}^{\mathcal{D}}C'f)\cong EC'F$. As $E\otimes F$ is a central idempotent of $\Lambda[\tilde{G}\times\tilde{L}^{\mathrm{opp}}]$ it follows that $\mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(EC'F)\cong E\mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')F$ by Lemma 1.31(b). $\square$

Suppose that we are in the situation of the previous lemma. Note that in most applications we have $Cf \cong eCf$. In this case the assumption ${}^{l}eCf \cong 0$ for all $l \in \tilde{L}\backslash L'$ can be dropped since it follows from the fact that the complex $C$ is $\mathcal{D}$-stable.

We can use this lifting result to prove the following proposition.

**Proposition 1.36.** *Let $C$ be a bounded complex of $\ell$-permutation modules inducing a splendid Rickard equivalence between the blocks $\Lambda Gb$ and $\Lambda Lc$. Let $(Q, c_Q)$ be a $c$-Brauer pair corresponding to the $b$-Brauer pair $(Q, b_Q)$ under the splendid Rickard equivalence given by the complex $C$ as in Proposition 1.16. Then the complex*

$$\operatorname{Ind}_{\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q)}^{\mathrm{N}_G(Q)\times \mathrm{N}_L(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(C))\operatorname{Tr}_{\mathrm{N}_L(Q,c_Q)}^{\mathrm{N}_L(Q)}(c_Q)$$

*induces a derived equivalence between the blocks $k\mathrm{N}_G(Q)\operatorname{Tr}_{\mathrm{N}_G(Q,b_Q)}^{\mathrm{N}_G(Q)}(b_Q)$ and $k\mathrm{N}_L(Q)\operatorname{Tr}_{\mathrm{N}_L(Q,c_Q)}^{\mathrm{N}_L(Q)}(c_Q)$.*

*Proof.* Recall that $\mathrm{Br}_{\Delta Q}(C)$ is a complex of $k\mathrm{N}_{G\times L^{\mathrm{opp}}}(Q)$-modules such that $b_Q \mathrm{Br}_{\Delta Q}(C)c_Q \cong \mathrm{Br}_{\Delta Q}(C)c_Q$ induces a Rickard equivalence between $k\,\mathrm{C}_G(Q)b_Q$ and $k\,\mathrm{C}_L(Q)c_Q$, see Proposition 1.16. Moreover, the groups $\mathrm{N}_G(Q,b_Q)/\mathrm{C}_G(Q)$ and $\mathrm{N}_L(L,c_Q)/\mathrm{C}_L(Q)$ are isomorphic by Corollary 1.18.

Thus, using the proof of Lemma 1.35 together with Remark 1.25, we conclude that the complex

$$\operatorname{Ind}_{\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q)}^{\mathrm{N}_G(Q)\times \mathrm{N}_L(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(C))\operatorname{Tr}_{\mathrm{N}_L(Q,c_Q)}^{\mathrm{N}_L(Q)}(c_Q)$$

induces a derived equivalence between the blocks $k\mathrm{N}_G(Q)\operatorname{Tr}_{\mathrm{N}_G(Q,b_Q)}^{\mathrm{N}_G(Q)}(b_Q)$ and $k\mathrm{N}_L(Q)\operatorname{Tr}_{\mathrm{N}_L(Q,c_Q)}^{\mathrm{N}_L(Q)}(c_Q)$. □

**Remark 1.37.** If the defect group $D$ of $b$ is abelian, then by Theorem 1.5 the group $\mathrm{N}_G(D,b_D)/\mathrm{C}_G(D)$ is of $\ell'$-order. In this case, the proof of Proposition 1.36 shows that the complex $\operatorname{Ind}_{\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta D)}^{\mathrm{N}_G(D)\times \mathrm{N}_L(D)^{\mathrm{opp}}}\mathrm{Br}_{\Delta D}(C)$ induces in fact a Rickard equivalence between $k\mathrm{N}_G(D)\,\mathrm{br}_D(b)$ and $k\mathrm{N}_L(D)\,\mathrm{br}_D(c)$ and not only a derived equivalence.

## 1.13 The Brauer functor and Clifford theory

In this section we recall some results of [Mar96, Section 3] and generalize them slightly. These results will be needed in Section 1.15.

Whenever $G$ is a finite group and $Q$, $R$ are subgroups of $G$, then we let

$$T_G(Q,R) := \{g \in G \mid Q^g \subseteq R\}.$$

In addition, we denote by $1_G$ the trivial $kG$-module. We recall the following lemma:

**Lemma 1.38.** *Let $R$ be a subgroup of $G$ and $Q$ an $\ell$-subgroup of $G$. Then*

$$\mathrm{Br}_Q^G(\mathrm{Ind}_R^G(1_R)) \cong \bigoplus_{g\in T} \mathrm{Ind}_{\mathrm{N}_{{}^gR}(Q)}^{\mathrm{N}_G(Q)}(1_{\mathrm{N}_{{}^gR}(Q)}),$$

*where $T$ is a complete set of representatives of the double cosets of* $\mathrm{N}_G(Q)\backslash T_G(Q,R)/R$.

*Proof.* See [Bro85, (1.4)]. □

The following lemma is a variant of [Mar96, Lemma 3.7]. We will also use this opportunity to provide some further details which help to clarify the proof.

**Lemma 1.39.** *Let $H$ be a subgroup of $G$ and $Q \subseteq P$ two $\ell$-subgroups of $H$. Suppose that $\mathrm{C}_G(Q)T_H(Q,P) = T_G(Q,P)$. Then for every relatively $P$-projective module $M \in kH$*-perm *there is a natural isomorphism*

$$\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\mathrm{Br}_Q^H(M)) \cong \mathrm{Br}_Q^G(\mathrm{Ind}_H^G(M)).$$

*of $k\mathrm{N}_G(Q)$-modules.*

*Proof.* Recall from Section 1.3 that for a $kH$-module $M$ we have

$$\mathrm{Br}_Q^H(M) = M^Q / \sum_{P<Q} \mathrm{Tr}_P^Q(M^P).$$

Observe that we have a natural homomorphism

$$\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(M^Q) \to (\mathrm{Ind}_H^G(M))^Q,\ n\otimes m \mapsto n \otimes m$$

of $k\mathrm{N}_G(Q)$-modules.

We show that $\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\sum_{R<Q}\mathrm{Tr}_R^Q(M^R))$ maps to $\sum_{R<Q}\mathrm{Tr}_R^Q((\mathrm{Ind}_H^G(M))^R)$ under this natural map. As a $k\mathrm{N}_G(Q)$-module $\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\sum_{R<Q}\mathrm{Tr}_R^Q(M^R))$ is generated by the set $1\otimes\sum_{R<Q}\mathrm{Tr}_R^Q(M^R)$. This set clearly maps to $\sum_{R<Q}\mathrm{Tr}_R^Q(1\otimes M^R)$, which is contained in $\sum_{R<Q}\mathrm{Tr}_R^Q((\mathrm{Ind}_H^G(M))^R)$. Thus, it follows that $\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\sum_{R<Q}\mathrm{Tr}_R^Q(M^R))$ maps to $\sum_{R<Q}\mathrm{Tr}_R^Q((\mathrm{Ind}_H^G(M))^R)$. Hence, we obtain a natural map

$$\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\mathrm{Br}_Q^H(M)) \to \mathrm{Br}_Q^G(\mathrm{Ind}_H^G(M)).$$

Since $M$ is a relatively $P$-projective $\ell$-permutation module it is a direct summand of modules of the form $\mathrm{Ind}_R^H(1_R)$, where $R$ is an $\ell$-subgroup of $P$. Thus, in order to show the statement in general we may assume that $M = \mathrm{Ind}_R^H(1_R)$, a permutation module with vertex $R$. In particular, $\mathrm{Br}_Q(M) \cong 0$ and $\mathrm{Br}_Q(\mathrm{Ind}_H^G(M)) \cong 0$ if $Q$ is not conjugate to a subgroup of $R$. Hence, we may additionally assume that $R$ contains $Q$.

By Lemma 1.38 there is an isomorphism

$$\mathrm{Br}_Q^G(\mathrm{Ind}_R^G(1_R)) \cong \bigoplus_{g \in T} \mathrm{Ind}_{\mathrm{N}_{{}^gR}(Q)}^{\mathrm{N}_G(Q)}(1_{\mathrm{N}_{{}^gR}(Q)}),$$

where $T$ is a complete set of representatives of $\mathrm{N}_G(Q)\backslash T_G(Q,R)/R$. On the other hand, Lemma 1.38 also yields an isomorphism

$$\mathrm{Ind}_{\mathrm{N}_H(Q)}^{\mathrm{N}_G(Q)}(\mathrm{Br}_Q^H(\mathrm{Ind}_R^H(1_R))) \cong \bigoplus_{g \in T'} \mathrm{Ind}_{\mathrm{N}_{{}^gR}(Q)}^{\mathrm{N}_G(Q)}(1_{\mathrm{N}_{{}^gR}(Q)}),$$

where $T'$ is a complete set of representatives of $\mathrm{N}_H(Q)\backslash T_H(Q,R)/R$. To prove the lemma, it is hence sufficient to prove that $T'$ is also a complete set of representatives of the double cosets of $\mathrm{N}_G(Q)\backslash T_G(Q,R)/R$. Firstly, if $x_1, x_2 \in T_H(Q,R)$ with $\mathrm{N}_G(Q)x_1R = \mathrm{N}_G(Q)x_2R$ then $x_1 = nx_2q$ for some $n \in \mathrm{N}_G(Q)$ and $q \in R$. Since $R$ is contained in $H$ it follows that $n \in \mathrm{N}_H(Q)$ and so $\mathrm{N}_H(Q)x_1R = \mathrm{N}_H(Q)x_2R$. Since $Q \subseteq R \subseteq P$ we obtain

$$\mathrm{C}_G(Q)T_H(Q,R) = T_G(Q,R)$$

by using our assumption. Hence, for $x \in T_G(Q,R)$ there exists some $h \in T_H(Q,R)$ and $n \in \mathrm{C}_G(Q)$ such that $x = nh$ which implies that $\mathrm{N}_G(Q)xR = \mathrm{N}_G(Q)hR$. □

The following remark is a variant of [Mar96, Corollary 3.9].

**Remark 1.40.** As in Section 1.9 we let $\tilde{L}$ be a subgroup of a finite group $\tilde{G}$ and $G$ be a normal subgroup of $\tilde{G}$. We set $L := G \cap \tilde{L}$ and we assume additionally that $\tilde{G} = \tilde{L}G$. Let $Q$ be an $\ell$-subgroup of $\tilde{L}$. In the following diagram, Ind and Res mean induction and restriction with respect to the subgroups of $\tilde{G} \times \tilde{L}^{\mathrm{opp}}$ involved.

$$\begin{array}{ccccc}
k[\tilde{G} \times \tilde{L}^{\mathrm{opp}}]\text{-perm} & \xrightarrow{\mathrm{Br}_{\Delta(Q)}} & k[\mathrm{N}_{\tilde{G} \times \tilde{L}^{\mathrm{opp}}}(\Delta Q)]\text{-perm} & \xrightarrow{\mathrm{Ind}} & k[\mathrm{N}_{\tilde{G}}(Q) \times \mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}]\text{-perm} \\
\uparrow{\scriptstyle \mathrm{Ind}} & & \uparrow{\scriptstyle \mathrm{Ind}} & & \uparrow{\scriptstyle \mathrm{Ind}} \\
k[G \times L^{\mathrm{opp}}\Delta(\tilde{L})]\text{-perm} & \xrightarrow{\mathrm{Br}_{\Delta(Q)}} & k[\mathrm{N}_{G \times L^{\mathrm{opp}}\Delta(\tilde{L})}(\Delta Q)]\text{-perm} & \xrightarrow{\mathrm{Ind}} & k[\mathrm{N}_G(Q) \times \mathrm{N}_L(Q)^{\mathrm{opp}}\Delta(\mathrm{N}_{\tilde{L}}(Q))]\text{-perm} \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\
k[G \times L^{\mathrm{opp}}]\text{-perm} & \xrightarrow{\mathrm{Br}_{\Delta(Q)}} & k[\mathrm{N}_{G \times L^{\mathrm{opp}}}(\Delta Q)]\text{-perm} & \xrightarrow{\mathrm{Ind}} & k[\mathrm{N}_G(Q) \times \mathrm{N}_L(Q)^{\mathrm{opp}}]\text{-perm}
\end{array}$$

We claim that the upper left square commutes for all relatively $\Delta\tilde{L}$-projective $\ell$-permutation $k[G \times L^{\text{opp}}\Delta(\tilde{L})]$-modules. In view of Lemma 1.39 it is sufficient to show that

$$\mathrm{C}_{\tilde{G}\times\tilde{L}^{\text{opp}}}(\Delta Q)\, T_{G\times L^{\text{opp}}\Delta\tilde{L}}(\Delta Q, \Delta R) = T_{\tilde{G}\times\tilde{L}^{\text{opp}}}(\Delta Q, \Delta R)$$

for all $\ell$-subgroups $R$ of $\tilde{L}$ containing $Q$. This is proved as in [Mar96, Corollary 3.9]: Let $(x,l) \in T_{\tilde{G}\times\tilde{L}^{\text{opp}}}(\Delta Q, \Delta R)$. Then ${}^{(x,l)}\Delta Q \subseteq \Delta R$ which implies that $xl \in \mathrm{C}_{\tilde{G}}(Q)$. On the other hand, ${}^{l}Q = Q$ which implies that $(l^{-1}, l) \in T_{G\times L^{\text{opp}}\Delta\tilde{L}}(\Delta Q, \Delta R)$. Therefore, $(x,l) = (xl,1)(l^{-1},l) \in T_{G\times L^{\text{opp}}\Delta\tilde{L}}(\Delta Q, \Delta R)$. This shows the equality.

The upper right and the bottom left square are clearly commutative. Moreover, the commutativity of the bottom right square is a consequence of Mackey's formula.

## 1.14 The Harris–Knörr correspondence

In this section we recall the notion of block induction. This will allow us to give a nice formulation of the important Harris–Knörr correspondence.

Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $b \in \mathrm{Z}(\Lambda G)$ a block of $G$. Then we say that the block idempotent $c \in \mathrm{Z}(\Lambda\tilde{G})$ *covers the block* $b$ if $cb \neq 0$. We write $\mathrm{Bl}(\tilde{G} \mid b)$ for the set of blocks of $\tilde{G}$ covering $b$.

We recall the definition of block induction, see [Nav98, Theorem 4.14].

**Definition 1.41.** Suppose that $H$ is a subgroup of $G$ and $b$ is a block idempotent of $G$. Furthermore, assume that there exists an $\ell$-subgroup $P$ of $G$ such that $P\mathrm{C}_G(P) \subseteq H \subseteq \mathrm{N}_G(P)$. Then we say that the block idempotent $c \in \mathrm{Z}(\mathcal{O}H)$ *induces to* $b$ if $\mathrm{br}_P(b)c \neq 0$. In this case we write $b = c^G$.

Note that the definition of block induction in [Nav98, page 87] is more general. However, we will not need this general definition and have therefore decided to use the characterisation of block induction in [Nav98, Theorem 4.14] as a definition.

Recall that for a subgroup $Q$ of the defect group $D$ of $b$ we denote $B_Q := \mathrm{Tr}_{\mathrm{N}_G(Q,b_Q)}^{\mathrm{N}_G(Q)}(b_Q)$, which is by Remark 1.6 a block idempotent of $\mathrm{N}_G(Q)$.

**Theorem 1.42** (Harris-Knörr)**.** *Let $G$ be a normal subgroup of a finite group $\tilde{G}$. Let $b$ be a block of $G$ with defect group $D$ and denote by $B_D$ its Brauer correspondent in $k\mathrm{N}_G(D)$. Then the map*

$$\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(D) \mid B_D) \to \mathrm{Bl}(\tilde{G} \mid b),\ c \mapsto c^{\tilde{G}}$$

*is a bijection.*

*Proof.* See [Nav98, Theorem 9.28]. □

If $Q$ is a characteristic subgroup of the defect group $D$ of $b$ we have $\mathrm{N}_G(D) \subseteq \mathrm{N}_G(Q)$. Brauer's first main theorem (see Remark 1.6) therefore yields a bijection

$$\mathrm{br}_D : \mathrm{Bl}(\mathrm{N}_G(Q) \mid D) \to \mathrm{Bl}(\mathrm{N}_G(D) \mid D).$$

After having established this notation we can now state the following lemma:

**Lemma 1.43.** *Let $Q$ be a characteristic subgroup of $D$. Then $B_Q \in \mathrm{Bl}(\mathrm{N}_G(Q))$ is the Brauer correspondent of $B_D \in \mathrm{Bl}(\mathrm{N}_G(D))$.*

*Proof.* By [Thé95, Theorem 40.4(b)] we have $\mathrm{br}_D(b_Q) = b_D$. Since $D \subseteq \mathrm{N}_G(Q)$ we can write $B_Q \in \mathrm{Z}(k\mathrm{C}_G(Q))$ as a sum $B_Q = \sum_{i=1}^s c_i$ of block idempotents of $k\mathrm{C}_G(Q)$. Note that each $c_i$ is a sum of idempotents which constitute a $D$-orbit on $\{{}^t b_Q \mid t \in \mathrm{N}_G(Q)\}$.

Assume first that $c_i$ comes from a $D$-orbit of length greater than 1. Let $t \in \mathrm{N}_G(Q)$ with ${}^t b_Q c_i \neq 0$. Then the block $c_i$ covers ${}^t b_Q$ and it follows that any defect group of $c_i$ is contained in $\mathrm{N}_G(Q, {}^t b_Q)$. Since ${}^t b_Q$ is not $D$-stable it follows that $D$ is not contained in $\mathrm{N}_G(Q, {}^t b_Q)$. Thus $D$ is not contained in a defect group of $c_i$. This implies that $\mathrm{br}_D(c_i) = 0$.

On the other hand, if $c_i = {}^t b_Q$ for some $t \in \mathrm{N}_G(Q)$ it follows that ${}^t b_Q$ is $D$-stable. Assume that $\mathrm{br}_D({}^t b_Q) \neq 0$. Then we have ${}^t(Q, b_Q) \trianglelefteq (D, b'_D)$ for some maximal $b$-subpair $(D, b'_D)$. Since also ${}^t(Q, b_Q) \trianglelefteq {}^t(D, b_D)$ it follows by [Thé95, Proposition 40.15(b)] that there exists some $x \in \mathrm{N}_G(Q, b_Q)$ such that $tx \in \mathrm{N}_G(D)$. From this we conclude that

$$\mathrm{br}_D({}^t b_Q) = \mathrm{br}_D({}^{tx} b_Q) = {}^{tx}\,\mathrm{br}_D(b_Q) = {}^{tx} b_D.$$

These calculations show that $\mathrm{br}_D(B_Q) B_D = B_D$. On the other hand $B_Q$ is an idempotent occurring in $\mathrm{br}_Q(b)$ and we have $\mathrm{br}_D(\mathrm{br}_Q(b)) = B_D$. Writing $\mathrm{br}_Q(b) = B_Q + C$ we obtain $B_D = \mathrm{br}_D(B_Q) + \mathrm{br}_D(C)$ a sum of orthogonal idempotents. Now observe that $B_D$ is a primitive central idempotent of $\mathrm{N}_G(D)$ and $\mathrm{br}_D(B_Q) B_D = B_D$. Therefore, $B_D = \mathrm{br}_D(B_Q)$. □

We obtain a version of the Harris–Knörr theorem for characteristic subgroups of defect groups.

**Corollary 1.44.** *With the notation of Theorem 1.42 assume that $Q$ is a characteristic subgroup of $D$. Let $(Q, b_Q)$ be a $b$-Brauer pair with $(Q, b_Q) \leq (D, b_D)$. Then block induction yields a bijection*

$$\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(Q) \mid B_Q) \to \mathrm{Bl}(\tilde{G} \mid b), c \mapsto c^{\tilde{G}}.$$

*Proof.* Brauer correspondence gives a bijection $\mathrm{br}_D : \mathrm{Bl}(G \mid D) \to \mathrm{Bl}(\mathrm{N}_G(D) \mid D)$ with $\mathrm{br}_D(b) = B_D$. Moreover, by Lemma 1.43 the map $\mathrm{br}_D : \mathrm{Bl}(\mathrm{N}_G(Q) \mid D) \to \mathrm{Bl}(\mathrm{N}_G(D) \mid D)$ is a bijection with $\mathrm{br}_D(B_Q) = B_D$.

By Theorem 1.42 we hence obtain bijections $\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(D) \mid B_D) \to \mathrm{Bl}(\tilde{G} \mid b)$ and $\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(D) \mid B_D) \to \mathrm{Bl}(\mathrm{N}_{\tilde{G}}(Q) \mid B_Q)$ both given by block induction. This yields a bijection

$$\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(Q) \mid B_Q) \to \mathrm{Bl}(\tilde{G} \mid b).$$

Moreover, if $c \in \mathrm{Bl}(\mathrm{N}_{\tilde{G}}(D) \mid B_D)$ then $c^{\mathrm{N}_{\tilde{G}}(Q)}$ and $c^{\tilde{G}}$ are both defined. By [Nav98, Problem 4.2] it follows that $c^{\tilde{G}} = (c^{\mathrm{N}_{\tilde{G}}(Q)})^{\tilde{G}}$. Hence, the bijection $\mathrm{Bl}(\mathrm{N}_{\tilde{G}}(Q) \mid B_Q) \to \mathrm{Bl}(\tilde{G} \mid b)$ is given by block induction. □

## 1.15 Splendid Rickard equivalences and Clifford theory

In Proposition 1.36 we have shown that a splendid Rickard equivalence induces a derived equivalence on the level of normalizers. Therefore, a natural question to ask is whether the so-obtained equivalences behave nicely with respect to the Clifford theory of Rickard equivalences and with the Brauer category of the involved blocks. These questions will be addressed in this section.

We first make the following useful observation.

**Lemma 1.45.** *Let $G$ be a normal subgroup of a finite group $\tilde{G}$. Let $b$ be a $\tilde{G}$-stable block of $G$ with defect group $D$ and $Q$ a characteristic subgroup of $D$. Then $B_Q$ is an $\mathrm{N}_{\tilde{G}}(Q)$-stable block of $\mathrm{N}_G(Q)$ and we have $\mathrm{N}_{\tilde{G}}(Q)/\mathrm{N}_G(Q) \cong \tilde{G}/G$.*

*Proof.* Recall that all defect groups of $b$ are $G$-conjugate. Since $b$ is a $\tilde{G}$-stable block of $G$ we thus obtain $\tilde{G} = G\mathrm{N}_{\tilde{G}}(D)$. Moreover, $Q$ is a characteristic subgroup of $D$ and so $\mathrm{N}_{\tilde{G}}(D) \subseteq \mathrm{N}_{\tilde{G}}(Q)$. From this we conclude that $\tilde{G}/G \cong \mathrm{N}_{\tilde{G}}(Q)/\mathrm{N}_G(Q)$. It remains to show that $B_Q$ is $\mathrm{N}_{\tilde{G}}(Q)$-stable. If $g \in \mathrm{N}_{\tilde{G}}(Q)$ then ${}^g(D, b_D)$ is a second maximal $b$-Brauer pair, so there exists some $x \in G$ with ${}^{gx}(D, b_D) = (D, b_D)$. In particular, $gx \in \mathrm{N}_{\tilde{G}}(D) \subseteq \mathrm{N}_{\tilde{G}}(Q)$ and thus $x \in \mathrm{N}_G(Q)$. Moreover, $(Q, b_Q) \leq (D, b_D)$ and $(Q, {}^{gx}b_Q) = {}^{gx}(Q, b_Q) \leq (D, b_D)$ are two $b$-Brauer pairs with first entry $Q$. Therefore, $gx \in \mathrm{N}_{\tilde{G}}(Q, b_Q)$ and so ${}^gB_Q = {}^{gx}B_Q = B_Q$. □

In the following, $\tilde{L}$ denotes a subgroup of a finite group $\tilde{G}$ and $G$ a normal subgroup of $\tilde{G}$. We set $L := G \cap \tilde{L}$ and assume that $\tilde{G} = \tilde{L}G$. As before we set $\mathcal{D} := (G \times L^{\mathrm{opp}})\Delta(\tilde{L})$. Furthermore, let $c \in \mathrm{Z}(kL)$ be a $\tilde{L}$-stable block of $L$ and $b \in \mathrm{Z}(kG)$.

**Lemma 1.46.** *Let $C$ be a bounded complex of $kGb$-$kLc$-bimodules inducing a splendid Rickard equivalence between the blocks $kGb$ and $kLc$. Assume that $C$ extends to a complex $C'$ of $k\mathcal{D}$-modules and denote $\tilde{C} := \mathrm{Ind}_{\mathcal{D}}^{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(C')$. Let $D$ be a defect group of $kLc$ and $Q$ a characteristic subgroup of $D$. Let $(Q, c_Q)$ be a $c$-Brauer pair corresponding to the $b$-Brauer pair $(Q, b_Q)$ as in Proposition 1.16. Set*

$$\tilde{\mathcal{C}} := \mathrm{Ind}_{\mathrm{N}_{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(\Delta Q)}^{\mathrm{N}_{\tilde{G}}(Q)\times\mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(\tilde{C}))C_Q \text{ and } \mathcal{C} := \mathrm{Ind}_{\mathrm{N}_{G\times L^{\mathrm{opp}}}(\Delta Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(C))C_Q.$$

*Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{D}^b(k\mathrm{N}_{\tilde{L}}(Q)C_Q) & \xrightarrow{\tilde{\mathcal{C}}\otimes^{\mathbb{L}}_{k\mathrm{N}_{\tilde{L}}(Q)} -} & \mathrm{D}^b(k\mathrm{N}_{\tilde{G}}(Q)B_Q) \\
\Big\downarrow \mathrm{Res}^{\mathrm{N}_{\tilde{L}}(Q)}_{\mathrm{N}_L(Q)} & & \Big\downarrow \mathrm{Res}^{\mathrm{N}_{\tilde{G}}(Q)}_{\mathrm{N}_G(Q)} \\
\mathrm{D}^b(k\mathrm{N}_L(Q)C_Q) & \xrightarrow{\mathcal{C}\otimes^{\mathbb{L}}_{k\mathrm{N}_L(Q)} -} & \mathrm{D}^b(k\mathrm{N}_G(Q)B_Q)
\end{array}$$

*where the horizontal maps induce equivalences of the derived categories.*

*Proof.* By the commutativity of the first two rows of the commutative diagram in Remark 1.40 we have a natural isomorphism

$$\tilde{\mathcal{C}} \cong \mathrm{Ind}_{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)\times\mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}}(\mathcal{C}'),$$

where $\mathcal{C}' := \mathrm{Ind}_{\mathrm{N}_{G\times L^{\mathrm{opp}}\Delta\tilde{L}}(\Delta Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{L}}(Q)}(\mathrm{Br}_{\Delta Q}(C'))C_Q$. Now by the commutativity of the second and the third row of the commutative diagram in Remark 1.40 we deduce that

$$\mathrm{Res}^{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{L}}(Q)}_{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}}(\mathcal{C}') \cong \mathcal{C}.$$

By Proposition 1.36 the complex $\mathcal{C}$ induces a derived equivalence between the blocks $k\mathrm{N}_G(Q)B_Q$ and $k\mathrm{N}_L(Q)C_Q$. By Lemma 1.45, the block $B_Q$ is $\mathrm{N}_{\tilde{G}}(Q)$-stable and $C_Q$ is $\mathrm{N}_{\tilde{L}}(Q)$-stable. Moreover, we have

$$\mathrm{N}_{\tilde{G}}(Q)/\mathrm{N}_G(Q) \cong \mathrm{N}_{\tilde{L}}(Q)/\mathrm{N}_L(Q).$$

It follows from Remark 1.25 that the complex $\tilde{\mathcal{C}} \cong \mathrm{Ind}_{\mathrm{N}_G(Q)\times\mathrm{N}_L(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)\times\mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}}(\mathcal{C}')$ induces a derived equivalence between $k\mathrm{N}_{\tilde{L}}(Q)C_Q$ and $k\mathrm{N}_{\tilde{G}}(Q)B_Q$. The commutativity of the diagram is now a consequence of Remark 1.26(a). □

In Corollary 1.44 we have established a Harris–Knörr correspondence for characteristic subgroups of the defect group of a block. It is therefore natural to ask whether the construction in Lemma 1.46 is compatible with this correspondence.

**Remark 1.47.** Assume that we are in the situation of Lemma 1.46. Let $c = c_1 + \cdots + c_r$ be a decomposition of $c$ into block idempotents of $k\tilde{L}$. We let $b = b_1 + \cdots + b_r$ be the decomposition of $b$ into block idempotents of $k\tilde{G}$ such that $b_i\tilde{C}c_i \neq 0$ in $\mathrm{Ho}^b(k[\tilde{G} \times \tilde{L}^{\mathrm{opp}}])$, see Lemma 1.9. We have a decomposition $\mathrm{br}_Q^{\tilde{G}}(b) = \mathrm{br}_Q^{\tilde{G}}(b_1) + \cdots + \mathrm{br}_Q^{\tilde{G}}(b_r)$ into orthogonal idempotents.

Denote by $B_{Q,i} := \mathrm{br}_Q(b_i)B_Q$ the Harris–Knörr correspondent of $b_i$, see Corollary 1.44. We deduce that

$$B_Q = B_{Q,1} + \cdots + B_{Q,r}$$

is a decomposition into block idempotents of $k\mathrm{N}_{\tilde{G}}(Q)$. Similarly, we have a decomposition

$$C_Q = C_{Q,1} + \cdots + C_{Q,r}$$

into block idempotents of $k\mathrm{N}_{\tilde{L}}(Q)$, where $C_{Q,i} := \mathrm{br}_Q(c_i)C_Q$. We have $\mathrm{Br}_{\Delta Q}(\tilde{\mathcal{C}}) \cong \bigoplus_{i=1}^r \mathrm{br}_Q(b_i)\,\mathrm{Br}_{\Delta Q}(\tilde{\mathcal{C}})\,\mathrm{br}_Q(c_i)$ and therefore we obtain

$$\tilde{\mathcal{C}} = \mathrm{Ind}_{\mathrm{N}_{\tilde{G}\times\tilde{L}^{\mathrm{opp}}}(\Delta Q)}^{\mathrm{N}_{\tilde{G}}(Q)\times\mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(\tilde{C})C_Q) \cong \bigoplus_{i=1}^{r} \tilde{\mathcal{C}}C_{Q,i}.$$

From this we conclude that the complex $\tilde{\mathcal{C}}C_{Q,i}$ induces a derived equivalence between the blocks $k\mathrm{N}_{\tilde{G}}(Q)B_{Q,i}$ and $k\mathrm{N}_{\tilde{L}}(Q)C_{Q,i}$. Thus, the local equivalences for the normalizer are compatible with the Harris–Knörr correspondence.

# Chapter 2

# Deligne–Lusztig theory and disconnected reductive groups

In this chapter we recall the necessary background in the representation theory of finite groups of Lie type. We will in particular discuss extensions of this theory to disconnected reductive groups. Then we will recall the Morita equivalence constructed by Bonnafé, Dat and Rouquier which can be seen as a starting point of this work.

## 2.1 Disconnected reductive algebraic groups

We assume that the reader is familiar with the notion of Levi subgroups and parabolic subgroups of connected reductive algebraic groups, see for instance [MT11, Chapter 12]. In this section, we will discuss a generalization of these notions to not necessarily connected reductive groups.

Fix a prime number $p$ and an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Let $\mathbf{G}$ denote a (not necessarily connected) reductive algebraic group defined over $\overline{\mathbb{F}_p}$. We denote by $\mathbf{G}^\circ$ the connected component of $\mathbf{G}$ containing the identity.

In the following, we recall some standard facts, which can for instance be found in [BDR17a, Section 2.D.] and [BDR17a, Section 3.A.]. A closed subgroup $\mathbf{P}$ of $\mathbf{G}$ is called *parabolic subgroup* if the variety $\mathbf{G}/\mathbf{P}$ is complete. One can show that a closed subgroup $\mathbf{P}$ of $\mathbf{G}$ is a parabolic subgroup of $\mathbf{G}$ if and only if $\mathbf{P}^\circ$ is a parabolic subgroup of $\mathbf{G}^\circ$. Moreover, we have $\mathbf{P}\cap\mathbf{G}^\circ = \mathbf{P}^\circ$ and the unipotent radicals of $\mathbf{P}$ and $\mathbf{P}^\circ$ coincide.

Suppose that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$. Let $\mathbf{L}_\circ$ be a Levi subgroup of $\mathbf{G}^\circ$ so that $\mathbf{P}^\circ = \mathbf{L}_\circ \ltimes \mathbf{U}$ is a Levi decomposition of the parabolic subgroup $\mathbf{P}^\circ$ in $\mathbf{G}^\circ$. Then we call $\mathbf{L} = \mathrm{N}_{\mathbf{P}}(\mathbf{L}_\circ)$ *a Levi subgroup* of $\mathbf{P}$ in $\mathbf{G}$. In addition, we have a decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $\mathbf{L}_\circ$ is the connected component of

$\mathbf{L}$, i.e. $\mathbf{L}^\circ = \mathbf{L}_\circ$.

**Example 2.1.** Let $\mathbf{G}$ be a reductive algebraic group. Let $\mathbf{P}_\circ = \mathbf{L}_\circ \ltimes \mathbf{U}$ be a parabolic subgroup with Levi decomposition in $\mathbf{G}^\circ$. Then $\mathbf{P} = \mathrm{N}_{\mathbf{G}}(\mathbf{P}_\circ)$ is a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{L} = \mathrm{N}_{\mathbf{G}}(\mathbf{L}_\circ, \mathbf{P}_\circ) = \mathrm{N}_{\mathbf{P}}(\mathbf{L}_\circ)$ such that $\mathbf{P}^\circ = \mathbf{P}_\circ$.

As we show in the next example, disconnected reductive groups arise naturally in the study of automorphisms of reductive groups.

**Example 2.2.** Let $\mathbf{G}_\circ$ be a connected reductive group and $\tau : \mathbf{G}_\circ \to \mathbf{G}_\circ$ an algebraic automorphism of $\mathbf{G}_\circ$ of finite order. Then the semidirect product $\mathbf{G} := \mathbf{G}_\circ \rtimes \langle \tau \rangle$ is again a reductive algebraic group but no longer connected. This situation was for instance considered in [Mal93]. Let $\mathbf{P}_\circ = \mathbf{L}_\circ \ltimes \mathbf{U}$ be a Levi decomposition of a parabolic subgroup $\mathbf{P}_\circ$ of $\mathbf{G}_\circ$. If both $\mathbf{L}_\circ$ and $\mathbf{P}_\circ$ are $\tau$-stable, then $\mathbf{P} := \mathbf{P}_\circ \rtimes \langle \tau \rangle$ is a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{L} := \mathbf{L}_\circ \rtimes \langle \tau \rangle$, see Example 2.1.

Disconnected reductive groups also appear naturally as local subgroups of (connected) reductive groups.

**Example 2.3.** Let $\mathbf{G}$ be a possibly disconnected reductive group, $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. In addition, we assume that $Q$ is a finite solvable $p'$-subgroup of $\mathbf{L}$. By [BDR17a, Remark 3.5] it follows that the normalizer $\mathrm{N}_{\mathbf{G}}(Q)$ is a reductive group. Moreover, $\mathrm{N}_{\mathbf{P}}(Q)$ is a parabolic subgroup of $\mathrm{N}_{\mathbf{G}}(Q)$ with Levi decomposition $\mathrm{N}_{\mathbf{P}}(Q) = \mathrm{N}_{\mathbf{L}}(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$. Similarly, $\mathrm{C}_{\mathbf{G}}(Q)$ is a reductive group with parabolic subgroup $\mathrm{C}_{\mathbf{P}}(Q)$ and Levi decomposition $\mathrm{C}_{\mathbf{P}}(Q) = \mathrm{C}_{\mathbf{L}}(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$, see [BDR17a, Proposition 3.4]. Note that $\mathrm{N}_{\mathbf{G}}(Q)/\mathrm{C}_{\mathbf{G}}(Q)$ is finite since it embeds under the natural map $\mathrm{N}_{\mathbf{G}}(Q)/\mathrm{C}_{\mathbf{G}}(Q) \hookrightarrow \mathrm{Aut}(Q)$ into the automorphism group of the finite group $Q$. Therefore, $\mathrm{N}_{\mathbf{G}}^\circ(Q) = \mathrm{C}_{\mathbf{G}}^\circ(Q)$ and we have a Levi decomposition $\mathrm{C}_{\mathbf{P}}^\circ(Q) = \mathrm{C}_{\mathbf{L}}^\circ(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$ in the connected reductive group $\mathrm{C}_{\mathbf{G}}^\circ(Q)$.

## 2.2 $\ell$-adic cohomology of Deligne–Lusztig varieties

From now on $\ell$ denotes a prime number with $p \neq \ell$ and $q$ is an integral power of $p$. Furthermore, by variety we always mean a quasi-projective variety defined over $\overline{\mathbb{F}_p}$.

Let $\mathbf{X}$ be a variety acted on by a finite group $G$. We denote by $R\Gamma_c(\mathbf{X}, \mathcal{O}) \in \mathrm{D}^b(\mathcal{O}G)$ the $\ell$-adic cohomology with compact support of the variety $\mathbf{X}$ with coefficients in $\mathcal{O}$, see [CE04, A.3.7] and [CE04, A.3.14]. For $A \in \{K, \mathcal{O}, k\}$ we define

$$R\Gamma_c(\mathbf{X}, A) := R\Gamma_c(\mathbf{X}, \mathcal{O}) \otimes^{\mathbb{L}}_{\mathcal{O}} A \in \mathrm{D}^b(AG).$$

Moreover, we denote by $H^d_c(\mathbf{X}, A) \in AG$-mod the $d$th cohomology module of the complex $R\Gamma_c(\mathbf{X}, A)$. For basic facts about $\ell$-adic cohomology which will be more than enough for this thesis we refer the reader to the expositions in [CE04, Appendix 3]. More supplementary material can be found in the appendix of [Car93] or [DM91, Chapter 10].

Let $\mathbf{G}$ be a reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, $\mathbf{P} = \mathbf{L}\mathbf{U}$ be a Levi decomposition and assume that $\mathbf{L}$ is $F$-stable. Consider the $\mathbf{G}^F$-$\mathbf{L}^F$-variety

$$\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}} := \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}F(\mathbf{U})\} \subseteq \mathbf{G}/\mathbf{U}.$$

If the ambient group $\mathbf{G}$ is clear from the context we will just write $\mathbf{Y}_{\mathbf{U}}$ instead of $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$. The cohomology of this variety provides us with a triangulated functor

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \mathrm{D}^b(\Lambda\mathbf{L}^F) \to \mathrm{D}^b(\Lambda\mathbf{G}^F),\ M \mapsto R\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \Lambda) \otimes^{\mathbb{L}}_{\Lambda\mathbf{L}^F} M.$$

This functor induces a map

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} := [\mathcal{R}^{\mathbf{G}}_{\mathbf{L}}] : G_0(\Lambda\mathbf{L}^F) \to G_0(\Lambda\mathbf{G}^F),\ [M] \mapsto \sum_i (-1)^i [H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \Lambda) \otimes_{\Lambda\mathbf{L}^F} M],$$

on Grothendieck groups (see Section 1.1) the so-called *Deligne–Lusztig induction.*

## 2.3 Properties of Deligne–Lusztig varieties

In this section we will study the following set-up: Let $\hat{\mathbf{G}}$ be a reductive group with Frobenius $F : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$. Moreover, assume that $\mathbf{G}$ is a closed $F$-stable normal subgroup of $\hat{\mathbf{G}}$. Suppose that $\mathbf{P} = \mathbf{L}\mathbf{U}$ and $\hat{\mathbf{P}} = \hat{\mathbf{L}}\mathbf{U}$ are two Levi decomposition of parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ and $\hat{\mathbf{P}}$ of $\hat{\mathbf{G}}$ such that $\hat{\mathbf{P}} \cap \mathbf{G} = \mathbf{P}$ and $\hat{\mathbf{L}} \cap \mathbf{G} = \mathbf{L}$. Assume that the Levi subgroup $\hat{\mathbf{L}}$ is $F$-stable. Let us denote

$$\mathcal{D} = \{(x, y) \in \hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}} \mid x\mathbf{G}^F = y^{-1}\mathbf{G}^F\} = (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\hat{\mathbf{L}}^F).$$

**Lemma 2.4.** *With the notation as above, the variety $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$ is a $\mathcal{D}$-stable subvariety of $\hat{\mathbf{G}}/\mathbf{U}$.*

*Proof.* Let $(x, y) \in \mathcal{D}$ and $g\mathbf{U} \in \mathbf{G}/\mathbf{U}$. Since $(x^{-1}, y^{-1}) \in \mathcal{D}$ we have $xy \in \mathbf{G}^F$ and $xgx^{-1} \in \mathbf{G}$. Since $\hat{\mathbf{L}}$ normalizes $\mathbf{U}$ we conclude that

$$xg\mathbf{U}y = xgy\mathbf{U} = xgx^{-1}xy\mathbf{U} \in \mathbf{G}/\mathbf{U}.$$

Hence, the group action of $\mathcal{D}$ stabilizes the subvariety $\mathbf{G}/\mathbf{U}$ of $\hat{\mathbf{G}}/\mathbf{U}$.

Now suppose that $g\mathbf{U} \in \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. Let us define $c = xgy$. It follows that

$$c^{-1}F(c) = y^{-1}g^{-1}F(g)y \in (\mathbf{U}F(\mathbf{U}))^y = \mathbf{U}F(\mathbf{U})$$

since $y \in \hat{\mathbf{L}}$ normalizes $\mathbf{U}$. Consequently, the Deligne–Lusztig variety $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ is a $\mathcal{D}$-stable subvariety of $\hat{\mathbf{G}}/\mathbf{U}$. □

We also consider the generalized Deligne–Lusztig varieties as introduced in [BDR17a, Section 6A]. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two parabolic subgroups of $\mathbf{G}$ with common $F$-stable Levi complement $\mathbf{L}$ and unipotent radicals $\mathbf{U}_1$ and $\mathbf{U}_2$ respectively. We define

$$\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}^{\mathbf{G}} = \{(g_1\mathbf{U}_1, g_2\mathbf{U}_2) \in \mathbf{G}/\mathbf{U}_1 \times \mathbf{G}/\mathbf{U}_2 \mid g_1^{-1}g_2 \in \mathbf{U}_1\mathbf{U}_2; g_2^{-1}F(g_1) \in \mathbf{U}_2F(\mathbf{U}_1)\}$$

which is a variety acted on diagonally by $\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}$. Similarily to Lemma 2.4 one proves the following.

**Lemma 2.5.** *Let $\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}$ act diagonally on $\hat{\mathbf{G}}/\mathbf{U}_1 \times \hat{\mathbf{G}}/\mathbf{U}_2$. Then $\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}^{\mathbf{G}}$ is a $\mathcal{D}$-stable subvariety of $\hat{\mathbf{G}}/\mathbf{U}_1 \times \hat{\mathbf{G}}/\mathbf{U}_2$.*

*Proof.* Let $(x, y) \in \mathcal{D}$ and $(g_1\mathbf{U}_1, g_2\mathbf{U}_2) \in \mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}^{\mathbf{G}}$. As in the proof of Lemma 2.4 we see that

$$x(g_1\mathbf{U}_1, g_2\mathbf{U}_2)y = (xg_1\mathbf{U}_1y, xg_2\mathbf{U}_2y) = (xg_1y\mathbf{U}_1, xg_2y\mathbf{U}_2) \in \mathbf{G}/\mathbf{U}_1 \times \mathbf{G}/\mathbf{U}_2.$$

Moreover, we have

$$(xg_1y)^{-1}xg_2y = y^{-1}g_1^{-1}g_2y \in (\mathbf{U}_1\mathbf{U}_2)^y = \mathbf{U}_1\mathbf{U}_2,$$

since $y \in \hat{\mathbf{L}}$ normalizes $\mathbf{U}_1$ and $\mathbf{U}_2$. Similarly,

$$(xg_2y)^{-1}F(xg_2y) = y^{-1}g_2^{-1}F(g_1)y \in (\mathbf{U}_2F(\mathbf{U}_1))^y = \mathbf{U}_2F(\mathbf{U}_2).$$

This shows that the subvariety $\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}^{\mathbf{G}}$ is $\mathcal{D}$-stable. □

**Notation 2.6.** Let $H$ be a finite group. If $\mathbf{X}$ is a right $H$-variety and $\mathbf{Y}$ a left $H$-variety we denote by $\mathbf{X} \times_H \mathbf{Y}$ the quotient of $\mathbf{X} \times \mathbf{Y}$ by the diagonal right action of the group $\Delta(H) = \{(h, h^{-1}) \mid h \in H\}$ given by

$$\mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}, (x, y) \mapsto (xh, h^{-1}y).$$

Now assume that $\mathbf{X}$ is a $G$-$H$-variety and $\mathbf{Y}$ an $H$-$L$-variety. Then $\mathbf{X}\times_H\mathbf{Y}$ becomes a $G$-$L$-variety. To compute the cohomology of this new variety one uses the following theorem:

**Theorem 2.7** (Künneth formula)**.** *If the stabilizers of points of* $\mathbf{X}\times\mathbf{Y}$ *under the diagonal action of* $H$ *are of invertible order in* $\Lambda$*, then we have*

$$R\Gamma_c(\mathbf{X},\Lambda)\otimes^{\mathbb{L}}_{\Lambda H} R\Gamma_c(\mathbf{Y},\Lambda)\cong R\Gamma_c(\mathbf{X}\times_H\mathbf{Y},\Lambda)$$

*in* $\mathrm{D}^b(\Lambda[G\times L^{\mathrm{opp}}])$.

*Proof.* See [BR03, Section 3.3]. □

The following geometric lemma describes two closely related decompositions of the Deligne–Lusztig variety $\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}$. The result is certainly well known, but it does not appear in this exact form in the literature, see also [CE04, Theorem 7.3].

**Lemma 2.8.** *We have two decompositions.*

*a)* $\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}=\coprod_{g\in\hat{\mathbf{G}}^F/\mathbf{G}^F} g\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}=\hat{\mathbf{G}}^F\times_{\mathbf{G}^F}\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ *as* $(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})$*-varieties.*

*b)* $\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}\cong(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})\times_{\mathcal{D}}\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ *as* $(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})$*-varieties.*

*Proof.* Firstly, observe that $\coprod_{g\in\hat{\mathbf{G}}^F/\mathbf{G}^F} g\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ is indeed a disjoint union of closed subvarieties of $\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}$.

Now, let $y\mathbf{U}\in\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}$. Then $y^{-1}F(y)\in\mathbf{U}F(\mathbf{U})\subseteq\mathbf{G}^\circ$. As $\mathbf{G}^\circ$ is connected the Lang map $\mathcal{L}:\mathbf{G}^\circ\to\mathbf{G}^\circ$, $g\mapsto g^{-1}F(g)$, is surjective. Consequently, there exists some $x\in\mathbf{G}^\circ$ such that $x^{-1}F(x)=y^{-1}F(y)$ and therefore $xy^{-1}\in\hat{\mathbf{G}}^F$. In particular, we have $y\mathbf{U}\in xy^{-1}\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. This proves part (a).

Let us now prove part (b). The map

$$\varphi:\hat{\mathbf{G}}^F\times\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}\to(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})\times\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}},\ (x,g\mathbf{U})\mapsto((x,1),g\mathbf{U}),$$

is a morphism of varieties. For $y\in\mathbf{G}^F$ the cosets of $\varphi(xy,y^{-1}g\mathbf{U})=((xy,1),y^{-1}g\mathbf{U})$ and $\varphi(x,g\mathbf{U})=((x,1),g\mathbf{U})$ are equal in the quotient variety $(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})\times_{\mathcal{D}}\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}$ since $(y^{-1},1)\in\mathcal{D}$. Therefore, the map $\varphi$ factors through the diagonal action of $\mathbf{G}^F$ and we obtain a $(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})$-equivariant morphism

$$\overline{\varphi}:\hat{\mathbf{G}}^F\times_{\mathbf{G}^F}\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}\to(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})\times_{\mathcal{D}}\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}.$$

We define a morphism of varieties

$$\psi:(\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}})\times\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}\to\hat{\mathbf{G}}^F\times\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}\,((x,y),g\mathbf{U})\mapsto(xy,y^{-1}g\mathbf{U}y).$$

For $(h, l) \in \mathcal{D}$ we have

$$\psi((xh, ly), h^{-1}gl^{-1}\mathbf{U}) = (xhly, (ly)^{-1}h^{-1}gl^{-1}ly\mathbf{U}) = (x(hly), (hly)^{-1}gy\mathbf{U}).$$

Since $hl \in \mathbf{G}^F$ we may write $hly = yx_0$ for some $x_0 \in \hat{\mathbf{G}}^F$. Thus

$$\psi((xh, ly), h^{-1}gl^{-1}\mathbf{U}) = (xyx_0, x_0^{-1}ygy\mathbf{U}),$$

which shows that $\psi$ factors through the diagonal action and we obtain a morphism

$$\overline{\psi} : (\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}) \times_{\mathcal{D}} \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} \to \hat{\mathbf{G}}^F \times_{\mathbf{G}^F} \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}.$$

We check that $\overline{\psi}$ and $\overline{\varphi}$ are inverse to each other. First note that clearly $\psi \circ \varphi = \mathrm{id}$. On the other hand $(\overline{\varphi} \circ \overline{\psi})(((x, y), g\mathbf{U})) = ((xy, 1), y^{-1}g\mathbf{U}y) = (x, y), g\mathbf{U})$, where the last equality follows from $(y^{-1}, y) \in \mathcal{D}$. □

**Remark 2.9.** Let $\mathcal{D}_0 := ((\mathbf{G}^\circ)^F \times ((\mathbf{L}^\circ)^F)^{\mathrm{opp}})\Delta(\mathbf{L}^F)$. Then the proof of Lemma 2.8 shows that the map

$$(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}) \times_{\mathcal{D}_0} \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}^\circ} \to \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\ ((x, y), g\mathbf{U}) \mapsto xg\mathbf{U}y,$$

is an isomorphism. Using this description of the isomorphism, it is clear that this is an isomorphism of $\mathcal{D}$-varieties.

**Corollary 2.10.** *Under the assumption of Lemma 2.8 we have*

$$R\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}, \Lambda) \cong \Lambda[\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}] \otimes_{\Lambda\mathcal{D}}^{\mathbb{L}} R\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)$$

*in* $\mathrm{D}^b(\Lambda[\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}])$.

*Proof.* By Lemma 2.8 we have

$$\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}} \cong (\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}) \times_{\mathcal{D}} \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$$

as $(\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}})$-varieties. The group $\mathcal{D}$ acts freely by right multiplication on $(\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}})$. Hence, it follows that $\mathcal{D}$ acts freely on $(\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}) \times \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. Thus, Theorem 2.7 is applicable and we obtain

$$R\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\hat{\mathbf{G}}}, \Lambda) \cong \Lambda[\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}] \otimes_{\Lambda\mathcal{D}}^{\mathbb{L}} R\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)$$

in $\mathrm{D}^b(\Lambda[\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}])$. □

## 2.4 Godement resolutions

Let $\mathbf{X}$ be a variety defined over an algebraic closure of $\mathbb{F}_p$ endowed with an action of a finite group $G$. By work of Rickard and Rouquier there exists an object $G\Gamma_c(\mathbf{X},\Lambda)$ in $\mathrm{Ho}^b(\Lambda G\text{-perm})$ which is a representative of $R\Gamma_c(\mathbf{X},\Lambda) \in \mathrm{D}^b(\Lambda G)$, see [Ric94] and [Rou02, Section 2]. The object $G\Gamma_c(\mathbf{X},\Lambda)$ is essentially obtained as the $\tau_{\leq 2\dim(\mathbf{X})}$ truncation of the Godement resolution of the variety $\mathbf{X}$. However, the exact construction is much more technically involved.

The advantage of the Rickard–Rouquier complex $G\Gamma_c(\mathbf{X},\Lambda)$ is that it is a complex of $\ell$-permutation modules which is compatible with the Brauer functor. More precisely, if $Q$ is an $\ell$-subgroup of $G$ then we have a canonical isomorphism

$$\mathrm{Br}_Q(G\Gamma_c(\mathbf{X},\Lambda)) \cong G\Gamma_c(\mathbf{X}^Q,k)$$

in $\mathrm{Ho}^b(k\mathrm{N}_G(Q))$, see [Rou02, Theorem 2.29]. Building on this fundamental result, Bonnafé–Dat–Rouquier show the following:

**Lemma 2.11.** *Let $\mathbf{G}$ be a (non-necessarily connected) reductive group with parabolic subgroup $\mathbf{P}$ and Levi decomposition $\mathbf{P}=\mathbf{L}\ltimes\mathbf{U}$ such that $F(\mathbf{L})=\mathbf{L}$. For an $\ell$-subgroup $Q$ of $\mathbf{L}^F$ we have*

$$\mathrm{Br}_{\Delta Q}(G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)) \cong G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},k)$$

*in* $\mathrm{Ho}^b(k[\mathrm{N}_{\mathbf{G}^F\times\mathbf{L}^{\mathrm{opp}}}(\Delta Q)])$,

*Proof.* See [BDR17a, Proposition 3.4(e)] and [BDR17a, Remark 3.5]. □

**Lemma 2.12.** *The components of the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)^{\mathrm{red}}$ of $\Lambda[\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}]$-modules are relatively $\Delta\mathbf{L}^F$-projective, i.e., the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)^{\mathrm{red}}$ is splendid.*

*Proof.* The complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)^{\mathrm{red}}$ can be endowed with a $\Lambda[\mathbf{G}^F\times\mathrm{N}_{\mathbf{G}^F}(\mathbf{P},\mathbf{L})^{\mathrm{opp}}]$-structure, see [BDR17a, Remark 2.2]. The indecomposable summands of the components of the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)^{\mathrm{red}}$ of $\Lambda[\mathbf{G}^F\times\mathrm{N}_{\mathbf{G}^F}(\mathbf{P},\mathbf{L})^{\mathrm{opp}}]$-modules have a vertex contained in $\Delta\mathrm{N}_{\mathbf{G}^F}(\mathbf{P},\mathbf{L})^{\mathrm{opp}}$, see [BDR17a, Corollary 3.8]. Consequently, the components of $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},\Lambda)^{\mathrm{red}}$ considered as $\Lambda[\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}]$-modules are relatively $\Delta\mathbf{L}^F$-projective by Lemma 1.13. □

## 2.5 Isogenies

Let $\mathbf{G}$ be a connected reductive group. Recall that an *isogeny of algebraic groups* $\varphi:\mathbf{G}\to\mathbf{G}$ is a surjective homomorphism of algebraic groups with finite kernel. Let $\varphi : \mathbf{G} \to \mathbf{G}$ be an isogeny stabilizing a maximal torus $\mathbf{T}_0$ of $\mathbf{G}$. We write $X(\mathbf{T}_0)$ for the character group of $\mathbf{T}_0$ and $Y(\mathbf{T}_0)$ for the cocharacter group of $\mathbf{T}_0$, see [MT11, Definition 3.4]. The morphism $\varphi$ induces a group homomorphism $\varphi : X(\mathbf{T}_0) \to X(\mathbf{T}_0)$ and its dual morphism $\varphi^\vee : Y(\mathbf{T}_0) \to Y(\mathbf{T}_0)$, $y \mapsto \varphi \circ y$, which preserve the set of roots $\Phi(\mathbf{T}_0)$ resp. coroots $\Phi^\vee(\mathbf{T}_0)$. Moreover, these group homomorphisms satisfy

1. $\varphi$ and $\varphi^\vee$ are injective.
2. There exists a bijection $\Phi(\mathbf{T}_0) \to \Phi(\mathbf{T}_0), \alpha \mapsto \alpha'$ and positive integers $q(\alpha)$, which are integral powers of $p$, such that $\varphi(\alpha') = q(\alpha)\alpha$ and $\varphi^\vee(\alpha^\vee) = q(\alpha)(\alpha')^\vee$.

We call any group homomorphism $f : X(\mathbf{T}_0) \to X(\mathbf{T}_0)$ with these two properties an *isogeny of the root datum* $(X(\mathbf{T}_0), \Phi(\mathbf{T}_0), Y(\mathbf{T}_0), \Phi^\vee(\mathbf{T}_0))$.

We recall the isogeny theorem:

**Theorem 2.13.** *Let $\mathbf{G}$ be a connected reductive group and $\mathbf{T}_0$ a maximal torus. Then for every isogeny $f : X(\mathbf{T}_0) \to \mathbf{X}(\mathbf{T}_0)$ of the root datum $(X(\mathbf{T}_0), \Phi(\mathbf{T}_0), Y(\mathbf{T}_0), \Phi^\vee(\mathbf{T}_0))$ there exists an isogeny $\varphi : \mathbf{G} \to \mathbf{G}$ inducing $f$ on $X(\mathbf{T}_0)$ which is unique up to inner automorphisms induced by $\mathbf{T}_0$.*

*Proof.* See [Spr09, Theorem 9.6.2]. □

## 2.6 Duality for connected reductive groups

The following material can be found in [DM91, Chapter 13]. Let $\mathbf{G}$ be a connected reductive group with maximal torus $\mathbf{T}_0$. Let $\mathbf{G}^*$ be a connected reductive group with maximal torus $\mathbf{T}_0^*$. We say that $(\mathbf{G}^*, \mathbf{T}_0^*)$ is *dual to* $(\mathbf{G}, \mathbf{T}_0)$ if there exists an isomorphism $\delta : X(\mathbf{T}_0^*) \to Y(\mathbf{T}_0)$ which induces an isomorphism between the root data $(X(\mathbf{T}_0), \Phi(\mathbf{T}_0), Y(\mathbf{T}_0), \Phi^\vee(\mathbf{T}_0))$ and $(X(\mathbf{T}_0^*), \Phi(\mathbf{T}_0^*), Y(\mathbf{T}_0^*), \Phi^\vee(\mathbf{T}_0^*))$, see [DM91, Definition 13.10].

Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism and assume that $\mathbf{T}_0$ is $F$-stable. By Theorem 2.13 there exists a Frobenius endomorphism $F^* : \mathbf{G}^* \to \mathbf{G}^*$ satisfying $\delta \circ F = (F^*)^\vee \circ \delta$ on $Y(\mathbf{T}_0^*)$. We then say that the triple $(\mathbf{G}, \mathbf{T}_0, F)$ is *in duality* with $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$, see [DM91, Definition 13.10]. Note that we will sometimes drop the $^*$ and write $F$ for both Frobenius endomorphisms.

In the following we write $W := \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ for the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$ and $W^* := \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)/\mathbf{T}_0^*$ for the Weyl group of $\mathbf{G}^*$ with respect to $\mathbf{T}_0^*$. Since $W$ stabilizes the torus $\mathbf{T}_0$ we have a natural action of $W$ on $X(\mathbf{T}_0)$ and a natural action on $Y(\mathbf{T}_0)$. For $w \in W$ we let $n_w \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$

be a representative of $w \in W$ and analogously $n_{w^*} \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)$ denotes a representative of $w^* \in W^*$.

Using the natural action of $W$ on $X(\mathbf{T}_0)$ and the action of $W^*$ on $Y(\mathbf{T}_0^*)$ one can show that the duality isomorphism $\delta : X(\mathbf{T}_0) \to Y(\mathbf{T}_0^*)$ induces an anti-isomorphism $^* : W \to W^*$. Moreover, this anti-isomorphism satisfies $w^* = F^*(F(w)^*)$ for all $w \in W$, see [Car93, Proposition 4.3.2].

Recall the following definition:

**Definition 2.14.** Let $G$ be a group and $\varphi : G \to G$ an automorphism of $G$. The action $G \times G \to G$, $(x, y) \mapsto xy\varphi(x)^{-1}$, is called *$\varphi$-conjugation* in $G$. We say that two elements $x, y \in G$ are *$\varphi$-conjugate* if $x\varphi$ and $y\varphi$ are conjugate (by an element of $G$) in $G \rtimes \langle \varphi \rangle$.

One can show that $^* : W \to W^*$ induces a bijection between the $F$-conjugacy classes of $W$ and the $F^*$-conjugacy classes of $W^*$, see [Car93, Proposition 4.3.4(ii)].

In the following, we write $(\mathbb{Q}/\mathbb{Z})_{p'}$ for the subgroup of elements of $p'$-order of the additive group $\mathbb{Q}/\mathbb{Z}$. We fix a group isomorphism

$$\iota : (\mathbb{Q}/\mathbb{Z})_{p'} \to \overline{\mathbb{F}_q}^\times,$$

an injective morphism

$$j : (\mathbb{Q}/\mathbb{Z})_{p'} \hookrightarrow \overline{\mathbb{Q}_\ell}^\times$$

and set

$$\kappa := j \circ \iota^{-1} : \overline{\mathbb{F}_q}^\times \hookrightarrow \overline{\mathbb{Q}_\ell}^\times.$$

We define a restriction map $\mathrm{res} : X(\mathbf{T}_0) \to \mathrm{Irr}(\mathbf{T}_0^F), \chi \mapsto \mathrm{Res}^{\mathbf{T}_0}_{\mathbf{T}_0^F}(\kappa \circ \chi)$. Let $n$ be an integer such that $\mathbf{T}_0^*$ is split over $\mathbb{F}_{q^n}$, i.e., this means that $(F^*)^n(t) = t^{q^n}$ for all $t \in \mathbf{T}_0^*$. We denote by

$$N_{F^{*n}/F^*} : \mathbf{T}_0^* \to \mathbf{T}_0^*,\ t \mapsto tF^*(t) \cdots (F^*)^{n-1}(t)$$

the norm map of the torus $\mathbf{T}_0^*$ with respect to $F^*$. Denote $\zeta = \iota(\frac{1}{q^n-1}) \in \overline{\mathbb{F}_q}^\times$ and define an evaluation map

$$\mathrm{ev} : Y(\mathbf{T}_0^*) \to (\mathbf{T}^*)^{F^*},\ y \mapsto N_{F^n/F}(y)(\zeta).$$

We then define $\delta_1 : \mathrm{Irr}(\mathbf{T}_0^F) \to (\mathbf{T}_0^*)^{F^*}$ to be the unique isomorphism which makes the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X(\mathbf{T}_0) & \xrightarrow{F-1} & X(\mathbf{T}_0) & \xrightarrow{\mathrm{res}} & \mathrm{Irr}(\mathbf{T}_0^F) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} & & \\
0 & \longrightarrow & Y(\mathbf{T}_0^*) & \xrightarrow{F^*-1} & Y(\mathbf{T}_0^*) & \xrightarrow{\mathrm{ev}} & (\mathbf{T}_0^*)^F & \longrightarrow & 1
\end{array}$$

We will now briefly explain how this construction can be generalized to arbitrary maximal $F$-stable tori of $\mathbf{G}$. Let $\mathbf{T}$ be an $F$-stable maximal torus of $\mathbf{G}$. Then there exists an element $g \in \mathbf{G}$ such that ${}^g\mathbf{T}_0 = \mathbf{T}$. We have $g^{-1}F(g) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ and we let $w \in W$ such that $w$ is the image of $g^{-1}F(g)$ under the map $\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0) \to W$. We say that $w$ is *the type* of the $F$-stable maximal torus $\mathbf{T}$. It follows that the map $\mathbf{T}^F \to \mathbf{T}_0^{wF}$, $t \mapsto t \mapsto {}^g t$, is an isomorphism of abelian groups. For the dual group, we let $h \in \mathbf{G}^*$ such that $h^{-1}F(h) = F^*(n_{w^*})$. It follows that the triple $(\mathbf{G}, \mathbf{T}_0, n_w F)$ is in duality with $(\mathbf{G}^*, \mathbf{T}_0^*, F^* n_{w^*})$ which yields a duality isomorphism $\delta_w : \mathrm{Irr}(\mathbf{T}_0^{wF}) \to (\mathbf{T}_0^*)^{F^* w^*}$. Furthermore, $\mathbf{T}^* := {}^h\mathbf{T}_0^*$ is a maximal $F^*$-stable torus of $\mathbf{G}^*$ of type $F^*(w^*)$ and we obtain that the triple $(\mathbf{G}, \mathbf{T}, F)$ is in duality with $(\mathbf{G}^*, \mathbf{T}^*, F^*)$. Hence the notion of duality does not depend on the choice of the maximal torus $\mathbf{T}_0$ which we made at the beginning of this section.

## 2.7 Levi subgroups, isogenies and duality

We recall the classification of $F$-stable Levi subgroups of a connected reductive group $\mathbf{G}$. Fix an $F$-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ contained in an $F$-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$. Let $\Phi$ be the root system of $\mathbf{G}$ relative to the torus $\mathbf{T}_0$ and $\Delta \subseteq \Phi$ the base of $\Phi$ associated to $\mathbf{T}_0 \subseteq \mathbf{B}_0$.

By [DM91, Proposition 4.3] the $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of $\mathbf{G}$ are classified by $F$-conjugacy classes of cosets $W_I w$, where $I \subseteq \Delta$ and $w \in W$ satisfies ${}^{wF}W_I = W_I$. More precisely, if $\mathbf{L}$ is an $F$-stable Levi subgroup of $\mathbf{G}$ of type $W_I w$ then there exists $g \in \mathbf{G}^F$ such that ${}^g\mathbf{L} = \mathbf{L}_I$ for some $I \subseteq \Delta$ and ${}^{g^{-1}}\mathbf{T}_0$ is a maximal torus of $\mathbf{L}$ of type $w = g^{-1}F(g)\mathbf{T}_0$. Here, $\mathbf{L}_I$ denotes the standard Levi subgroup of $\mathbf{G}$ associated to a subset $I$ of the base $\Delta$, see [MT11, Section 12.2].

An important property of duality is that it extends to Levi subgroups.

**Lemma 2.15.** *Suppose that $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ is in duality with $(\mathbf{G}, \mathbf{T}_0, F)$. Then the map which sends a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ of type $W_I w$ to a Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ of type $W_I^* F^*(w^*)$ induces a bijection between the $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of $\mathbf{G}$ and the $(\mathbf{G}^*)^{F^*}$-conjugacy classes of $F^*$-stable Levi subgroups of $\mathbf{G}^*$.*

*Proof.* See [CE04, Section 8.2]. □

The following remark which is taken from [CS13, Section 2.3] describes how Levi subgroups in duality with each other can be described without explicit reference to a maximally $F$-split torus.

**Remark 2.16.** Assume that $\mathbf{L}$ is an $F$-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{L}^*$ is an $F^*$-stable Levi subgroup of $\mathbf{G}^*$. Then the Levi subgroups $\mathbf{L}$ and $\mathbf{L}^*$ are in duality with each other if and only if there exists a maximal $F$-stable maximal torus $\mathbf{T} \subseteq \mathbf{L}$ and a maximal $F^*$-stable maximal torus $\mathbf{T}^* \subseteq \mathbf{L}^*$ such that $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ is in duality with $(\mathbf{G}, \mathbf{T}, F)$ and $\Phi(\mathbf{L}, \mathbf{T})$ corresponds to $\Phi^\vee(\mathbf{L}^*, \mathbf{T}^*)$ under the duality isomorphism $X(\mathbf{T}) \cong Y(\mathbf{T}^*)$.

The bijection of Lemma 2.15 has important properties, see [CE04, Section 8.2] for the following. Suppose that $\mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$ corresponding to a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ under the bijection in Lemma 2.15. Then it follows that $(\mathbf{L}, F)$ is in duality with $(\mathbf{L}^*, F^*)$. Moreover, the map $^* : W \to W^*$ induces a group anti-isomorphism between the corresponding Weyl groups of the Levi subgroups. This in turn induces an anti-isomorphism

$$\mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \cong \mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*,$$

which satisfies $w^* = F^*(F(w)^*)$ for all $w \in \mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$.

We will now define what it means for isogenies to be in duality with each other.

**Definition 2.17.** Suppose that $(\mathbf{G}, \mathbf{T}_0, F)$ is in duality with $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$. We say that isogenies $\sigma : \mathbf{G} \to \mathbf{G}$ and $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ are *in duality with each other* if there exist $g \in \mathbf{G}$ and $h \in \mathbf{G}^*$ such that $\sigma_0 := g\sigma$ stabilizes $\mathbf{T}_0$ (resp. $\sigma_0^* := h\sigma^*$ stabilizes $\mathbf{T}_0^*$) and $\delta \circ \sigma_0 = (\sigma_0^*)^\vee \circ \delta$ on $Y(\mathbf{T}_0^*)$.

Note that this means that dual isogenies are only defined up to inner automorphisms of $\mathbf{G}$ respectively $\mathbf{G}^*$.

The following remark is crucial for working with automorphisms of finite groups of Lie type, see also [NTT08, Section 2] and the proof of [CS13, Proposition 2.2].

**Remark 2.18.** Let $\sigma : \mathbf{G} \to \mathbf{G}$ be a bijective morphism commuting with the action of $F$. We want to show that there exists a bijective morphism $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ in duality with $\sigma$ which commutes with $F^*$.

Recall that we have fixed a pair $(\mathbf{T}_0, \mathbf{B}_0)$ consisting of an $F$-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ contained in an $F$-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$. Since $(\sigma(\mathbf{T}_0), \sigma(\mathbf{B}_0))$ is again such a pair it follows that ${}^g(\mathbf{T}_0, \mathbf{B}_0) = (\sigma(\mathbf{T}_0), \sigma(\mathbf{B}_0))$ for some $g \in \mathbf{G}^F$. Hence, we may assume that $\sigma$ stabilizes the pair $(\mathbf{T}_0, \mathbf{B}_0)$. Thus, Theorem 2.13 together with [Tay18, Lemma 5.5] shows that there exists a bijective morphism $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ in duality with $\sigma$. Moreover, by the uniqueness statement of Theorem 2.13 we can choose $\sigma^*$ such that $\sigma^* F^* = F^* \sigma^*$. (We first have $t\sigma^* F^* = F^* \sigma^*$ for some $t \in \mathbf{T}_0^*$. Then by Lang's theorem there exists $t_0 \in \mathbf{T}_0^*$ such that $t_0 \sigma^*$ commutes with $F^*$.) The isogeny $\sigma^*$ with these properties is then unique up to $(\mathbf{G}^*)^{F^*}$-conjugation.

**Corollary 2.19.** *Let $\sigma : \mathbf{G} \to \mathbf{G}$ be a bijective morphism $\sigma \circ F = F \circ \sigma$ and $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ be a dual isogeny with $\sigma^* \circ F^* = F^* \circ \sigma^*$. Under the bijection in Lemma 2.15, the set of $\sigma$-stable $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of $\mathbf{G}$ corresponds to the set of $\sigma^*$-stable $(\mathbf{G}^*)^{F^*}$-conjugacy classes of $F^*$-stable Levi subgroups of $\mathbf{G}^*$.*

*Proof.* As in Remark 2.18 we may assume without loss of generality that $\sigma$ stabilizes the pair $(\mathbf{T}_0, \mathbf{B}_0)$. We may also assume that $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ satisfies $\delta \circ \sigma = (\sigma^*)^\vee \circ \delta$ on $Y(\mathbf{T}_0^*)$, see Definition 2.17. In particular, this yields $w^* = \sigma^*(\sigma(w)^*)$ for all $w \in W$ (same proof as in [Car93, Proposition 4.3.2]).

Observe that the $\mathbf{G}^F$-conjugacy class of an $F$-stable Levi subgroup of type $W_I w$ is $\sigma$-stable if and only if $\sigma(W_I w)$ is $F$-conjugate to $W_I w$. This is equivalent to $\sigma^*(W_I^* F^*(w^*))$ being $F^*$-conjugate to $W_I^* F^*(w^*)$. The latter is now equivalent to the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $F^*$-stable Levi subgroups of $\mathbf{G}^*$ associated to $W_I^* F^*(w^*)$ being $\sigma^*$-stable. This gives the claim. □

## 2.8 Rational Lusztig series for connected reductive groups

We continue our discussion on duality from Section 2.6 and keep the same notation. The following results can be found in [DM91, Chapter 13] and [Bon06, Chapter 9, Chapter 11]. Our presentation follows [Tay18, Section 6].

We denote by $\nabla(\mathbf{G}, F)$ the set of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an $F$-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is an irreducible character of $\mathbf{T}^F$. The group $\mathbf{G}^F$ acts by conjugation on the set $\nabla(\mathbf{G}, F)$. We denote by $\nabla(\mathbf{T}_0, W, F)$ the set of pairs $(w, \theta)$ where $w \in W$ and $\theta \in \mathrm{Irr}(\mathbf{T}_0^{wF})$. Then the Weyl group $W$ acts on $\nabla(\mathbf{T}_0, W, F)$ via $z \cdot (w, \theta) = (zwF(z)^{-1}, {}^z\theta)$. Let $(w, \theta) \in \nabla(\mathbf{T}_0, W, F)$. By Lang's theorem there exists $g_w \in \mathbf{G}$ such that $g_w^{-1} F(g_w) = n_w$ and we have $({}^{g_w}\mathbf{T}_0, {}^{g_w}\theta) \in \nabla(\mathbf{G}, F)$. One can now show that the map

$$\nabla(\mathbf{T}_0, W, F)/W \to \nabla(\mathbf{G}, F)/\mathbf{G}^F,\ (w, \theta) \mapsto ({}^{g_w}\mathbf{T}_0, {}^{g_w}\theta),$$

is a bijection, see [Tay18, Lemma 6.2].

We denote by $\mathcal{S}(\mathbf{G}^*, F^*)$ the set of pairs $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an $F^*$-stable maximal torus of $\mathbf{G}^*$ and $s \in (\mathbf{T}^*)^{F^*}$. Clearly, $(\mathbf{G}^*)^{F^*}$ acts by conjugation on the set $\mathcal{S}(\mathbf{G}^*, F)$. We let $\mathcal{S}(\mathbf{T}_0^*, W^*, F)$ be the set of pairs $(w, s)$ where $w \in W^*$ and $s \in (\mathbf{T}_0^*)^{F^* w^*}$. As before we have an action of $W^*$ on $\mathcal{S}(\mathbf{T}_0^*, W^*, F^*)$ by setting $z \cdot (w, s) = (zwF^*(z)^{-1}, {}^{F^*(z)}s)$. Given $w \in W^*$ we obtain $({}^{h_w}\mathbf{T}_0^*, {}^{h_w}s) \in \mathcal{S}(\mathbf{G}^*, F^*)$ where $h_w^{-1} F^*(h_w) = F^*(n_w)$. Then the map

$$\mathcal{S}(\mathbf{T}_0^*, W^*, F^*)/W^* \to \mathcal{S}(\mathbf{G}^*, F^*)/(\mathbf{G}^*)^{F^*},\ (w, s) \mapsto ({}^{h_w}\mathbf{T}_0^*, {}^{h_w}s),$$

is a bijection, see [Tay18, Lemma 6.4].

By Section 2.6 we obtain for $w \in W$ a bijection $\delta_w : \mathrm{Irr}(\mathbf{T}_0^{wF}) \to (\mathbf{T}_0^*)^{w^*F^*}$. From this one concludes that the map

$$\nabla(\mathbf{T}_0, W, F)/W \to \mathcal{S}(\mathbf{T}_0^*, W^*, F^*)/W^*, \ (w, \theta) \mapsto (w^*, \delta_w(\theta))$$

is a bijection, see [Tay18, Lemma 6.6]. In particular, we obtain a bijection

$$\nabla(\mathbf{G}, F)/\mathbf{G}^F \to \mathcal{S}(\mathbf{G}^*, F^*)/(\mathbf{G}^*)^{F^*}.$$

We can now define the notion of rational series.

**Definition 2.20.** Fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$. The subset $\mathcal{X} \subseteq \nabla(\mathbf{G}, F)$ consisting of all pairs $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ which correspond to some $(\mathbf{T}^*, t) \in \mathcal{S}(\mathbf{G}^*, F^*)/(\mathbf{G}^*)^{F^*}$, where $t$ is $(\mathbf{G}^*)^{F^*}$-conjugate to $s$, under the bijection above is called the *rational series associated to the* $(\mathbf{G}^*)^{F^*}$*-conjugacy class of* $s \in (\mathbf{G}^*)^{F^*}$.

Let $\mathbf{T}$ be an $F$-stable maximal torus of $\mathbf{G}$ and let $\mathbf{B}$ be a Borel subgroup containing $\mathbf{T}$ such that $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. Then we write $\mathbf{Y_B} = \mathbf{Y_U}$ for the corresponding Deligne–Lusztig variety. Moreover we write

$$R_\mathbf{T}^\mathbf{G} := [\mathcal{R}_{\mathbf{T}\subseteq\mathbf{B}}^\mathbf{G}] : G_0(K\mathbf{T}^F) \to G_0(K\mathbf{G}^F), \ [M] \mapsto \sum_i (-1)^i [H_c^i(\mathbf{Y}_\mathbf{B}^\mathbf{G}, \Lambda) \otimes_{\Lambda\mathbf{T}^F} M],$$

for the corresponding Deligne–Lusztig induction. Note that the map $R_\mathbf{T}^\mathbf{G}$ does not depend on the choice of the Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$ by [DM91, Theorem 11.13] and the remarks following said theorem.

We can now define the notion of rational Lusztig series.

**Definition 2.21.**

(a) Let $s \in (\mathbf{G}^*)^F$ be semisimple. Then we define $\mathcal{E}(\mathbf{G}^F, s)$ to be the set of all irreducible characters occurring in a $R_\mathbf{T}^\mathbf{G}(\theta)$, where $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ is in the rational series associated to the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $s \in (\mathbf{G}^*)^{F^*}$.

(b) Let $s \in (\mathbf{G}^*)^F$ be a semisimple element of $\ell'$-order. We set

$$\mathcal{E}_\ell(\mathbf{G}^F, s) := \coprod_t \mathcal{E}(\mathbf{G}^F, t),$$

where $t$ runs over a set of representatives of conjugacy classes of semisimple elements of $(\mathbf{G}^*)^{F^*}$ such that $s = t_{\ell'}$.

(c) Let $s \in (\mathbf{G}^*)^F$ be a semisimple element of $\ell'$-order and define

$$e_s^{\mathbf{G}^F} := \sum_{\chi \in \mathcal{E}_\ell(\mathbf{G}^F, s)} e_\chi,$$

where $e_\chi \in \mathrm{Z}(K\mathbf{G}^F)$ is the central idempotent corresponding to $\chi$.

**Theorem 2.22** (Broué–Michel)**.** *Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order. Then we have $e_s^{\mathbf{G}^F} \in \mathrm{Z}(\mathcal{O}\mathbf{G}^F)$ and hence $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ is a sum of blocks.*

*Proof.* See [CE04, Theorem 9.12]. □

In [BR03] the previous result is formulated in the language of derived categories and in [BDR17a] using this language generalized to disconnected reductive groups. We will recall this result in the next section.

## 2.9 Lusztig series for disconnected reductive groups

We give an elementary description of Lusztig series for disconnected reductive groups introduced in [BDR17a].

Let $\mathbf{G}$ be a non-necessarily connected reductive group. Note that the maximal tori of $\mathbf{G}$ are the maximal tori of $\mathbf{G}^\circ$. As in the case of connected reductive groups, we denote by $\nabla(\mathbf{G}, F)$ the set of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an $F$-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is an irreducible character of $\mathbf{T}^F$.

We denote by $\nabla_{\ell'}(\mathbf{G}, F)$ the subset of $\nabla(\mathbf{G}, F)$ consisting of the pairs $(\mathbf{T}, \theta)$ such that the order of $\theta$ is coprime to $\ell$. Note that $\mathrm{Irr}(\mathbf{T}^F)_{\ell'}$ can be identified with the set of characters $\mathbf{T}^F \to \Lambda^\times$. We denote by $e_\theta \in \mathrm{Z}(\Lambda\mathbf{T}^F)$ the unique central primitive idempotent of $\Lambda\mathbf{T}^F$ with $\theta(e_\theta) \neq 0$.

**Definition 2.23.** We say that two pairs $(\mathbf{T}_1, \theta_1) \in \nabla(\mathbf{G}, F)$ and $(\mathbf{T}_2, \theta_2) \in \nabla(\mathbf{G}, F)$ are *rationally conjugate* if there exists some $t \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{T}_1)$ such that $(\mathbf{T}_1, {}^t\theta_1)$ and $(\mathbf{T}_2, \theta_2)$ are rationally conjugate in $\mathbf{G}^\circ$. We write $\nabla(\mathbf{G}, F)/\equiv$ for the set of equivalence classes under rational conjugation.

For the following, we need to recall some of the standard definitions related to triangulated categories. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ be a full triangulated subcategory of $\mathcal{T}$. We say that $\mathcal{S}$ is a *thick subcategory* of $\mathcal{T}$ if it is closed under taking direct summands, see [Ric89, Proposition 1.3]. If $S$ is a set of objects in $\mathcal{T}$ we say that $\mathcal{S}$ is the *subcategory generated* *by S* if it is the smallest full thick triangulated subcategory containing $S$, see [BR03, Section 2.3]. For instance, if $A$ is a $\Lambda$-algebra which is free and of finite type over $\Lambda$, then one can show that $A$-perf, see Section 1.2, is the thick subcategory of $\mathrm{D}^b(A)$ generated by the regular $A$-module $A$ (either by direct calculations or by using [BR03, Lemma 9.1]).

With this in mind, we can now state the following definition from [BDR17a, 4.D.].

**Definition 2.24.** Let $\mathcal{X} \subseteq \nabla_{\ell'}(\mathbf{G}, F)$ be a rational series of $\mathbf{G}$. We denote by $\mathcal{C}_{\mathcal{X}}$ the thick subcategory of $\Lambda\mathbf{G}^F$-perf generated by the complexes $R\Gamma_c(\mathbf{Y_B})e_\theta$, with $(\mathbf{T}, \theta) \in \mathcal{X}$ and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}^\circ$ with maximal torus $\mathbf{T}$. We denote by $e_{\mathcal{X}} \in \Lambda\mathbf{G}^F$ the central idempotent such that $\mathcal{C}_{\mathcal{X}} = \Lambda\mathbf{G}^F e_{\mathcal{X}}$-perf.

Note that the existence of the idempotents $e_{\mathcal{X}} \in \mathrm{Z}(\Lambda\mathbf{G}^F)$ is ensured by [BDR17a, Theorem 4.12].

**Remark 2.25.** Suppose that $\mathbf{G}$ is a connected reductive group. We let $s \in (\mathbf{G}^*)^F$ be a semisimple element of $\ell'$-order such that its $(\mathbf{G}^*)^F$-conjugacy class is associated to the rational series $\mathcal{X} \in \nabla_{\ell'}(\mathbf{G}, F)$. Then we have $e_s^{\mathbf{G}^F} = e_{\mathcal{X}}$, see [BR03, Remark 9.3]. In other words, Definition 2.24 is consistent with Definition 2.21.

**Lemma 2.26.** *Let $\mathcal{X} \subseteq \nabla_{\ell'}(\mathbf{G}, F)$ be a rational series of $\mathbf{G}$ and choose a rational series $\mathcal{X}^\circ$ of $\mathbf{G}^\circ$ such that $\mathcal{X}^\circ \subseteq \mathcal{X}$. Then we have $e_{\mathcal{X}} = \mathrm{Tr}^{\mathbf{G}^F}_{\mathrm{N}_{\mathbf{G}^F}(e_{\mathcal{X}^\circ})}(e_{\mathcal{X}^\circ})$.*

*Proof.* Write $e'_{\mathcal{X}} := \sum_{g \in \mathbf{G}^F/\mathrm{N}_{\mathbf{G}^F}(e_{\mathcal{X}^\circ})} {}^g e_{\mathcal{X}^\circ}$. Let $(\mathbf{T}, \theta) \in \mathcal{X}$ and $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}^\circ$ with maximal torus $\mathbf{T}$. By [BDR17a, (3.1)] we have

$$\mathrm{Ind}^{\mathbf{G}^F}_{(\mathbf{G}^\circ)^F}(R\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{\mathbf{B}})e_\theta) \cong R\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}})e_\theta \cong \mathrm{Ind}^{\mathbf{G}^F}_{(\mathbf{G}^\circ)^F}(R\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{{}^t\mathbf{B}})e_{{}^t\theta})$$

for any $t \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{T})$. Therefore, the generators of $\mathcal{C}_{\mathcal{X}}$ lie inside $\Lambda\mathbf{G}^F e'_{\mathcal{X}}$-perf. Thus, $\mathcal{C}_{\mathcal{X}}$ is a subcategory of $\Lambda\mathbf{G}^F e'_{\mathcal{X}}$-perf and we have $e_{\mathcal{X}} e'_{\mathcal{X}} \neq 0$. By [BDR17a, Theorem 4.12] it follows that we have two decompositions

$$1 = \sum_{\mathcal{Z} \in \nabla_{\ell'}(\mathbf{G},F)/\equiv} e_{\mathcal{Z}} = \sum_{\mathcal{Z} \in \nabla_{\ell'}(\mathbf{G},F)/\equiv} e'_{\mathcal{Z}}.$$

into orthogonal central idempotents. From this we deduce that $e_{\mathcal{X}} = e'_{\mathcal{X}}$. □

We recall the definition of (super)-regular rational series, see [BR03, Section 11.4] and in particular [BR03, Lemma 11.6].

**Definition 2.27.** Let $\mathbf{G}$ be a connected reductive group and $\mathbf{L}$ be a $F$-stable Levi subgroup of $\mathbf{G}$. We say that the rational series of $(\mathbf{L}, F)$ associated to the conjugacy class of the semisimple element $s \in (\mathbf{L}^*)^{F^*}$ is $(\mathbf{G}, \mathbf{L})$-*regular* (respectively *superregular*) if $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ (respectively $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$).

This notation can now be naturally extended to rational series of disconnected reductive groups. Let $\mathbf{G}$ be a reductive group. If $\mathcal{X}$ is a rational series of $(\mathbf{L}, F)$ we say that $\mathcal{X}$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-*regular* (respectively *superregular*) if any (and hence every) rational series of $(\mathbf{L}^\circ, F)$ contained in $\mathcal{X}$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-regular (respectively superregular). We then say that the central idempotent $e_\mathcal{X}$ associated to $\mathcal{X}$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-*(super)-regular*.

## 2.10 Lusztig series and Brauer morphism

Let $\mathbf{G}$ be a reductive group and $Q$ a finite $\ell$-subgroup of $\mathbf{G}^F$. Then we consider the map

$$i_Q^{\mathbf{G}} : \nabla_{\ell'}(\mathrm{C}_{\mathbf{G}}(Q), F)/\equiv \to \nabla_{\ell'}(\mathbf{G}, F)/\equiv$$

as defined in [BDR17a, Theorem 4.14]. By [BDR17a, Theorem 4.14] for any rational series $\mathcal{Y} \subseteq \nabla_{\ell'}(\mathbf{G}, F)$ we have

$$\mathrm{br}_Q^{\mathbf{G}^F}(e_\mathcal{Y}) = \sum_{\mathcal{Z} \in (i_Q^{\mathbf{G}})^{-1}(\mathcal{Y})} e_\mathcal{Z}.$$

**Lemma 2.28.** *Let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ and let $\mathcal{X} \subseteq \nabla_{\ell'}(\mathbf{L}, F)$ be a $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-(super)-regular rational series. Then for any $\ell$-subgroup $Q$ of $\mathbf{L}^F$ we have that*

$$\mathrm{br}_Q^{\mathbf{L}^F}(e_\mathcal{X}) = \sum_{\mathcal{Z} \in (i_Q^{\mathbf{L}})^{-1}(\mathcal{X})} e_\mathcal{Z}$$

*is a decomposition into orthogonal $(\mathrm{C}^\circ_{\mathbf{G}}(Q), \mathrm{C}^\circ_{\mathbf{L}}(Q))$-(super-)regular idempotents.*

*Proof.* See [BDR17a, Proposition 4.11]. □

We gather some useful facts.

**Lemma 2.29.** *Let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. In addition, let $\mathcal{X} \subseteq \nabla_{\ell'}(\mathbf{L}, F)$ be a (super)-regular rational series of $(\mathbf{G}^\circ, \mathbf{L}^\circ)$.*

*(a) There exists a unique rational series $\mathcal{Y} \subseteq \nabla_{\ell'}(\mathbf{G}, F)$ containing $\mathcal{X}$.*

*(b) Deligne–Lusztig induction restricts to a functor* $\mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} : \mathrm{D}^b(\Lambda\mathbf{L}^F e_{\mathcal{X}}) \to \mathrm{D}^b(\Lambda\mathbf{G}^F e_{\mathcal{Y}})$.

*(c) Given* $\mathcal{X}' \in (i_Q^{\mathbf{L}})^{-1}(\mathcal{X})$ *let* $\mathcal{Y}'$ *be the unique rational series of* $\nabla_{\ell'}(\mathrm{C}_{\mathbf{G}}(Q), F)$ *containing* $\mathcal{X}'$. *Then we have* $i_Q^{\mathbf{G}}(\mathcal{Y}) = \mathcal{Y}'$.

*Proof.* Let $\mathcal{X}^\circ$ be a rational series of $(\mathbf{L}^\circ, F)$ contained in $\mathcal{X}$. Then $\mathcal{X}^\circ$ is associated to the conjugacy class of a semisimple element $s \in ((\mathbf{L}^\circ)^*)^{F^*}$ of $\ell'$-order, see Definition 2.20. Then the rational series $\mathcal{Y}^\circ$ of $(\mathbf{G}^\circ, F)$ associated to $s \in ((\mathbf{G}^\circ)^*)^{F^*}$ is the unique rational series containing $\mathcal{Y}^\circ$. Thus, $\mathcal{Y}$ is the unique rational series of $(\mathbf{G}, F)$ containing $\mathcal{Y}^\circ$. This shows part (a).

Deligne–Lusztig induction restricts to a functor $\mathcal{R}_{\mathbf{L}^\circ\subseteq\mathbf{P}^\circ}^{\mathbf{G}^\circ} : \mathrm{D}^b(\Lambda(\mathbf{L}^\circ)^F e_s^{\mathbf{L}^F}) \to \mathrm{D}^b(\Lambda(\mathbf{G}^\circ)^F e_s^{\mathbf{G}^F})$ by [BR03, Theorem 11.4]. For part (b) it suffices to show that $\mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(M) \in \mathrm{D}^b(\Lambda\mathbf{G}^F e_{\mathcal{Y}})$ for $M = \Lambda\mathbf{L}^F e_{\mathcal{X}}$. By [BDR17a, (3.1)] we have

$$\mathrm{Ind}_{(\mathbf{G}^\circ)^F}^{\mathbf{G}^F} \circ \mathcal{R}_{\mathbf{L}^\circ\subseteq\mathbf{P}^\circ}^{\mathbf{G}^\circ} = \mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} \circ \mathrm{Ind}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F}.$$

By Lemma 2.26 we have $M = \mathrm{Ind}_{(\mathbf{L}^\circ)^F}^{\mathbf{L}^F}(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F})$ and we can conclude that $\mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(M) \in \mathrm{D}^b(\Lambda\mathbf{G}^F e_{\mathcal{Y}})$. This implies that $\mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$ restricts to a functor $\mathcal{R}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} : \mathrm{D}^b(\Lambda\mathbf{L}^F e_{\mathcal{X}}) \to \mathrm{D}^b(\Lambda\mathbf{G}^F e_{\mathcal{Y}})$.

We now prove part (c). By Lemma 2.11 we have

$$\mathrm{Br}_{\Delta Q}(G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})e_{\mathcal{X}}) \cong G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}), \Lambda)\mathrm{br}_Q(e_{\mathcal{X}}) = \mathrm{br}_Q(e_{\mathcal{Y}})G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}), \Lambda)\mathrm{br}_Q(e_{\mathcal{X}}).$$

On the other hand, by part (b) $\mathcal{R}_{\mathrm{C}_{\mathbf{L}}(Q)\subseteq\mathrm{C}_{\mathbf{P}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}$ restricts to a functor

$$\mathcal{R}_{\mathrm{C}_{\mathbf{L}}(Q)\subseteq\mathrm{C}_{\mathbf{P}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)} : \mathrm{D}^b(\Lambda\,\mathrm{C}_{\mathbf{L}^F}(Q)e_{\mathcal{X}'}) \to \mathrm{D}^b(\Lambda\,\mathrm{C}_{\mathbf{G}^F}(Q)e_{\mathcal{Y}'}).$$

This implies that $\mathrm{br}_Q(e_{\mathcal{Y}})e_{\mathcal{Y}'} \neq 0$ which shows that $e_{\mathcal{Y}'}$ appears in the decomposition into central idempotents of $\mathrm{br}_Q(e_{\mathcal{Y}})$ from Lemma 2.28. Therefore, we necessarily have $i_Q^{\mathbf{G}}(\mathcal{Y}) = \mathcal{Y}'$. □

## 2.11 Regular embedding and Lusztig series

We recall the following definition, see [CE04, Section 15.1]

**Definition 2.30.** Let $\mathbf{G}$ be a connected reductive group. A *regular embedding* of $\mathbf{G}$ is a morphism $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ of algebraic groups, where $\tilde{\mathbf{G}}$ is a connected reductive group such that $\mathrm{Z}(\tilde{\mathbf{G}})$ is connected and $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] \subseteq \iota(\mathbf{G})$.

In the presence of a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ we always assume that the regular embedding $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ is chosen in a way such that there exists a Frobenius endomorphism $F : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ satisfying $F \circ \iota = \iota \circ F$. Moreover, we will identify $\mathbf{G}$ with its image $\iota(\mathbf{G})$ in $\tilde{\mathbf{G}}$.

**Remark 2.31.** A standard way to define a regular embedding is the following (see [CE04, Section 15.1]): Let $\mathbf{G}$ be a connected reductive group and $\mathbf{S}$ be a torus of $\mathbf{G}$ containing $\mathrm{Z}(\mathbf{G})$. Then $\tilde{\mathbf{G}} := \mathbf{S} \times_{\mathrm{Z}(\mathbf{G})} \mathbf{G}$ is a connected reductive group with connected center $\mathbf{S}$ and the natural map $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ is a regular embedding. Moreover, if $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ one can choose $\mathbf{S}$ to be an $F$-stable torus of $\mathbf{G}$. Then the Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ extends in a natural way to $\tilde{\mathbf{G}}$ by defining $F(s, g) := F(s)F(g)$ for $(s, g) \in \tilde{\mathbf{G}} = \mathbf{S} \times_{\mathrm{Z}(\mathbf{G})} \mathbf{G}$.

Now let $\mathbf{G}$ be a connected reductive group with Frobenius $F : \mathbf{G} \to \mathbf{G}$ and $F$-stable maximal torus $\mathbf{T}_0$. Let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding of $(\mathbf{G}, F)$. We denote by $\tilde{\mathbf{T}}_0 := \mathrm{Z}(\tilde{\mathbf{G}})\iota(\mathbf{T}_0)$ the unique maximal torus of $\tilde{\mathbf{G}}$ containing $\mathbf{T}_0$. Let $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ be in duality with $(\mathbf{G}, \mathbf{T}_0, F)$. By [CE04, Section 15.1] there exists a surjective morphism $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ of dual groups with kernel a connected central torus of $\tilde{\mathbf{G}}^*$.

Suppose that $\mathbf{L}$ is an $F$-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Then $\tilde{\mathbf{L}} := \mathrm{Z}(\tilde{\mathbf{G}})\mathbf{L}$ is an $F$-stable Levi subgroup of $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{P}} := \mathrm{Z}(\tilde{\mathbf{G}})\mathbf{P}$ is a parabolic subgroup of $\tilde{\mathbf{G}}$ with Levi decomposition $\tilde{\mathbf{P}} = \tilde{\mathbf{L}} \ltimes \mathbf{U}$. By Lemma 2.8 we have a natural isomorphism

$$\mathrm{Res}^{\tilde{\mathbf{G}}^F}_{\mathbf{G}^F} \circ \mathcal{R}^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{L}}} \cong \mathcal{R}^{\mathbf{G}}_{\mathbf{L}} \circ \mathrm{Res}^{\tilde{\mathbf{L}}^F}_{\mathbf{L}^F}.$$

**Lemma 2.32.** *Let $\mathbf{G}$ be a connected reductive group and let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding of $\mathbf{G}$. Let $J$ be a set of representatives of the $(\tilde{\mathbf{G}}^*)^F$-conjugacy classes of $\ell'$-elements $\tilde{t} \in (\tilde{\mathbf{G}}^*)^F$ with $\iota^*(\tilde{t}) = s$. Then*

$$e_s^{\mathbf{G}^F} = \sum_{\tilde{t} \in J} e_{\tilde{t}}^{\tilde{\mathbf{G}}^F}.$$

*Proof.* This is part of [BDR17a, Lemma 7.4]. □

## 2.12 The Bonnafé–Dat–Rouquier Morita equivalence

Let $\mathbf{G}$ be a connected reductive group defined over an algebraic closure of $\mathbb{F}_p$, where $p$ is a prime number. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism of $\mathbf{G}$ defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$ as in Section 2.6. Fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order. Let $\mathbf{L}^*$ be an $F^*$-stable Levi subgroup of $\mathbf{G}^*$ which satisfies $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ and

$$\mathbf{L}^* \, \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} = \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \mathbf{L}^*.$$

This assumption is for instance satisfied if $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s)))$ is the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ or if $\mathbf{L}^*$ is any Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}$. Then we define

$$\mathbf{N}^* := \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \mathbf{L}^*$$

which is a subgroup of $\mathbf{G}^*$ by the property above. Note that $\mathbf{N}^*$ is an $F^*$-stable subgroup of $\mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)$. Let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ in duality with the Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. We let $\mathbf{N}$ be the subgroup of $\mathrm{N}_{\mathbf{G}}(\mathbf{L})$ corresponding to $\mathbf{N}^*$ under the isomorphism of the relative Weyl groups

$$\mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \cong \mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$$

induced by duality. The closed subgroup $\mathbf{N}$ of $\mathbf{G}$ is $F$-stable and it holds that $\mathbf{N}^F = \mathrm{N}_{\mathbf{G}^F}(\mathbf{L}, e_s^{\mathbf{L}^F})$ by [BDR17a, (7.1)]. We let $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. In addition, we let $d := \dim(\mathbf{Y}_{\mathbf{U}})$. By [BR03, Theorem 11.7] we have

$$H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \Lambda) e_s^{\mathbf{L}^F} = 0 \text{ for } i \neq d.$$

Hence, we are interested only in the $d$th cohomology group of the variety $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}$. For convenience, we will therefore use the following definition.

**Notation 2.33.** Let $\mathbf{X}$ be a variety of dimension $n$. Then we write $R\Gamma^{\dim}_c(\mathbf{X}, \Lambda) := R\Gamma_c(\mathbf{X}, \Lambda)[n]$ and $H^{\dim}_c(\mathbf{X}, \Lambda) := H^n_c(\mathbf{X}, \Lambda)$.

Let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Set $\tilde{\mathbf{L}} = \mathbf{L}\mathrm{Z}(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{N}} = \mathbf{N}\tilde{\mathbf{L}}$.

**Assumption 2.34.** *Suppose that the* $k[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta\tilde{\mathbf{L}}^F]$*-module*

$$H^{\dim}_c(\mathbf{Y}_{\mathbf{U}}, k) e_s^{\mathbf{L}^F}$$

*extends to a* $k[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta\tilde{\mathbf{N}}^F]$*-module.*

This assumption is for instance satisfied if $\mathbf{N}^F/\mathbf{L}^F$ is cyclic, see Lemma 1.32.

We have the following theorem, see [BDR17a, Theorem 7.7]:

**Theorem 2.35** (Bonnafé–Dat–Rouquier)**.** *Suppose that Assumption 2.34 holds. Then there exists an* $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$*-bimodule extending* $H^d_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \mathcal{O})e_s^{\mathbf{L}^F}$ *and for any such bimodule* $M$ *there exists a complex* $C$ *of* $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$*-bimodules extending* $G\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \Lambda)^{\mathrm{red}} e_s^{\mathbf{L}^F}$ *such that* $H^d(C) \cong M$*. The complex* $C$ *induces a splendid Rickard equivalence between* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ *and the bimodule* $M$ *induces a Morita equivalence between* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$*.*

*Proof.* In the proof of [BDR17a, Theorem 7.5] apply Assumption 2.34 instead of [BDR17a, Proposition 7.3]. The rest of the proof of the theorem is as in [BDR17a, Section 7]. □

Note that the assumption previously made that $\mathbf{L}^*$ normalizes $\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}$ is not necessary for the following theorem. This means we only assume that $\mathbf{L}^*$ is an $F^*$-stable Levi subgroup containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$.

**Theorem 2.36.** *Let* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *be two parabolic subgroups of* $\mathbf{G}$ *with common Levi complement* $\mathbf{L}$ *and unipotent radical* $\mathbf{U}_1$ *respectively* $\mathbf{U}_2$*. Then we have*

$$H_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1}, \Lambda)e_s^{\mathbf{L}^F} \cong H_c^{\dim}(\mathbf{Y}_{\mathbf{U}_2}, \Lambda)e_s^{\mathbf{L}^F}$$

*as* $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$*-bimodules.*

*Proof.* This is proved in [BDR17a, Theorem 7.2]. We sketch how the isomorphism of the theorem is obtained. All mentioned statements are proved in loc. cit. We define

$$\mathbf{Y}^{\mathrm{cl}}_{\mathbf{U}_1,\mathbf{U}_2} := \{(g_1\mathbf{U}_1, g_2\mathbf{U}_2) \in \mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2} \mid g_1\mathbf{U}_1 \in \mathbf{Y}_{\mathbf{U}_1}\},$$

which is a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-stable closed subvariety of $\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}$. We have a closed immersion $i_{\mathbf{U}_1,\mathbf{U}_2} : \mathbf{Y}^{\mathrm{cl}}_{\mathbf{U}_1,\mathbf{U}_2} \to \mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}$ and a natural projection map $\pi_{\mathbf{U}_1,\mathbf{U}_2} : \mathbf{Y}^{\mathrm{cl}}_{\mathbf{U}_1,\mathbf{U}_2} \to \mathbf{Y}_{\mathbf{U}_1}$.

We have an isomorphism $\pi^*_{\mathbf{U}_1,\mathbf{U}_2} : R\Gamma_c(\mathbf{Y}_{\mathbf{U}_1}, \Lambda)[-2d] \to R\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda)$ where $d = \dim(\mathbf{U}_1 \cap F(\mathbf{U}_1)) - \dim(\mathbf{U}_1 \cap \mathbf{U}_2 \cap F(\mathbf{U}_1))$. Moreover, we have a morphism $i^*_{\mathbf{U}_1,\mathbf{U}_2} : R\Gamma_c(\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda) \to R\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda)$. The resulting map

$$\psi_{\mathbf{U}_1,\mathbf{U}_2} = (\pi^*_{\mathbf{U}_1,\mathbf{U}_2})^{-1} \circ i^*_{\mathbf{U}_1,\mathbf{U}_2} : R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda) \to R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1}, \Lambda)$$

induces a quasi-isomorphism

$$\psi_{\mathbf{U}_1,\mathbf{U}_2,s} : R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda)e_s^{\mathbf{L}^F} \to R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1}, \Lambda)e_s^{\mathbf{L}^F}$$

of $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$-complexes. Similarily, the map $\psi_{\mathbf{U}_2,F(\mathbf{U}_1)}$ induces a quasi-isomorphism

$$\psi_{\mathbf{U}_2,F(\mathbf{U}_1),s} : R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_2,F(\mathbf{U}_1)}, \Lambda)e_s^{\mathbf{L}^F} \to R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_2}, \Lambda)e_s^{\mathbf{L}^F}$$

of $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$-complexes. However, the shift map

$$\mathrm{sh} : \mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2} \to \mathbf{Y}_{\mathbf{U}_2,F(\mathbf{U}_1)}$$

given by $(g_1\mathbf{U}_1, g_2\mathbf{U}_2) \mapsto (g_2\mathbf{U}_2, F(g_1\mathbf{U}_1))$ is $\mathbf{G}^F$-$\mathbf{L}^F$-equivariant and induces an equivalence of étale sites. In particular this map induces a quasi-isomorphism

$$\mathrm{sh}^* : R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_1,\mathbf{U}_2}, \Lambda) \to R\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{U}_2,F(\mathbf{U}_1)}, \Lambda)$$

of $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$-complexes. Consequently, we have a quasi-isomorphism

$$\Theta_{\mathbf{U}_2,\mathbf{U}_1} := \psi_{\mathbf{U}_2,F(\mathbf{U}_1),s} \circ \mathrm{sh}^* \circ \psi^{-1}_{\mathbf{U}_1,\mathbf{U}_2,s} : R\Gamma_c(\mathbf{Y}_{\mathbf{U}_1}, \Lambda)e_s^{\mathbf{L}^F} \to R\Gamma_c(\mathbf{Y}_{\mathbf{U}_2}, \Lambda)e_s^{\mathbf{L}^F}$$

of $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$-complexes. □

We single out a special case of Theorem 2.35.

**Theorem 2.37** (Bonnafé–Dat–Rouquier)**.** *Let $\mathbf{L}^*$ be an $F^*$-stable Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)\,\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}$. Then the complex $C = G\Gamma_c(\mathbf{Y}_\mathbf{U}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}$ of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$-$\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$ bimodules induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{G}^F}$. The bimodule $H^{\dim(\mathbf{Y}_\mathbf{U})}(C)$ induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$.*

# Chapter 3

# On the Bonnafé, Dat and Rouquier Morita equivalence

In this chapter we extend Theorem 2.35 to a case which is not covered by the arguments in [BDR17a]. Specifically we consider the situation for semisimple elements in type $D$ whose centralizer has non-cyclic component group. Some arguments in this chapter use considerations already present in an unpublished note by Bonnafé, Dat and Rouquier [BDR17b]. The material and the results of this chapter can also be found in the author's article [Ruh18].

## 3.1 A remark on Clifford theory

In this section we construct a counterexample to the statement of [BDR17a, Proposition 7.3]. Let us recall the assumptions of this proposition:

**Assumption 3.1.** *Let $\tilde{Y}$ be a finite group and $\tilde{X}$ and $Y$ be normal subgroups of $\tilde{Y}$. Assume that $\tilde{Y} = \tilde{X}Y$ and denote $X = \tilde{X} \cap Y$. Assume that $\mathbf{k}$ is a field with $[Y : X] \in \mathbf{k}^\times$. Let $M$ be a $Y$-invariant, finitely generated $kX$-module. Suppose that*

$$\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))/J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))) \cong \mathbf{k}^n$$

*for some $n$. Assume that the $\mathbf{k}\tilde{X}$-module $\mathrm{Ind}_X^{\tilde{X}}(M)$ extends to $\tilde{Y}$.*

Under Assumption 3.1 the authors claim in [BDR17a, Proposition 7.3] that the $\mathbf{k}X$-module $M$ extends to $Y$.

In their proof they show that the natural injection

$$\mathrm{End}_{\mathbf{k}X}^\times(M)/1+J(\mathrm{End}_{\mathbf{k}X}(M)) \hookrightarrow \mathrm{End}_{\mathbf{k}\tilde{X}}^\times(\mathrm{Ind}_X^{\tilde{X}}(M))/1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))$$

splits. However, it is not clear under these assumptions that this injection splits in a way compatible with the action of the quotient group $Y/X$. This is however needed in an essential way in the proof of [BDR17a, Proposition 7.3].

We will now construct an explicit counterexample to their statement. We are very much indebted to Gabriel Navarro for pointing this out to us.

**Example 3.2.** Assume that $\mathbf{k} = \mathbb{C}$ is the field of complex numbers. We consider the group $\tilde{Y}$ generated by the elements $t_1, t_2, n_1, n_2$ subject to the following defining relations:

- $t_1^4 = t_2^4 = 1 = [t_1, t_2]$ and $t_1^2 = t_2^2$.
- $t_1^{n_1} = t_1^{-1}$, $t_2^{n_1} = t_2$, $t_1^{n_2} = t_1$ and $t_2^{n_2} = t_2^{-1}$.
- $[n_1, n_2] = n_1^2 = n_2^2 = t_1^2$.

Using the computer program GAP [GAP19] one can check that $\tilde{Y}$ is a finite group of order 32. It is actually isomorphic to the extraspecial group of order 32 of type "-". However, we won't use this description in what follows. Moreover, one checks $\tilde{X} := \langle t_1, t_2 \rangle$ is an abelian normal subgroup of $\tilde{Y}$. In addition, $Y := \langle n_1, n_2 \rangle$ is a normal subgroup of $\tilde{Y}$ isomorphic to the dihedral group $D_8$. The group $X := \tilde{X} \cap Y = \langle t_1^2 \rangle$ is the center of $\tilde{Y}$ and has order 2.

Now let $M$ be a module affording the unique non-trivial irreducible complex character of $X$. Since $\tilde{X}$ is abelian it follows that $M$ extends to a $\mathbf{k}\tilde{X}$-module. By Clifford theory, it follows that

$$\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M)) \cong \mathbf{k}^{[\tilde{X}:X]} = \mathbf{k}^4.$$

The module $M$ is $\tilde{Y}$-stable since it is the unique non-trivial irreducible $\mathbf{k}X$-module. Furthermore, we have

$$\mathrm{Ind}_X^{\tilde{X}}(M) \cong M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

where the $M_i$ are pairwise non-isomorphic simple $\mathbf{k}\tilde{X}$-modules which restrict to $M$. From the explicit description of these modules, we conclude that the conjugation action of the quotient group $\tilde{Y}/\tilde{X}$ acts regularily on the set of isomorphism classes of $M_1, \ldots, M_4$. We deduce that $\mathrm{Ind}_{\tilde{X}}^{\tilde{Y}}(M_1)$ is (isomorphic to) an extension of $\mathrm{Ind}_X^{\tilde{X}}(M)$. Therefore, Assumption 3.1 is satisfied. However, the non-trivial character of the center of $D_8$ does not extend to $D_8$. Therefore, the module $M$ does not extend to $Y$. This contradicts the statement of [BDR17a, Proposition 7.3].

## 3.2 Steinberg relation

In this section we describe the Steinberg presentation of a simple algebraic group of simply connected type as introduced in [Ste16]. This will allow us to perform explicit computations in these groups.

Let $\Phi$ be an abstract indecomposable root system in a finite dimensional Euclidean vector space. Let $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base of the root system $\Phi$. We denote by $\Phi^+$ the set of positive roots, i.e. the subset of the root system $\Phi$ which consists of the roots which can be written as a linear combination of the simple roots with natural numbers as coefficients. We write $\Phi^\vee$ for the set of coroots of $\Phi$ with base given by $\Delta^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}$. We assume that $\Phi$ has at least rank 2 ,i.e., $\Phi$ is not of type $A_1$. We consider the group $\mathbf{G}$ generated by the set of symbols $\{\mathbf{x}_\alpha(t) \mid \alpha \in \Phi, t \in \overline{\mathbb{F}_p}\}$ subject to the following relations:

1. $\mathbf{x}_\alpha(t_1)\mathbf{x}_\alpha(t_2) = \mathbf{x}_\alpha(t_1 + t_2)$ for all $t_1$, $t_2 \in \overline{\mathbb{F}_p}$ and $\alpha \in \Phi$.

2. Let $\alpha, \beta \in \Phi$ with $\alpha \pm \beta \neq 0$. Then

$$[\mathbf{x}_\alpha(t_1), \mathbf{x}_\beta(t_2)] = \prod_{i,j>0,\, i\alpha+j\beta\in\Phi} \mathbf{x}_{i\alpha+j\beta}(c_{i,j,\alpha,\beta} t_1^i t_2^j),$$

   where the product is taken over a fixed order of the roots $\Phi$ and $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ are as in [Ste16, Lemma 15] (where the $c_{i,j,\alpha,\beta}$ are structure constants possibly depending on the chosen order).

3. $\mathbf{h}_\alpha(t_1)\mathbf{h}_\alpha(t_2) = \mathbf{h}_\alpha(t_1 t_2)$ for all $t_1$, $t_2 \in \overline{\mathbb{F}_p}^\times$, where $\mathbf{h}_\alpha(t) := \mathbf{n}_\alpha(t)\mathbf{n}_\alpha(-1)$ and $\mathbf{n}_\alpha(t) := \mathbf{x}_\alpha(t)\mathbf{x}_{-\alpha}(-t^{-1})\mathbf{x}_\alpha(t)$ for $t \in \overline{\mathbb{F}_p}^\times$.

Steinberg shows that the abstract group $\mathbf{G}$ is the universal Chevalley group constructed from $\Phi$ and $\overline{\mathbb{F}_p}$, see [Ste16, Theorem 8]. Furthermore, he shows that $\mathbf{G}$ can be given the structure of an algebraic group in a unique way such that the maps $\mathbf{x}_\alpha : (\overline{\mathbb{F}_p}, +) \to \mathbf{G}$, $t \mapsto \mathbf{x}_\alpha(t)$ for $\alpha \in \Phi$ are isomorphisms onto their image. The algebraic group $\mathbf{G}$ is then a simple algebraic group of simply connected type with root system isomorphic to $\Phi$, see [Ste16, Theorem 6] and the Existence Theorem in [Ste16, Chapter 5]. Moreover, $\mathbf{T}_0 = \{\mathbf{h}_\alpha(t) \mid \alpha \in \Phi^+, t \in \overline{\mathbb{F}_p}^\times\}$ is a maximal torus of $\mathbf{G}$ and we will (by abuse of notation) identify the root system of $\mathbf{G}$ with respect to the torus $\mathbf{T}_0$ with the abstract root system $\Phi$.

Note that $\mathbf{x}_\alpha(t)$, $\mathbf{h}_\alpha(t)$ and $\mathbf{n}_\alpha(t)$ are not uniquely defined and their relations depend on the choice of certain structure constants. However, the relations simplify in the case where the involved roots are orthogonal. For the following remark for $\alpha, \beta \in \Phi$ we define

$$p_{\alpha,\beta} := \max\{i \in \mathbb{Z} \mid -i\alpha + \beta \in \Phi\}$$

and

$$q_{\alpha,\beta} := \max\{i \in \mathbb{Z} \mid i\alpha + \beta \in \Phi\}.$$

**Remark 3.3.** Let $\alpha, \beta \in \Phi$ with $\alpha \perp \beta$, $u \in \overline{\mathbb{F}_p}$ and $t \in \overline{\mathbb{F}_p}^\times$. Then we have:

a) $\mathbf{x}_\beta(u)^{\mathbf{h}_\alpha(t)} = \mathbf{x}_\beta(u)$ and $\mathbf{n}_\beta(u)^{\mathbf{h}_\alpha(t)} = \mathbf{n}_\beta(u)$,

b) $\mathbf{x}_\beta(u)^{\mathbf{n}_\alpha(t)} = \mathbf{x}_\beta(u)$ and $\mathbf{n}_\beta(u)^{\mathbf{n}_\alpha(t)} = \mathbf{n}_\beta(u)$ if $p_{\alpha,\beta} = q_{\alpha,\beta} = 0$.

c) $\mathbf{x}_\beta(u)^{\mathbf{n}_\alpha(t)} = \mathbf{x}_\beta(-u)$ and $\mathbf{n}_\beta(u)^{\mathbf{n}_\alpha(t)} = \mathbf{n}_\beta(-u)$ if $p_{\alpha,\beta} = q_{\alpha,\beta} = 1$.

*Proof.* See [Spä06, Remark 2.1.7]. □

## 3.3 Notation

We introduce the notation which will be in force until the last section of this chapter. Let $\mathbf{G}^*$ be a simple, adjoint algebraic group of type $D_n$ with $n$ even and $F^* : \mathbf{G}^* \to \mathbf{G}^*$ be a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}^*$ such that $(\mathbf{G}^*)^{F^*}$ is of untwisted type $D_n$. Fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order. Then $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is an $F^*$-stable connected reductive group. Thus, there exists an $F^*$-stable maximal torus $\mathbf{T}_0^*$ of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ contained in an $F^*$-stable Borel subgroup $\mathbf{B}(s)$ of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$.

The dual group $\mathbf{G}$ of $\mathbf{G}^*$ is a simple simply connected group of type $D_n$. Therefore, there exists a surjective morphism $\pi : \mathbf{G} \to \mathbf{G}^*$ with kernel $\mathrm{Z}(\mathbf{G})$. Note that the existence of such a morphism $\pi$ is specific to the situation in type $D$ and does not exist in general for groups in duality with each other. We let $\mathbf{T}_0$ be the maximal torus of $\mathbf{G}$ such that $\mathbf{T}_0^* = \pi(\mathbf{T}_0)$. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism stabilizing $\mathbf{T}_0$ such that $(\mathbf{G}, \mathbf{T}_0, F)$ is in duality with $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$.

We denote by $W$ the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$ and by $W^*$ the Weyl group of $\mathbf{G}^*$ with respect to $\mathbf{T}_0^*$. The map $\pi$ induces an isomorphism $\pi : W \to W^*$ which allows $W$ to be identified with $W^*$. Under this identification, the anti-isomorphism $^* : W \to W^*$, induced by duality, is then given by inversion, i.e., $w^* = w^{-1}$ for all $w \in W$.

The root system of $\mathbf{G}$ can be described more explicitly as follows. Let $\overline{\Phi}$ be a root system of type $B_n$, $n$ even, with base $\{e_1, e_i - e_{i-1} \mid 2 \leq i \leq n\}$ where $\{e_i \mid 1 \leq i \leq n\}$ is the canonical orthonormal basis with respect to the standard scalar product on $\mathbb{R}^n$.

Consider the root system $\Phi \subseteq \overline{\Phi}$ consisting of all long roots of $\overline{\Phi}$. Recall that $\Phi$ is a root system of type $D_n$. Let $\overline{\mathbf{G}}$ be the simple, simply connected algebraic group defined over $\overline{\mathbb{F}_p}$ with root system $\overline{\Phi}$. By [MS16, Section 2.C] there exists an embedding $\mathbf{G} \hookrightarrow \overline{\mathbf{G}}$ such that the image of $\mathbf{T}_0$ is a maximal torus of $\overline{\mathbf{G}}$. In particular, we can identify $\Phi$ with the root system of $\mathbf{G}$ with respect to the torus $\mathbf{T}_0$ and $\overline{\Phi}$ with the root system of $\overline{\mathbf{G}}$ with respect to $\mathbf{T}_0$.

For $\bar{\alpha} \in \overline{\Phi}$ let $\mathbf{x}_{\bar{\alpha}}(r)$, $r \in \overline{\mathbb{F}_p}$, and $\mathbf{n}_{\bar{\alpha}}(r)$, $\mathbf{h}_{\bar{\alpha}}(r)$, $r \in \overline{\mathbb{F}_p}^{\times}$, be the Chevalley generators associated to the maximal torus $\overline{\mathbf{T}_0}$ of $\overline{\mathbf{G}}$ as in Section 3.2.

Using the embedding of $\mathbf{G}$ into $\overline{\mathbf{G}}$ we obtain a surjective group homomorphism

$$(\overline{\mathbb{F}_p}^{\times})^n \to \mathbf{T}_0,\, (\lambda_1, \dots, \lambda_n) \mapsto \prod_{i=1}^{n} \mathbf{h}_{e_i}(\lambda_i),$$

with kernel $\{(\lambda_1, \dots, \lambda_n) \in \{\pm 1\}^n \mid \prod_{i=1}^n \lambda_i = 1\}$. Hence we can write an element $\lambda \in \mathbf{T}_0$ (in a non-unique way) as $\lambda = \prod_{i=1}^n \mathbf{h}_{e_i}(\lambda_i)$ for suitable $\lambda_i \in \overline{\mathbb{F}_p}^{\times}$. For a subset $\mathcal{A} \subseteq \overline{\mathbb{F}_p}^{\times}$ with $\mathcal{A} = -\mathcal{A}$ we define

$$I_{\mathcal{A}}(\lambda) := \{j \in \{1, \dots, n\} \mid \lambda_j \in \mathcal{A}\}.$$

Note that this does not depend on the choice of the sequence $(\lambda_1, \dots, \lambda_n)$ but only on the element $\lambda \in \mathbf{T}_0$. Let $\omega_4 \in \overline{\mathbb{F}_p}^{\times}$ be a primitive 4th root of unity. By [MS16, Section 2.C] we have $\mathrm{Z}(\mathbf{G}) = \langle z_1, z_2 \rangle$, where $z_1 = \mathbf{h}_{e_1}(-1)$ and $z_2 = \prod_{i=1}^n \mathbf{h}_{e_i}(\omega_4)$.

We also fix a tuple $(t_1, \dots, t_n) \in (\overline{\mathbb{F}_p}^{\times})^n$ such that $t = \prod_{i=1}^n \mathbf{h}_{e_i}(t_i) \in \mathbf{T}_0$ satisfies $\pi(t) = s$.

Let $F_0 : \mathbf{G} \to \mathbf{G}$ be the Frobenius endomorphism defined by $\mathbf{x}_\alpha(t) \mapsto \mathbf{x}_\alpha(t^q)$, for $t \in \overline{\mathbb{F}_p}$ and $\alpha \in \Phi$. We let $F_0^* : \mathbf{G}^* \to \mathbf{G}^*$ be defined as the unique morphism satisfying $\pi \circ F_0 = F_0^* \circ \pi$. Then the map $\delta : X : (\mathbf{T}_0) \to Y(\mathbf{T}_0^*)$, $\chi \mapsto \pi \circ \chi^\vee$ induces a duality isomorphism between the triples $(\mathbf{G}^*, \mathbf{T}_0^*, F_0^*)$ and $(\mathbf{G}, \mathbf{T}_0, F_0)$. There exists an element $v \in W$ with preimage $\mathbf{m}_v \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ of $v$ such that $F = \mathbf{m}_v F_0$. Since $(\mathbf{G}, \mathbf{T}_0, F)$ is in duality with $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ there exists some $\mathbf{m}_{v^*} \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)$, a preimage of $v^*$ in $\mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)$, such that $F^* = F_0^* \mathbf{m}_{v^*}$ (Note that $F_0^*$ acts trivially on $W^*$, so $F_0^*(v^*) = v^*$).

## 3.4 Classifying semisimple conjugacy classes

As in Section 2.12 we let $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s)))$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ and $\mathbf{N}^* = \mathrm{C}_{\mathbf{G}^*}(s)\mathbf{L}^*$. Let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ containing the maximal torus $\mathbf{T}_0$ which is in duality with the Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. We set $\mathbf{N}$ to be the subgroup of $\mathrm{N}_{\mathbf{G}}(\mathbf{L})$ such that $\mathbf{N}/\mathbf{L} \cong \mathbf{N}^*/\mathbf{L}^*$ under the canonical isomorphism between $\mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ and $\mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ induced by duality.

Our aim is to prove the following proposition:

**Proposition 3.4.** *Let $\mathbf{G}$ be a simple, simply connected algebraic group such that $\mathbf{G}^F$ is of type $D_n$ with even $n \geq 4$. If $\ell \nmid (q^2-1)$ then the $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F$-bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule.*

We first observe the following:

**Lemma 3.5.** *In order to prove Proposition 3.4 we can assume that $\mathbf{N}^F/\mathbf{L}^F$ is non-cyclic.*

*Proof.* If $\mathbf{N}^F/\mathbf{L}^F$ is cyclic then Lemma 1.32 shows that $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule. □

The overall aim of the next two sections is to construct two commuting elements $\mathbf{n}_1, \mathbf{n}_2 \in \mathbf{N}^F$ such that $\mathbf{L}^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle = \mathbf{N}^F$ satisfying certain properties which will allow us to understand the Clifford theory between $\mathbf{L}^F$ and $\mathbf{N}^F$.

Firstly, let us give a more explicit description of the quotient group $\mathbf{N}^F/\mathbf{L}^F$. By definition we have an injective morphism

$$\mathbf{N}^*/\mathbf{L}^* \hookrightarrow \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s).$$

As in [Bon05, Lemma 2.6] we consider the morphism

$$\omega_s : \mathrm{C}_{\mathbf{G}^*}(s) \to \mathrm{Z}(\mathbf{G})$$

with $\omega_s(x) := [y, t]$ where $y \in \mathbf{G}$ satisfies $\pi(y) = x$. By [Bon05, Corollary 2.8] this induces an injection

$$\mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s) \hookrightarrow \mathrm{Z}(\mathbf{G}).$$

Thus, we have an embedding $\mathbf{N}/\mathbf{L} \hookrightarrow \mathrm{Z}(\mathbf{G})$, which induces a map $\mathbf{N}^F/\mathbf{L}^F \hookrightarrow \mathrm{Z}(\mathbf{G})^F$ on fixed points. As $\mathrm{Z}(\mathbf{G})^F \cong C^2_{\gcd(2,q-1)}$ we can assume by Lemma 3.5 that $q$ is odd and that $\mathbf{N}^F/\mathbf{L}^F \cong \mathrm{Z}(\mathbf{G})^F$. Let $W(s)$ (resp. $W^\circ(s)$) be the Weyl group of $\mathrm{C}_{\mathbf{G}^*}(s)$ (resp. $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$) with respect to $\mathbf{T}_0^*$. By [DM91, Remark 2.4] we have a canonical isomorphism

$$W(s)/W^\circ(s) \to \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s).$$

Recall that $\mathbf{T}_0^*$ is contained in a maximal $F^*$-stable Borel subgroup $\mathbf{B}(s)$ of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Let $\Phi(s)$ be the root system of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ with set of positive roots

$\Phi^+(s)$ associated to this choice. According to [Bon05, Proposition 1.3] we have $W(s) = W^\circ(s) \rtimes A(s)$, where $A(s) := \{w \in W(s) \mid w(\Phi^+(s)) = \Phi^+(s)\}$. Since $A(s)$ is $F^*$-stable this shows that the map

$$W(s)^{F^*}/W^\circ(s)^{F^*} \to (\mathrm{C}_{\mathbf{G}^*}(s))^{F^*}/(\mathrm{C}^\circ_{\mathbf{G}^*}(s))^{F^*}$$

is again an isomorphism. As the morphism $\omega_s$ induces an isomorphism

$$(\mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s))^{F^*} \cong \mathrm{Z}(\mathbf{G})^F \cong C_2 \times C_2$$

we conclude that there exist $w_1^*, w_2^* \in (W^*)^{F^*}$ with ${}^{w_1}t = tz_1$ and ${}^{w_2}t = tz_2$. Since $(W^*)^{F^*} = \mathrm{C}_{W^*}(v^*)$ we have $w_1, w_2 \in \mathrm{C}_W(v)$.

**Remark 3.6.** The set $I_{\{\pm 1, \pm\omega_4\}}(t)$ is non-empty.

*Proof.* Suppose that $I_{\{\pm 1, \pm\omega_4\}}(t) = \emptyset$. Write ${}^{w_1}t = \prod_{i=1}^n \mathbf{h}_{e_i}(s_i)$ for suitable $s_i \in \overline{\mathbb{F}_p}^\times$. Then ${}^{w_1}tt^{-1}\mathbf{h}_{e_1}(-1) = 1$ implies that $s_i t_i^{-1} \in \{\pm 1\}$ for all $i$. Now note that

$$\emptyset = I_{\{\pm 1, \pm\omega_4\}}(t) = I_{\{\pm 1, \pm\omega_4\}}(tz_1) = I_{\{\pm 1, \pm\omega_4\}}({}^{w_1}t).$$

Thus, $s_i, t_i \notin \{\pm 1, \pm\omega_4\}$ and so $s_i = t_i$ for all $i$. This leads to the contradiction ${}^{w_1}t = t$. $\square$

Recall that the Weyl group $\overline{W} = \mathrm{N}_{\overline{\mathbf{G}}}(\mathbf{T}_0)/\mathbf{T}_0$ can be identified with the subgroup

$$\{\sigma \in S_{\{\pm 1, \ldots, \pm n\}} \mid \sigma(-i) = -\sigma(i) \text{ for all } i = 1, \ldots, n\}$$

of $S_{\{\pm 1, \ldots, \pm n\}}$. By [GP00, Proposition 1.4.10] it follows that the natural map $W \hookrightarrow \overline{W}$ identifies the Weyl group $W$ as the kernel of the group homomorphism

$$\varepsilon : \overline{W} \to \{\pm 1\},\ \sigma \mapsto (-1)^{|\{i \in \{1, \ldots, n\} \mid \sigma(i) < 0\}|}.$$

**Lemma 3.7.** *In order to prove Proposition 3.4 we may assume that $t$ is of the form $t = \prod_{i=1}^n \mathbf{h}_{e_i}(t_i)$ such that $t_i = t_j$ whenever $t_j \in \{\pm t_i, \pm t_i^{-1}\}$.*

*Proof.* Let $\underline{n} := \{1, \ldots, n\}$. We define the equivalence relation $\sim$ on $\underline{n}$ by saying that $i \sim j$ if $t_j \in \{\pm t_i, \pm t_i^{-1}\}$. Let $K$ be a set of representatives for the equivalence classes of $\underline{n}$ under $\sim$. We let $K' := \{i \in \underline{n} \mid t_i^{-1} \in \{\pm t_k \mid k \in K\}\}$.

Let $x \in I_{\{\pm 1, \pm\omega_4\}}(t)$. Under the identification of the Weyl group with a subgroup of $S_{\{\pm 1, \ldots, \pm n\}}$ we set

$$w := (x, -x)^{|K'|} \prod_{k \in K'} (k, -k) \in W.$$

Since $\mathbf{h}_{e_i}(-1) = \mathbf{h}_{e_1}(-1) = z_1$ for all $i \in \{1, \ldots, n\}$ we see that either ${}^w t$ or ${}^w tz_1$ is of the desired form. We let $t' \in \{{}^w t, {}^w tz_1\}$ be said element. In order to prove Proposition 3.4 it is therefore harmless to replace $s$ by its conjugate $s' := {}^w s \in \mathbf{T}_0^*$. Since $\pi(t') = s'$ this element has a preimage $t' \in \mathbf{T}_0$ which is of the form as announced in the lemma. $\square$

From now on we assume that the element $t$ has the form given in Lemma 3.7. Recall that

$$\mathrm{C}_{\mathbf{G}}(t) = \langle \mathbf{T}_0, \mathbf{x}_\alpha(r) \mid \alpha \in \Phi \text{ with } \alpha(t) = 1, r \in \overline{\mathbb{F}_p}^\times \rangle.$$

Let $\alpha = e_i \pm e_j \in \Phi$ with $\alpha(t) = 1$. Then $\alpha(t) = (t_i t_j^{\pm 1})^2 = 1$ and therefore $t_i = \varepsilon t_j^{\mp 1}$ for some $\varepsilon \in \{\pm 1\}$. By the form of $t$ given in Lemma 3.7, this implies $t_i = t_j$. In addition, we have $\alpha = e_i - e_j$ if $t_i$ is not a 4th root of unity. Therefore, the root system $\Phi(t)$ of $\mathrm{C}_{\mathbf{G}}(t)$ is given by

$$\Phi(t) = (\{\pm e_i \pm e_j \mid i,j \in I_{\{\pm 1\}}(t)\} \cup \{\pm e_i \pm e_j \mid i,j \in I_{\{\pm \omega_4\}}(t)\} \cup \{e_i - e_j \mid t_i = t_j\}) \cap \Phi.$$

We write $W(t)$ for the Weyl group of $\mathrm{C}_{\mathbf{G}}(t)$ relative to the torus $\mathbf{T}_0$.

**Lemma 3.8.** *We have* $|I_{\{\pm 1\}}(t)| = |I_{\{\pm \omega_4\}}(t)| = 1$.

*Proof.* Recall that $w_2 \in W$ satisfies ${}^{w_2} t = tz_2$ where $z_2 = \prod_{i=1}^n \mathbf{h}_{e_i}(\omega_4)$. Therefore, we have

$$I_{\{\pm 1\}}({}^{w_2} t) = I_{\{\pm 1\}}(tz_2) = I_{\{\pm \omega_4\}}(t).$$

Thus, $w_2$ swaps the sets $I_{\{\pm 1\}}(t)$ and $I_{\{\pm \omega_4\}}(t)$. Hence, $|I_{\{\pm 1\}}(t)| = |I_{\{\pm \omega_4\}}(t)|$. Note that $I_{\{\pm 1\}}(t) = I_{\{\pm 1\}}({}^{w_1} t)$ and $I_{\{\pm \omega_4\}}(t) = I_{\{\pm \omega_4\}}({}^{w_1} t)$.

Suppose that $|I_{\{\pm 1\}}(t)| > 1$ and let $a, b \in I_{\{\pm 1\}}(t)$ with $a \neq b$. Fix $c, d \in I_{\{\pm \omega_4\}}(t)$ with $c \neq d$ and let $w_1' := (a, -a)(d, -d) \in W$. It follows that ${}^{w_1'} t = tz_1$.

Recall that

$$\mathrm{Z}(\mathrm{C}_{\mathbf{G}}(t)) = \bigcap_{\alpha \in \Phi(t)} \mathrm{Ker}(\alpha).$$

Let $\lambda = \prod_{i=1}^n \mathbf{h}_{e_i}(\lambda_i) \in \mathrm{Z}(\mathrm{C}_{\mathbf{G}}(t))$ be arbitrary. Since $e_a + e_b, e_a - e_b \in \Phi(t)$ we have $(\lambda_a \lambda_b^{\pm 1})^2 = 1$. This implies that $\lambda_a$ and $\lambda_b$ are 4th roots of unity. An analogue argument shows that $\lambda_c$ and $\lambda_d$ are also 4th roots of unity. We conclude that ${}^{w_1'} \lambda = \lambda z_1$ or ${}^{w_1'} \lambda = \lambda$ in this case.

Note that $\pi(\mathrm{C}_{\mathbf{G}}(t)) = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ by [Bon05, (2.2)]. From this we can conclude that

$$\pi(\mathrm{Z}^\circ(\mathrm{C}_{\mathbf{G}}(t))) = \mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s)).$$

As ${}^{w_1'}\pi(\lambda) = \pi(\lambda)$ for all $\lambda \in \mathrm{Z}(\mathrm{C}_{\mathbf{G}}(t))$ we conclude that $w_1' \in \mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s)))$. Since ${}^{w_1^{-1}w_1'}t = t$ it follows that $w_1^{-1}w_1' \in W(t)$. From this we deduce that $w_1 \in \mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s)))$. This contradicts the assumption $\mathbf{N}^*/\mathbf{L}^* \cong \mathrm{Z}(\mathbf{G})$.

We conclude that $|I_{\{\pm 1\}}(t)| \leq 1$. By Remark 3.6 we must have $|I_{\{\pm 1\}}(t)| = 1$. □

By the previous lemma, up to a change of coordinates, we may assume that $I_{\{\pm 1\}}(t) = \{1\}$ and $I_{\{\pm \omega_4\}}(t) = \{n\}$.

In the following remark (which will not be needed anymore) we relate our calculations to the classification of quasi-isolated elements in [Bon05].

**Remark 3.9.** According to the classification in [Bon05, Table II] there exist three $\mathbf{G}^*$-conjugacy classes of semisimple elements $s$ such that

$$A(s) = \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s) \cong C_2 \times C_2.$$

In two of these cases the minimal Levi subgroup $\mathbf{L}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ satisfies $\mathrm{C}_{\mathbf{G}^*}(s) \cap \mathbf{L}^* \neq \mathrm{C}^\circ_{\mathbf{G}^*}(s)$, i.e. $\mathrm{C}_{\mathbf{L}^*}(s)/\mathrm{C}^\circ_{\mathbf{L}^*}(s)$ is cyclic. In the other case we have that $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is of type $A_{n-3}$.

Let us now explain how we obtain this semisimple conjugacy class with our methods. If we set

$$t := \mathbf{h}_{e_n}(\omega_4) \prod_{i=2}^{n-1} \mathbf{h}_{e_i}(\omega)$$

for some $\omega \in \overline{\mathbb{F}_p}^\times \setminus \{\pm 1, \pm \omega_4\}$ then the element $s := \pi(t)$ is a representative of this third conjugacy class.

## 3.5 Computations in the Weyl group

Let us collect the information we have obtained so far. The root system $\Phi(t)$ of $\mathrm{C}_{\mathbf{G}}(t)$ is given by

$$\Phi(t) = \{e_i - e_j \mid t_i = t_j\} \setminus \{0\}.$$

Observe that $\mathrm{C}_{\mathbf{G}}(t)$ is an $F$-stable Levi subgroup of $\mathbf{G}$ which by Remark 2.16 is in duality with $\mathbf{L}^*$ so that $\mathbf{L} = \mathrm{C}_{\mathbf{G}}(t)$. In particular, since $\pi(t) = s$ we obtain $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$.

Let us introduce some further notation.

**Definition 3.10.** Let $I = \{2, \ldots, n-1\}$ and define $\overline{\Phi}' := \{\pm e_i \pm e_j \mid i, j \in I\} \setminus \{0\}$. Let

$$\mathbf{T}_1 := \langle \mathbf{h}_\alpha(r) \mid r \in \overline{\mathbb{F}_p}^\times, \alpha \in \{e_1 \pm e_n\}\rangle$$

and

$$\mathbf{G}_2 := \langle \mathbf{h}_{\bar{\alpha}}(\bar{r}), \mathbf{x}_\alpha(r) \mid \bar{\alpha} \in \overline{\Phi}', \alpha \in \Phi(t), \bar{r} \in \overline{\mathbb{F}_p}^\times, r \in \overline{\mathbb{F}_p}\rangle.$$

The roots $\{e_1 \pm e_n\}$ are orthogonal to those in $\overline{\Phi}'$ and no non-trivial linear combination of $\{e_1 \pm e_n\}$ and $\Phi(t)$ is a root in $\Phi$. Therefore, we have $\mathbf{T}_1 \subseteq \mathrm{Z}(\mathbf{L})$. For $\mathbf{T}_2 := \mathbf{G}_2 \cap \mathbf{T}_0$ we have $\mathbf{T}_0 = \mathbf{T}_1\mathbf{T}_2$. This implies $\mathbf{L} = \mathbf{T}_1\mathbf{G}_2$ and $\mathbf{T}_1 \cap \mathbf{T}_2 = \langle z_1 \rangle$.

**Lemma 3.11.** *Consider the restriction map*

$$\mathrm{Res} : \{\sigma \in S_{\{\pm 1, \ldots, \pm n\}} \mid \sigma(1), \sigma(n) \in \{\pm 1, \pm n\}\} \to S_{\{\pm 1, \pm n\}}.$$

*We have* $\mathrm{Res}(v) \in \langle (1,-1)(n,-n), (1,-n)(-1,n)\rangle$.

*Proof.* Firstly, note that ${}^vF_0(s) = s$ which implies that $I_{\{\pm 1, \pm\omega_4\}}({}^vt) = I_{\{\pm 1, \pm\omega_4\}}(t)$. Therefore, $\mathrm{Res}(v)$ is well-defined.

Since $w_2$ permutes the sets $I_{\{\pm 1\}}(t) = \{1\}$ and $I_{\{\pm\omega_4\}}(t) = \{n\}$ we have $\mathrm{Res}(w_2) = (1,-n)(-1,n)$. Let $w_1' = (1,-1)(n,-n) \in W$. Then we have ${}^{w_1'}t = tz_1$. This implies that $w_1' w_1^{-1} \in W(t)$. Since $W(t) \subseteq \mathrm{Ker}(\mathrm{Res})$ we must have $\mathrm{Res}(w_1) = (1,-1)(n,-n)$.

As $w_1, w_2 \in \mathrm{C}_W(v)$ we have $[\mathrm{Res}(w_i), \mathrm{Res}(v)] = 1$ for $i = 1, 2$. Thus,

$$\mathrm{Res}(v) \in \mathrm{C}_{S_{\{\pm 1, \pm n\}}}(\langle (1,-1)(n,-n), (1,-n)(-1,n)\rangle).$$

A short calculation shows that the subgroup $\langle (1,-1)(n,-n), (1,-n)(-1,n)\rangle$ is self-centralizing in $S_{\{\pm 1, \pm n\}}$. □

**Lemma 3.12.** *Let $A, B \subseteq \overline{\Phi}$ such that $A \perp B$. Let $x = \prod_{\alpha \in A} \mathbf{n}_\alpha(r_\alpha)$ and $y = \prod_{\beta \in B} \mathbf{n}_\beta(r_\beta)$ for $r_\alpha, r_\beta \in \overline{\mathbb{F}_p}^\times$. If $x, y \in \mathbf{G}$ then $x$ and $y$ commute.*

*Proof.* Recall that the inclusion map $\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0) \hookrightarrow \mathrm{N}_{\overline{\mathbf{G}}}(\mathbf{T}_0)$ induces the embedding $W \hookrightarrow \overline{W}$ such that $W = \mathrm{Ker}(\varepsilon)$. We note that $\varepsilon(\mathbf{n}_\alpha(1)\mathbf{T}_0) = -1$ for $\alpha \in \overline{\Phi}$ if and only if $\alpha$ is a short root. As $x, y \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ we deduce that the number of short roots in $A$ resp. $B$ is even.

Let $\alpha \in A$ and $\beta \in B$. By Remark 3.3(b) we have $\mathbf{n}_\alpha(r_\alpha)^{\mathbf{n}_\beta(r_\beta)} = \mathbf{n}_\alpha(r_\alpha)$, if either $\alpha$ or $\beta$ is a long root. On the other hand by Remark 3.3(c), we have $\mathbf{n}_\alpha(r_\alpha)^{\mathbf{n}_\beta(r_\beta)} = \mathbf{h}_\alpha(-1)\mathbf{n}_\alpha(r_\alpha)$ if both $\alpha$ and $\beta$ are short roots. Note that if $\alpha$ is a short root then $\mathbf{h}_\alpha(-1) = \mathbf{h}_{e_1}(-1)$. The result follows from this. □

In the following, we will consider the element

$$\mathbf{n}_1 := \mathbf{n}_{e_1}(1)\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)} \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0),$$

which is a preimage of $w_1' = (1,-1)(n,-n) \in W$. By the proof of Lemma 3.11 it is possible to find $\mathbf{n}_2' \in \langle \mathbf{n}_{e_i}(r)\mathbf{n}_{e_j}(s), \mathbf{n}_{e_i-e_j}(r) \mid i \neq j, i,j \in I, r,s \in \overline{\mathbb{F}_p}^{\times} \rangle$ such that the element

$$\mathbf{n}_2 := \mathbf{n}_{e_1-e_n}(1)\mathbf{n}_2' \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$$

is a preimage of $w_2 \in W$.

**Lemma 3.13.** *The elements* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *commute. In addition, we have* $\mathbf{n}_1 \in \mathrm{C}_{\mathbf{G}}(\mathbf{G}_2)$.

*Proof.* Let us first prove that $\mathbf{n}_1$ and $\mathbf{n}_2$ commute. By Remark 3.3(c) we have $\mathbf{n}_{e_j}(u)^{\mathbf{n}_{e_i}(1)} = \mathbf{n}_{e_j}(-u) = \mathbf{h}_{e_j}(-1)\mathbf{n}_{e_j}(u)$ for $u \in \overline{\mathbb{F}_p}$, whenever $i \neq j$. By the relation in [Spä06, Theorem 2.1.6(b)] we have $\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)} \in \{\mathbf{n}_{e_n}(1), \mathbf{n}_{e_n}(-1)\}$. By Lemma 3.12,

$$\mathbf{n}_1^{\mathbf{n}_2} = \mathbf{n}_1^{\mathbf{n}_{e_1-e_n}(1)} = \mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)^2} = \mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)^{\mathbf{h}_{e_1-e_n}(-1)}.$$

According to [Spä06, Remark 2.1.8] we have $\mathbf{h}_{e_1-e_n}(-1) = \mathbf{h}_{e_1}(\omega_4)\mathbf{h}_{e_n}(\omega_4^{-1})$, where $\omega_4 \in \overline{\mathbb{F}_p}^{\times}$ is a fourth root of unity. Using Remark 3.3(a), we obtain

$$\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)^{\mathbf{h}_{e_1-e_n}(-1)} = \mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)^{\mathbf{h}_{e_1}(\omega_4)} = \mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)z_1.$$

Since $\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)} \in \{\mathbf{n}_{e_n}(1), \mathbf{n}_{e_n}(-1)\}$ we deduce that

$$\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)}\mathbf{n}_{e_1}(1)z_1 = \mathbf{n}_{e_1}(1)\mathbf{n}_{e_1}(1)^{\mathbf{n}_{e_1-e_n}(1)} = \mathbf{n}_1.$$

Therefore, $\mathbf{n}_1^{\mathbf{n}_2} = \mathbf{n}_1$ and we conclude that $\mathbf{n}_1$ and $\mathbf{n}_2$ commute.

Finally, note that $\mathbf{n}_1 \in \mathrm{C}_{\mathbf{G}}(\mathbf{G}_2)$ by Remark 3.3(b) and [Spä06, Theorem 2.1.6(c)]. □

**Lemma 3.14.** *We can assume that the elements* $\mathbf{n}_1, \mathbf{n}_2$ *are* $F$*-stable.*

*Proof.* Firstly, recall that if $y \in \mathbf{T}_0$ there exists $t \in \mathbf{T}_0$ with $y = tF(t)^{-1}$ and we have an isomorphism $\mathbf{G}^F \to \mathbf{G}^{yF}$, $g \mapsto {}^t g$, which yields isomorphic fixed-point structures for all relevant subgroups. We may thus fix a nice representative of $v \in W$ in $\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ which we will construct now.

By Lemma 3.11 there exist $\mathbf{m}_1 \in \langle \mathbf{n}_1, \mathbf{n}_{e_1-e_n}(1) \rangle$ and $\mathbf{m}_2 \in \langle \mathbf{n}_{e_i}(r)\mathbf{n}_{e_j}(s), \mathbf{n}_{e_i-e_j}(r) \mid i \neq j, i,j \in I, r,s \in \overline{\mathbb{F}_p}^{\times} \rangle$ such that

$$\mathbf{m} := \mathbf{m}_1\mathbf{m}_2$$

satisfies $\mathbf{m}\mathbf{T}_0 = v$ in $W$.

By Lemma 3.13 it follows that $\mathbf{m}_1 \in \langle \mathbf{n}_1, \mathbf{n}_{e_1-e_n}(1)\rangle$ commutes with $\mathbf{n}_1$. Moreover, by the proof of Lemma 3.12 we conclude that $\mathbf{m}_2$ and $\mathbf{n}_1$ commute. As $\mathbf{n}_1$ is $F_0$-stable it therefore follows that $(\mathbf{m}F_0)(\mathbf{n}_1) = \mathbf{n}_1$.

Since $w_2 \in \mathrm{C}_W(v)$ we necessarily have $(\mathbf{m}F_0)(\mathbf{n}_2)\mathbf{n}_2^{-1} \in \mathbf{T}_0$. By Lemma 3.12 it follows that $\mathbf{m}_2$ commutes with $\mathbf{n}_{e_1-e_n}(1)$ and $\mathbf{m}_1$ commutes with $\mathbf{n}_2'$. From this we deduce that

$$(\mathbf{m}F_0)(\mathbf{n}_2)\mathbf{n}_2^{-1} = {}^{\mathbf{m}}\mathbf{n}_2\mathbf{n}_2^{-1} = {}^{\mathbf{m}_2}\mathbf{n}_2'\mathbf{n}_2'^{-1}.$$

Since ${}^{\mathbf{m}_2}\mathbf{n}_2'\mathbf{n}_2'^{-1}$ is purely an expression in the roots $e_i, e_i - e_j$ with $i, j \in I$ we can deduce that

$$(\mathbf{m}F_0)(\mathbf{n}_2)\mathbf{n}_2^{-1} \in \mathbf{T}_2 = \langle \mathbf{h}_{e_i}(r) \mid i \in I, r_i \in \overline{\mathbb{F}_p}^{\times}\rangle.$$

By Lang's theorem applied to the Frobenius endomorphism $\mathbf{m}F_0 : \mathbf{T}_2 \to \mathbf{T}_2$ there exists $t_2 \in \mathbf{T}_2$ such that $(\mathbf{m}F_0)(t_2\mathbf{n}_2) = t_2\mathbf{n}_2$. Replacing $\mathbf{n}_2$ by $t_2\mathbf{n}_2 \in \langle \mathbf{n}_{e_i}(r)\mathbf{n}_{e_j}(s), \mathbf{n}_{e_i-e_j}(r) \mid i \neq j, i, j \in I, r, s \in \overline{\mathbb{F}_p}^{\times}\rangle$ we can henceforth assume that $(\mathbf{m}F_0)(\mathbf{n}_2) = \mathbf{n}_2$. Note that the statement of Lemma 3.13 remains valid since $\mathbf{n}_1$ centralizes the subtorus $\mathbf{T}_2$. This completes the proof of the lemma. □

We are now ready to prove the main result of this section.

**Proposition 3.15.** *We have $\mathbf{L}^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle = \mathbf{N}^F$.*

*Proof.* The elements $\mathbf{n}_1, \mathbf{n}_2 \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ satisfy ${}^{\mathbf{n}_1}t = tz_1$ and ${}^{\mathbf{n}_2}t = tz_2$. From this we deduce that $\pi(\mathbf{n}_1), \pi(\mathbf{n}_2) \in \mathrm{C}_{\mathbf{G}^*}(s)$. By duality we have an isomorphism $\mathbf{N}^F/\mathbf{L}^F \cong (\mathbf{N}^*)^{F^*}/(\mathbf{L}^*)^{F^*}$ from which we can conclude that $\mathbf{L}^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle = \mathbf{N}^F$. □

In the next section we will consider the subgroup $L_0$ of $\mathbf{L}^F$ defined by $L_0 = \mathbf{T}_1^F\mathbf{G}_2^F$. As $\mathbf{T}_1 \subseteq \mathrm{C}_{\mathbf{L}}(\mathbf{G}_2)$ it follows that $L_0$ is a central product of $\mathbf{T}_1^F$ and $\mathbf{G}_2^F$. The following lemma shows that $\mathbf{L}^F/L_0 \cong C_2$.

**Lemma 3.16.** *Let $\mathcal{L} : \mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$, denote the Lang map of $\mathbf{G}$. There exists $x_1 \in \mathbf{T}_1$ and $x_2 \in \mathbf{T}_2$ such that $\mathcal{L}(x_1) = \mathcal{L}(x_2) = \mathbf{h}_{e_1}(-1)$ and $x := x_1x_2$ satisfies $\mathbf{L}^F = \mathbf{T}_1^F\mathbf{G}_2^F\langle x\rangle$.*

*Proof.* The existence of $x_1$ and $x_2$ follows by applying Lang's theorem. Since $\mathbf{T}_1 \cap \mathbf{G}_2 = \mathbf{T}_1 \cap \mathbf{T}_2 = \langle \mathbf{h}_{e_1}(-1)\rangle$ the second claim follows. □

## 3.6 Representation theory

At the beginning of Section 3.5 we have established that $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}}(s)$, so in particular $s \in \mathrm{Z}(\mathbf{L}^*)^{F^*}$. Let $\hat{s} : \mathbf{L}^F \to \mathcal{O}^\times$ be the character of $\mathbf{L}^F$ corresponding to the central element $s \in \mathrm{Z}(\mathbf{L}^*)^{F^*}$, see [CE04, Equation 8.19].

**Lemma 3.17.** *The linear character $\hat{s} : \mathbf{L}^F \to \mathcal{O}^\times$ extends to $\mathbf{N}^F$.*

*Proof.* By [Spä10, Theorem 1.1] the character $\lambda := \mathrm{Res}^{\mathbf{L}^F}_{\mathbf{T}_0^F}(\hat{s})$ extends to its inertia group in $\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}_0^F)$. However, $\mathbf{n}_1, \mathbf{n}_2 \in \mathbf{N}_{\mathbf{G}^F}(\mathbf{T}_0^F)$ and $\lambda$ is $\mathbf{N}^F$-invariant which implies that $\lambda$ extends to a character $\lambda'$ of $\mathbf{T}_0^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle$. We define a character $\hat{s}' : \mathbf{N}^F \to \mathcal{O}^\times$ by $\hat{s}'(x) := \hat{s}(l)\lambda'(n)$ where $x \in \mathbf{N}^F$ with $x = ln$ for $l \in \mathbf{L}^F$ and $n \in \mathbf{T}_0^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle$. Note that this character is well-defined as $\hat{s}$ and $\lambda$ agree on the intersection $\mathbf{L}^F \cap \mathbf{T}_0^F\langle \mathbf{n}_1, \mathbf{n}_2\rangle = \mathbf{T}_0^F$. □

The following lemma is a module theoretic generalization of [Spä10, Lemma 4.1].

**Lemma 3.18.** *Let $\widetilde{Y}$ be a finite group with normal subgroup $\widetilde{X}$ and subgroup $Y$ such that $\widetilde{Y} = Y\widetilde{X}$. Denote $X := Y \cap \widetilde{X}$ and suppose that $\ell \nmid [Y : X]$. Suppose that $M$ is an absolutely indecomposable $\mathcal{O}X$-module which extends to an $\mathcal{O}Y$-module and suppose that $\widetilde{M}$ is an $\mathcal{O}\widetilde{X}$-module such that $M = \mathrm{Res}^{\widetilde{X}}_X(\widetilde{M})$. If $\widetilde{M}$ is $\widetilde{Y}$-invariant then $\widetilde{M}$ extends to $\widetilde{Y}$.*

*Proof.* Let us recall some basic facts about Clifford theory, see [BDR17a, Section 7.B] (over $k$) and [Dad84] (over $\mathcal{O}$). We follow the notation in [BDR17a, Section 7.B].

Firstly, for $y \in Y$, define

$$N_y := \{\phi \in \mathrm{End}^\times_{\mathcal{O}}(M) \mid \phi(xm) = yxy^{-1}\phi(m) \text{ for all } x \in X, m \in M\}$$

and let $N := \cup_{y\in Y} N_y$. Note that $N$ is a group with normal subgroup $N_1$. Since $M$ is $Y$-invariant we have a surjective morphism $Y \to N/N_1$ given by $y \mapsto yN_1$. We form the group

$$\widehat{Y} := Y \times_{N/N_1} N = \{(y, \varphi) \in Y \times N \mid \varphi \in N_y\}.$$

We let $A := \mathrm{End}_{\mathcal{O}X}(M)$. Consider the following exact sequence:

$$1 \to A^\times \to \widehat{Y} \to Y \to 1.$$

The $\mathcal{O}X$-module $M$ extends to an $\mathcal{O}Y$-module if and only if this sequence splits, see [Dad84, 1.7]

The action of $X$ on $M$ defines an element $\phi_x \in N_x$ for every $x \in X$. We identify $X$ with its image under the diagonal embedding $X \hookrightarrow \widehat{Y}, x \mapsto (x, \phi_x)$. As $\ell \nmid [Y : X]$ it follows (see [Dad84, Theorem 4.5]) that $M$ extends to an $\mathcal{O}Y$-module if and only if the following exact sequence splits:

$$1 \to A^\times/(1 + J(A)) \to \widehat{Y}/X(1 + J(A)) \to Y/X \to 1.$$

Similarly, we can look at $\widetilde{M}$ instead of $M$. We denote the corresponding objects with a tilde. Analogously, the module $\widetilde{M}$ extends to an $\mathcal{O}\widetilde{Y}$ module if and only if the following exact sequence splits:

$$1 \to \widetilde{A}^\times/(1 + J(\widetilde{A})) \to \widehat{\widetilde{Y}}/\widetilde{X}(1 + J(\widetilde{A})) \to \widetilde{Y}/\widetilde{X} \to 1.$$

Let $\pi : \widetilde{Y}/\widetilde{X} \to Y/X$ be the inverse map of the natural isomorphism $Y/X \to \widetilde{Y}/\widetilde{X}$. Restriction defines a homomorphism $\widetilde{A}^\times \to A^\times$.

Now we define a map $\widehat{\widetilde{Y}} \to \widehat{Y}$ as follows. For $(\widetilde{y}, \phi) \in \widehat{\widetilde{Y}}$ we let $\tilde{x} \in \widetilde{X}$ such that $y := \tilde{y}\tilde{x} \in Y$. Let $\phi_{\tilde{x}}$ be the natural action of $\widetilde{x}$ on $\widetilde{M}$. Then it follows that $\phi\phi_{\tilde{x}} \in \widetilde{N}_y \subseteq N_y$. We define

$$\widehat{\pi} : \widehat{\widetilde{Y}}/\widetilde{X} \to \widehat{Y}/X,\ (\widetilde{y}, \phi) \mapsto (y, \phi\phi_{\widetilde{x}}).$$

Note that if $\tilde{x}' \in \widetilde{X}$ with $y' := \tilde{y}\tilde{x}' \in Y$ then $x := \tilde{x}^{-1}\tilde{x}' \in X$ and we have $y' = yx$. From this we deduce that $(y', \phi\phi_{\tilde{x}'}) = (y, \phi\phi_x)$ in $\widehat{Y}/X$ which shows that the map $\widehat{\pi}$ is well-defined. We can therefore consider the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{A}^\times/(1 + J(\widetilde{A})) & \longrightarrow & \widehat{\widetilde{Y}}/\widetilde{X}(1 + J(\widetilde{A})) & \longrightarrow & \widetilde{Y}/\widetilde{X} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \widehat{\pi} & & \downarrow \pi & & \\
1 & \longrightarrow & A^\times/(1 + J(A)) & \longrightarrow & \widehat{Y}/X(1 + J(A)) & \longrightarrow & Y/X & \longrightarrow & 1
\end{array}$$

Now note that $\pi$ is an isomorphism. Moreover, as $M$ and $\widetilde{M}$ are absolutely indecomposable we have $A^\times/(1 + J(A)) \cong k^\times$ and $\widetilde{A}^\times/(1 + J(\widetilde{A})) \cong k^\times$. Thus, the first and the third vertical map are isomorphisms. By the five lemma, it follows that $\widehat{\pi} : \widehat{\widetilde{Y}}/\widetilde{X}(1 + J(\widetilde{A})) \to \widehat{Y}/X(1 + J(A))$ is also an isomorphism. Therefore, the first and the second row of the diagram above are isomorphic group extensions. However, by assumption we already know that $M$ extends to an $\mathcal{O}Y$-module which implies that the sequence in the second row splits. Thus, also the sequence of the first row splits and $\widetilde{M}$ extends to an $\mathcal{O}\widetilde{Y}$-module. □

We are now ready to prove the main statement of this section.

**Proposition 3.19.** *Let $M$ be an $\mathbf{N}^F$-invariant indecomposable $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$-bimodule. If $\ell$ does not divide $q^2-1$ then $M$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule.*

*Proof.* By Lemma 3.17, it follows that $M$ extends to $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}$ if and only if $M \otimes_{\mathcal{O}} \hat{s}^{-1}$ extends to $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}$. We may therefore assume from now on that $M$ is an indecomposable $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$-bimodule.

Since $\ell \nmid [\mathbf{L}^F : L_0]$ there exists an indecomposable $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}L_0$-bimodule $M_0$ such that $M$ is a direct summand of $\mathrm{Ind}_{\mathbf{G}^F \times L_0^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M_0)$. As $1 \times (\mathbf{T}_1^F)^{\mathrm{opp}}$ is central in $\mathbf{G}^F \times L_0^{\mathrm{opp}}$ we deduce that

$$\mathrm{Res}_{1 \times (\mathbf{T}_1^F)_{\ell'}^{\mathrm{opp}}}^{\mathbf{G}^F \times L_0^{\mathrm{opp}}}(M_0) = S^{\dim(M_0)}$$

for some simple $\mathcal{O}(\mathbf{T}_1)_{\ell'}^F$-module $S$. Let $\lambda : (\mathbf{T}_1)_{\ell'}^F \to \mathcal{O}^\times$ be the character corresponding to $S$. Since $\mathrm{Res}_{1 \times (\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M)$ is an $\mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$-module it follows that $\lambda$ is a character in a unipotent block, which implies that $\lambda$ is the trivial character.

Note that $|\mathbf{T}_1^F| \in \{(q-1)^2, q^2-1\}$ and therefore $\ell \nmid |\mathbf{T}_1^F|$ by assumption. We conclude that

$$\mathrm{Res}_{1 \times (\mathbf{T}_1^F)^{\mathrm{opp}}}^{\mathbf{G}^F \times L_0^{\mathrm{opp}}}(M_0) = \mathcal{O}^{\dim(M_0)},$$

where $\mathcal{O}$ is the trivial $\mathcal{O}(\mathbf{T}_1^F)^{\mathrm{opp}}$-module. Since $L_0/\mathbf{T}_1^F \cong \mathbf{G}_2^F/\langle z_1 \rangle$ we may consider $M_0$ as an indecomposable $\mathcal{O}[\mathbf{G}^F \times (\mathbf{G}_2^F/\langle z_1\rangle)^{\mathrm{opp}}]$-module.

The element $\mathbf{n}_1$ centralizes $\mathbf{G}_2^F$ and hence we can extend $M_0$ to an $\mathcal{O}[\mathbf{G}^F \times (L_0\langle \mathbf{n}_1\rangle)^{\mathrm{opp}}]$-module by letting $\mathbf{n}_1$ act trivially on $M_0$. We denote this extension by $M_1$.

Since $M$ is a direct summand of $\mathrm{Ind}_{\mathbf{G}^F \times L_0^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M_0)$ it follows that its restriction $\mathrm{Res}_{\mathbf{G}^F \times L_0^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M)$ is a direct summand of

$$\mathrm{Res}_{\mathbf{G}^F \times L_0^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}} \mathrm{Ind}_{\mathbf{G}^F \times L_0^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M_0) \cong M_0 \oplus M_0^x,$$

where $x = x_1 x_2 \in \mathbf{L}^F$ as in Lemma 3.16. As the quotient group $\mathbf{L}^F/L_0$ is cyclic of $\ell'$-order it follows by [Rou98, Lemma 10.2.13] that either

$$\mathrm{Res}_{\mathbf{G}^F \times (L_0)^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M) \cong M_0 \text{ or } \mathrm{Res}_{\mathbf{G}^F \times (L_0)^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M) \cong M_0 \oplus M_0^x.$$

We treat these two cases separately.

**Case 1:** Assume that $\operatorname{Res}^{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times L_0^{\mathrm{opp}}}(M)\cong M_0$.
Since $M$ is $\mathbf{N}^F$-invariant it follows that $M_0$ is $\mathbf{N}^F$-invariant.

We have $[\mathbf{n}_1,\mathbf{n}_2]=1$. Thus, the action of $\mathbf{n}_1$ on $M_1^{\mathbf{n}_2}$ is equal to the action of $\mathbf{n}_2\mathbf{n}_1\mathbf{n}_2^{-1}=\mathbf{n}_1$ on $M_1$. However, $\mathbf{n}_1$ acts trivially on $M_0$. Since $M_0$ is $\mathbf{n}_2$-invariant there exists an isomorphism $\phi: M_0\to M_0^{\mathbf{n}_2}$ of $\mathbf{G}^F\times(L_0)^{\mathrm{opp}}$-modules. Recall that $\mathbf{n}_1\in \mathrm{C}_{\mathbf{G}}(\mathbf{G}_2)$, so $\mathbf{n}_1$ acts trivially on $M_1$. It follows that $\phi: M_1\to M_1^{\mathbf{n}_2}$ is an isomorphism of $\mathcal{O}\mathbf{G}^F\times(L_0\langle\mathbf{n}_1\rangle)^{\mathrm{opp}}$-modules or in other words $M_1$ is $\mathbf{n}_2$-invariant. From this we conclude that $M_1$ extends to $\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\mathbf{n}_2\rangle)^{\mathrm{opp}}$.

Applying Lemma 3.18 to $\tilde{X}=\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}$ and $Y=\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\mathbf{n}_2\rangle)^{\mathrm{opp}})$ implies that $M$ extends to a $\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}$-module.

**Case 2:** Assume that $\operatorname{Res}^{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times L_0^{\mathrm{opp}}}(M)\cong M_0\oplus {}^xM_0$.
We note that $M_0^{\mathbf{n}_1}\cong M_0$. On the other hand, we either have $M_0^{\mathbf{n}_2}\cong M_0$ or $M_0^{\mathbf{n}_2x}\cong M_0$.

Suppose that $M_0^{\mathbf{n}_2}\cong M_0$. Then $M_0$ is $\mathbf{N}^F$-invariant. Using the same proof as in case 1 we deduce that $M_0$ extends to a $\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\mathbf{n}_2\rangle)^{\mathrm{opp}}$-module.

Suppose that $M_0^{\mathbf{n}_2x}\cong M_0$. We have $\mathbf{h}_{e_1}(-1)^x=\mathbf{h}_{e_1}(-1)$ as $\mathcal{L}(x_1)=\mathcal{L}(x_2)=\mathbf{h}_{e_1}(-1)$. Since $x_2\in\mathbf{G}_2$ we conclude that $\mathbf{n}_1^{x_2}=\mathbf{n}_1$. Therefore, $\mathbf{n}_1^{\mathbf{n}_2x}=\mathbf{n}_1^x=\mathbf{n}_1^{x_1}$.

Clearly, $x_1x_1^{\mathbf{n}_1^{-1}}\in\mathbf{T}_1^F$ which implies that $\mathbf{n}_1^{x_1}\mathbf{n}_1^{-1}\in\mathbf{T}_1^F$. From this we deduce that $\mathbf{n}_1^{\mathbf{n}_2x}\mathbf{n}_1^{-1}\in\mathbf{T}_1^F$. Now $\mathbf{n}_1$ acts on $M_1^{\mathbf{n}_2x}$ as $\mathbf{n}_1^{\mathbf{n}_2x}$ acts on $M_1$. Since $\mathbf{T}_1^F$ and $\mathbf{n}_1$ act trivially on $M_1$ it follows that $\mathbf{n}_1$ acts trivially on $M_1^{\mathbf{n}_2x}$. Since $M_0$ is $\mathbf{n}_2x$-invariant it follows that $M_1$ is $\mathbf{n}_2x$-invariant. Thus, $M_0$ extends to a $\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\mathbf{n}_2x\rangle)^{\mathrm{opp}}$-module.

It follows that $M_0$ extends to a $\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\hat{\mathbf{n}}_2\rangle)^{\mathrm{opp}}$-module $M'$, where $\hat{\mathbf{n}}_2\in\{\mathbf{n}_2,\mathbf{n}_2x\}$. By Mackey's formula we deduce that

$$\operatorname{Res}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}\operatorname{Ind}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\hat{\mathbf{n}}_2\rangle)^{\mathrm{opp}}}(M')\cong\operatorname{Ind}^{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times L_0^{\mathrm{opp}}}\operatorname{Res}^{\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\hat{\mathbf{n}}_2\rangle)^{\mathrm{opp}}}_{\mathbf{G}^F\times L_0^{\mathrm{opp}}}(M')\cong M.$$

Thus, $\operatorname{Ind}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F\times(L_0\langle\mathbf{n}_1,\hat{\mathbf{n}}_2\rangle)^{\mathrm{opp}}}(M')$ is an extension of $M$ to $\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}$. This finishes the proof. □

Using a standard argument in Clifford theory we can now deduce Proposition 3.4 from the previous proposition.

*Proof of Proposition 3.4.* According to [BDR17a, Theorem 7.2] the bimodule $H^d_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}},\mathcal{O})e_s^{\mathbf{L}^F}$ is $\mathcal{O}\mathbf{N}^F$-invariant. Let $H^d_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}},\mathcal{O})e_s^{\mathbf{L}^F}=\bigoplus_{i=1}^k M_i$ be a decomposition into $\mathbf{N}^F$-orbits of indecomposable direct summands of $H^d(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}},\mathcal{O})e_s^{\mathbf{L}^F}$. Let $N_i$ be an indecomposable direct summand of $M_i$ and $T_i$ be its inertia group in $\mathbf{N}^F$. If $T_i$ is a proper subgroup of $\mathbf{N}^F$ then $T_i/\mathbf{L}^F$ is cyclic of $\ell'$-order so that $N_i$ extends to $\mathbf{G}^F \times (T_i)^{\mathrm{opp}}$. If $T_i = \mathbf{N}^F$ then $N_i$ extends to $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}$ by Proposition 3.19. Let $N_i'$ be an extension of $N_i$ to $\mathbf{G}^F \times (T_i)^{\mathrm{opp}}$. By Clifford theory, it follows that $\mathrm{Ind}_{\mathbf{G}^F \times (T_i)^{\mathrm{opp}}}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}(N_i')$ is an extension of $M_i$. This shows that $H_c^d(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}$. □

## 3.7 Proof of Morita equivalence

In Proposition 3.4 we have proved that the $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F$-bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule. We prove now that the extended bimodule induces a Morita equivalence. In the original proof of [BDR17a, Theorem 7.5] the authors use that $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, k)e_s^{\mathbf{L}^F}$ extends to a $k\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{N}}^F)$-module. Since we were not able to show this in our situation we need to use a different approach. In order to remedy this problem, we borrow arguments from [BDR17b].

From now on let $\mathbf{G}$ be a connected reductive group. We keep the notation of [BDR17a, Section 7.C]. In particular we fix a regular embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$. We denote $\tilde{\mathbf{L}} = \mathbf{L}\mathrm{Z}(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{N}} = \mathbf{N}\tilde{\mathbf{L}}$.

**Proposition 3.20.** *Suppose that the $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F$-bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule $M'$. Then the bimodule $M'$ induces a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.*

*Proof.* Let $M'$ be an $\mathcal{O}\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}$-bimodule extending $M := H_c^d(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$. Recall that $M$ is projective as $\mathcal{O}\mathbf{G}^F$-module and projective as $\mathcal{O}\mathbf{L}^F$-module. As $\ell \nmid [\mathbf{N}^F : \mathbf{L}^F]$ it follows that $M'$ is projective as $\mathcal{O}\mathbf{N}^F$-module. Note that $\mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} M$ is a faithful $\mathcal{O}\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$-module, see proof of [BDR17a, Theorem 7.5]. Thus, $M$ is a faithful $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$-module.

By Theorem 1.29 it suffices to prove that $M' \otimes_{\mathcal{O}} K$ induces a bijection between irreducible characters of $K\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $K\mathbf{G}^F e_s^{\mathbf{G}^F}$. As $M$ is a faithful $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$-module it suffices to prove that the natural map $K\mathbf{N}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{K\mathbf{G}^F}(M' \otimes_{\mathcal{O}} K)$ is an isomorphism. As in the proof of [BDR17a, Theorem 7.5] we consider the $\mathcal{O}[\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}]$-module $\widetilde{M} = \mathrm{Ind}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}^{\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}}(M)$. We have $\mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F}(M') \cong \widetilde{M}$ as $\tilde{\mathbf{G}}^F$-modules. Since $M$ is $\tilde{\mathbf{G}}^F$-invariant this implies

$$\dim(\mathrm{End}_{K\mathbf{G}^F}(M)) = [\tilde{\mathbf{G}}^F : \mathbf{G}^F] \dim(\mathrm{End}_{K\tilde{\mathbf{G}}^F}(\widetilde{M})).$$

In addition, the bimodule $\widetilde{M}$ extends to an $\mathcal{O}[\tilde{\mathbf{G}}^F \times (\widetilde{\mathbf{N}}^F)^{\mathrm{opp}}]$-bimodule $\widetilde{M}'$, see proof of [BDR17a, Theorem 7.5], which induces a Morita equivalence between $\mathcal{O}\widetilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\widetilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$. This shows that $\dim(\mathrm{End}_{K\tilde{\mathbf{G}}^F}(\widetilde{M})) = \dim(K\widetilde{\mathbf{N}}^F e_s^{\mathbf{L}^F})$. Moreover, we have

$$\dim(K\widetilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}) = [\widetilde{\mathbf{N}}^F : \mathbf{N}^F] \dim(K\mathbf{N}^F e_s^{\mathbf{L}^F}).$$

From these calculations using $\tilde{\mathbf{G}}^F/\mathbf{G}^F \cong \widetilde{\mathbf{N}}^F/\mathbf{N}^F$ we deduce that

$$\dim(\mathrm{End}_{K\mathbf{G}^F}(M)) = \dim(K\mathbf{N}^F e_s^{\mathbf{L}^F}).$$

To complete the proof we show the following lemma:

**Lemma 3.21.** *The natural map $K\mathbf{N}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{K\mathbf{G}^F}(M')$ is injective.*

*Proof.* Let $\dot{n}$ be a representative of $n \in \mathbf{N}^F/\mathbf{L}^F$ in $\mathbf{N}^F$. Let $\alpha_n \in \mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$, $n \in \mathbf{N}^F/\mathbf{L}^F$, such that $\sum_{n \in \mathbf{N}^F/\mathbf{L}^F} \alpha_n \dot{n} = 0$ on $M'$. Denote $\hat{M} := \mathrm{Ind}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}} M$. Let $\theta_{\dot{n}}$ be the automorphism on $\hat{M}$ induced by the action of $\dot{n}$ on $M'$. More concretely, we have

$$\theta_{\dot{n}}(m \otimes l) = m\dot{n} \otimes {}^n l$$

for $l \in (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}$ and $m \in M'$.

For $l \in \tilde{\mathbf{L}}^F$ we have $\theta_{\dot{n}} \circ l \circ \theta_{\dot{n}}^{-1} = {}^n l$ on $\hat{M}$. Let $e \in \mathrm{Z}(\mathcal{O}\tilde{\mathbf{L}}^F)$ be the central idempotent as in [BDR17a, Theorem 7.5] such that $e_s^{\mathbf{L}^F} = \sum_{n \in \mathbf{N}^F/\mathbf{L}^F} {}^n e$. We have

$$\hat{M} = \bigoplus_{n \in \mathbf{N}^F/\mathbf{L}^F} \hat{M}\, {}^n e.$$

For $m \in \hat{M}e$ we therefore have

$$\sum_{n \in \mathbf{N}^F/\mathbf{L}^F} \alpha_n \theta_{\dot{n}}(m) = 0.$$

As $\theta_{\dot{n}}(m) \in \hat{M}\, {}^n e$ we have $\alpha_n \theta_{\dot{n}}(m) = 0$ for all $n \in \mathbf{N}^F/\mathbf{L}^F$ and $m \in \hat{M}e$. This means that $\alpha_n \theta_{\dot{n}}$ vanishes on $\hat{M}e$. Composing with $\theta_{\dot{y}}^{-1}$ for $y \in \mathbf{N}^F/\mathbf{L}^F$ shows that $\alpha_n \theta_{\dot{n}}$ vanishes on $\hat{M}\, {}^y e$ as well. We conclude that $\alpha_n \theta_n = 0$ on $\hat{M}$. As $\theta_{\dot{n}}$ is an isomorphism we must have $\alpha_n = 0$ on $\hat{M}$. Since $\hat{M} = \mathrm{Res}_{\mathbf{G}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}}^{\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}}(\widetilde{M})$ and $\widetilde{M}$ is a faithful $\mathcal{O}\tilde{\mathbf{L}}^F e_s^{\mathbf{L}^F}$-module we deduce that $\alpha_n = 0$. It follows that $K\mathbf{N}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{K\mathbf{G}^F}(M')$ is injective. □

Now let us finish the proof of Proposition 3.20. Since $\dim(\mathrm{End}_{K\mathbf{G}^F}(M)) = \dim(K\mathbf{N}^F e_s^{\mathbf{L}^F})$ it follows that the natural map

$$K\mathbf{N}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{K\mathbf{G}^F}(M')$$

is an isomorphism. Thus, the bimodule $M' \otimes_{\mathcal{O}} K$ induces a Morita equivalence between $K\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $K\mathbf{G}^F e_s^{\mathbf{L}^F}$. As we have argued above this implies that $M'$ induces a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$. □

We can now prove the main theorem of this chapter.

**Theorem 3.22.** *Suppose that $\mathbf{G}$ is a simple algebraic group. If $\ell \nmid (q^2-1)$ or if $\mathbf{N}^F/\mathbf{L}^F$ is cyclic then the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}$ of $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F$-bimodules extends to a complex $C$ of $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodules. The complex $C$ induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ and the bimodule $H^{\dim(\mathbf{Y}_{\mathbf{U}})}(C)$ induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

*Proof.* The quotient $\mathbf{N}^F/\mathbf{L}^F$ is of $\ell'$-order and embeds into $\mathrm{Z}(\mathbf{G})^F$, see for instance [CE04, Lemma 13.16(i)]. Thus, the quotient $\mathbf{N}^F/\mathbf{L}^F$ is cyclic of $\ell'$-order unless possibly if $\mathbf{G}$ is simply connected and $\mathbf{G}^F$ is of untwisted type $D_n$, $n \geq 4$ even. If $\mathbf{N}^F/\mathbf{L}^F$ is cyclic (of $\ell'$-order) then it follows by [Rou98, Lemma 10.2.13] that the $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{L}^F$-bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, \mathcal{O}) e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule $M'$. In the remaining cases Proposition 3.4 asserts that the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O}) e_s^{\mathbf{L}^F}$ extends to an $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodule $M'$.

By Proposition 3.20 the extended bimodule $M'$ induces a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.

Let $e_s^{\mathbf{G}^F} = \sum_{i=1}^r b_i$ be a decomposition into blocks. Since $M'$ induces a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ it follows that $M' = \oplus_{i=1}^r b_i M'$ is a decomposition into indecomposable $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$-$\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$-bimodules. By Proposition 3.19 it follows that $\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(b_i M')$ decomposes into pairwise non-isomorphic indecomposable summands. Since $b_i \mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M')$ and $b_j \mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M')$ have no non-zero direct summand in common for $i \neq j$, it follows that $\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(M')$ decomposes into pairwise non-isomorphic indecomposable summands.

In particular, one observes that the conclusion of [BDR17a, Theorem 7.6] and therefore of [BDR17a, Theorem 7.7] holds true in this case. This proves Theorem 3.22. □

It might be worth mentioning that even though we have not managed to prove Theorem 3.22 without a restriction on the prime $\ell$ our proof yields a bijection between ordinary characters.

**Corollary 3.23.** *Suppose that $\mathbf{G}$ is a simple algebraic group. Then the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, K) e_s^{\mathbf{L}^F}$ extends to a $K\mathbf{G}^F$-$K\mathbf{N}^F$-bimodule $M'$. This bimodule induces a character bijection $\mathrm{Irr}(\mathbf{N}^F, e_s^{\mathbf{L}^F}) \to \mathrm{Irr}(\mathbf{G}^F, e_s^{\mathbf{G}^F})$.*

# Chapter 4

# Equivariant Morita equivalence and local equivalences

In the first part of this chapter we recall the classification of automorphisms of finite simple groups of Lie type. A careful analysis of these automorphisms will be necessary to study the "Clifford theory" with respect to these automorphisms of the Bonnafé–Dat–Rouquier Morita equivalence.

In the second part, we will discuss some extensions of the Bonnafé–Dat–Rouquier Morita equivalence to local subgroups. This will require working with disconnected reductive groups.

## 4.1 Automorphisms of simple groups of Lie type

We briefly recall the classification of automorphisms of finite simple groups of Lie type. Let $\mathbf{G}$ be a simple algebraic group of simply connected type. Fix a maximal torus $\mathbf{T}_0$ and a Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ containing $\mathbf{T}_0$. We let $\Phi$ be the root system relative to $\mathbf{T}_0$ and $\Delta$ be the base of $\Phi$ relative to $\mathbf{T}_0 \subseteq \mathbf{B}_0$. For every $\alpha \in \Phi$ we fix a one-parameter subgroup $\mathbf{x}_\alpha : (\overline{\mathbb{F}_p}, +) \to \mathbf{G}$ as in Section 3.2.

We consider the following bijective morphisms of $\mathbf{G}$:

- The *field endomorphism* $\phi_0 : \mathbf{G} \to \mathbf{G}$, $\mathbf{x}_\alpha(t) \mapsto \mathbf{x}_\alpha(t^p)$ for every $t \in \overline{\mathbb{F}_p}$ and $\alpha \in \Phi$.

- For any length-preserving symmetry $\gamma$ of the Dynkin diagram associated to the root system $\Phi$ we consider the *graph automorphism* $\gamma : \mathbf{G} \to \mathbf{G}$ given by $\gamma(\mathbf{x}_\alpha(t)) = \mathbf{x}_{\gamma(\alpha)}(t)$ for every $t \in \overline{\mathbb{F}_p}$ and $\alpha \in \pm\Delta$.

For any fixed prime power $q = p^f$ of $p$ and a graph automorphism $\gamma$ we consider the Frobenius endomorphism $F = \phi_0^f \gamma : \mathbf{G} \to \mathbf{G}$. Note that any Frobenius endomorphism of $\mathbf{G}$ is (up to inner automorphisms of $\mathbf{G}$) of this form by [MT11, Theorem 22.5]. We say that $(\mathbf{G}, F)$ is *untwisted* if $\gamma$ is the identity and *twisted* otherwise. Moreover, let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Then the automorphisms of $\mathbf{G}^F$ obtained by conjugation with $\tilde{\mathbf{G}}^F$ are called *diagonal automorphisms* of $\mathbf{G}^F$.

**Lemma 4.1.** *There exists a regular embedding $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ such that the bijective morphisms $\gamma : \mathbf{G} \to \mathbf{G}$ and $\phi_0 : \mathbf{G} \to \mathbf{G}$ extend to bijective morphisms $\tilde{\gamma}$ and $\tilde{\phi_0}$ of $\tilde{\mathbf{G}}$ such that:*

*(a) The morphisms $\tilde{\gamma}$ and $\tilde{\phi_0}$ commmute.*

*(b) Let $\tilde{F} := \tilde{\phi_0}^f \tilde{\gamma}$ be an extension of the Frobenius $F = \phi_0^f \gamma$ of $\mathbf{G}$. Then the order of $\tilde{\phi_0}_{|\tilde{\mathbf{G}}^{\tilde{F}}}$ (resp. $\tilde{\gamma}_{|\tilde{\mathbf{G}}^{\tilde{F}}}$) and of $\phi_{0|\mathbf{G}^F}$ (resp. $\gamma_{|\mathbf{G}^F}$) coincide.*

*Proof.* Such a regular embedding is for instance constructed in [MS16, Section 2.B]. Alternatively, as in Remark 2.31, one can define $\tilde{\mathbf{G}} := \mathbf{T}_0 \times_{\mathrm{Z}(\mathbf{G})} \mathbf{G}$ and for $(t_0, g) \in \tilde{\mathbf{G}}$ define $\tilde{\gamma}(t_0, g) := \gamma(t_0)\gamma(g)$ and $\tilde{\phi_0}(t_0, g) := \tilde{\phi}(t_0)\tilde{\phi}(g)$ respectively. $\square$

Let $\phi : \mathbf{G} \to \mathbf{G}$ be a bijective morphism as in Lemma 4.1. Then this lemma shows that we can construct suitable extensions of $\phi$ to $\tilde{\mathbf{G}}$ such that all relevant relations are preserved. Therefore, we will also denote by $\phi : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ such an extension to $\tilde{\mathbf{G}}$.

**Proposition 4.2.** *Let $\mathbf{G}$ be a simple algebraic group of simply connected type such that $\mathbf{G}^F / \mathrm{Z}(\mathbf{G})^F$ is simple and non-abelian. If $(\mathbf{G}, F)$ is untwisted then any automorphism of $\mathbf{G}^F$ is a product of a graph automorphism, a field automorphism and a diagonal automorphism. Otherwise, any automorphism of $\mathbf{G}^F$ is a product of a field and a diagonal automorphism.*

*Proof.* See [MT11, Theorem 24.24]. $\square$

To avoid cumbersome notation we will use the same letter for bijective morphisms of $\mathbf{G}$ commuting with $F$ and their restriction to $\mathbf{G}^F$:

**Notation 4.3.** Let $\sigma : \mathbf{G} \to \mathbf{G}$ be a bijective morphism of algebraic groups with $\sigma \circ F = F \circ \sigma$. Then we also denote by $\sigma : \mathbf{G}^F \to \mathbf{G}^F$ the automorphism of $\mathbf{G}^F$ obtained by restricting $\sigma$ to $\mathbf{G}^F$. In particular, the expression $\mathbf{G}^F \rtimes \langle \sigma \rangle$ always denotes the semidirect product of finite groups obtained by letting $\sigma : \mathbf{G}^F \to \mathbf{G}^F$ act on $\mathbf{G}^F$.

This allows us to give a description of the stabilizer of $e_s^{\mathbf{G}^F}$ in terms of the automorphisms of Lemma 4.1.

**Corollary 4.4.** *Let* $\mathbf{G}$ *be a simple algebraic group of simply connected type not of type* $D_4$ *with Frobenius* $F$*. Let* $s \in (\mathbf{G}^*)^{F^*}$ *be a semisimple element of* $\ell'$*-order. There exists a Frobenius endomorphism* $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ *with* $F_0^r = F$ *for some positive integer* $r$ *and a bijective morphism* $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ *such that the image of* $\tilde{\mathbf{G}}^F \rtimes \langle F_0, \sigma \rangle$ *in* $\mathrm{Out}(\mathbf{G}^F)$ *is the stabilizer of* $e_s^{\mathbf{G}^F}$ *in* $\mathrm{Out}(\mathbf{G}^F)$*.*

*Proof.* Let $\mathrm{Diag}_{\mathbf{G}^F}$ be the image of the set of diagonal automorphisms in $\mathrm{Out}(\mathbf{G}^F)$. The stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$ contains $\mathrm{Diag}_{\mathbf{G}^F}$ by Lemma 2.32.

Suppose that $(\mathbf{G}, F)$ is twisted. In this case, the group $\mathrm{Out}(\mathbf{G}^F)/\mathrm{Diag}_{\mathbf{G}^F}$ is cyclic and the statement of the corollary can be deduced from this.

Now suppose that $(\mathbf{G}, F)$ is untwisted. If the Dynkin diagram of $\mathbf{G}$ admits a non-trivial symmetry we let $\gamma : \mathbf{G} \to \mathbf{G}$ be a graph automorphism associated to such a symmetry. Then the classification of simple groups of Lie type shows that $\mathrm{Out}(\mathbf{G}^F)/\mathrm{Diag}_{\mathbf{G}^F} \cong \langle \gamma, \phi_0 \rangle \cong C_t \times C_m$, where $t \leq 3$. Thus, every subgroup of $\mathrm{Out}(\mathbf{G}^F)/\mathrm{Diag}_{\mathbf{G}^F}$ is either cyclic or isomorphic to $\langle \gamma \rangle \times \langle F_0 \rangle$, where $F_0 = \phi_0^i : \mathbf{G} \to \mathbf{G}$ for some $i$ with $i \mid f$. Lemma 4.1 now yields the claim. □

## 4.2 Equivariance of Deligne–Lusztig induction

In this section we establish some elementary results on the action of group automorphisms on Deligne–Lusztig varieties. Most of the results in this section are known, see [NTT08, Section 2].

Let $\mathbf{G}$ be a reductive group and $\sigma : \mathbf{G} \to \mathbf{G}$ be a bijective morphism of algebraic groups which commutes with the action of the Frobenius endomorphism $F$, i.e. we have $\sigma \circ F = F \circ \sigma$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ such that $F(\mathbf{L}) = \mathbf{L}$. Note that $\sigma(\mathbf{P})$ is a parabolic subgroup of $\mathbf{G}$ with $F$-stable Levi $\sigma(\mathbf{L})$ and unipotent radical $\sigma(\mathbf{U})$.

**Lemma 4.5.** *Let* $\mathbf{G}$ *be a not necessarily connected reductive group. Let* $\sigma : \mathbf{G} \to \mathbf{G}$ *be a bijective automorphism of* $\mathbf{G}$ *commuting with the action of* $F$ *and stabilizing* $\mathbf{L}$*. Then* $\sigma$ *induces an isomorphism*

$$\sigma^* : R\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\sigma(\mathbf{U})}, \Lambda) \to {}^{\sigma}R\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}}, \Lambda)^{\sigma}$$

*in* $\mathrm{D}^b(\Lambda[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$*.*

*Proof.* The variety $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}^\circ}$ is smooth, see for instance [CE04, Theorem 7.2]. By Lemma 2.8, we have $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}} = \coprod_{g\in\mathbf{G}^F/(\mathbf{G}^\circ)^F} g\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}^\circ}$ which implies that the variety $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ is smooth as well. Thus, as in the proof of [NTT08, Proposition 2.1] it follows that the morphism $\sigma : {}^\sigma\mathbf{Y}_{\mathbf{U}}^{\sigma^{-1}} \to \mathbf{Y}_{\sigma(\mathbf{U})}$ given by $g\mathbf{U} \mapsto \sigma(g)\sigma(\mathbf{U})$ is $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-equivariant and induces an isomorphism of étale sites. □

The statement of the previous lemma can be refined.

**Lemma 4.6.** *In the situation of Lemma 4.5 we have an isomorphism*

$$\sigma^* : G\Gamma_c(\mathbf{Y}_{\sigma(\mathbf{U})}^{\mathbf{G}}, \Lambda) \to {}^\sigma G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^\sigma$$

*in* $\mathrm{Ho}^b(\Lambda[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$.

*Proof.* As we have seen in the proof of Lemma 4.5, the morphism $\sigma : {}^\sigma(\mathbf{Y}_{\mathbf{U}})^\sigma \to \mathbf{Y}_{\sigma(\mathbf{U})}$ induces an isomorphism of étale sites. Now [Rou02, Theorem 2.12] shows that the map $G\Gamma_c(\mathbf{Y}_{\sigma(\mathbf{U})}^{\mathbf{G}}, \Lambda) \to {}^\sigma G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^\sigma$ is an isomorphism. □

From now on let $\mathbf{G}$ be a connected reductive group and $s \in (\mathbf{G}^*)^{F^*}$ a semisimple element of $\ell'$-order. Using Lemma 4.5 one can show that ${}^\sigma\mathcal{E}(\mathbf{G}^F, t) = \mathcal{E}(\mathbf{G}^F, (\sigma^*)^{-1}(t))$ for every semisimple element $t \in (\mathbf{G}^*)^{F^*}$, see [NTT08, Corollary 2.4] and also [Tay18, Proposition 7.2].

**Lemma 4.7.** *Let* $\mathbf{G}$ *be a connected reductive group and* $s \in (\mathbf{G}^*)^{F^*}$ *a semisimple element of* $\ell'$*-order. Then* $\sigma(e_s^{\mathbf{G}^F}) = e^{\mathbf{G}^F}_{(\sigma^*)^{-1}(s)}$.

*Proof.* Recall that $e_s^{\mathbf{G}^F} = \sum_{\chi\in\mathcal{E}_\ell(\mathbf{G}^F,s)} e_\chi$. Moreover, we have

$${}^\sigma\mathcal{E}_\ell(\mathbf{G}^F, s) = \coprod_{t\in{\mathbf{G}^*}^{F^*}:t_{\ell'}=s} {}^\sigma\mathcal{E}(\mathbf{G}^F, t) = \coprod_{t\in{\mathbf{G}^*}^{F^*}:t_{\ell'}=s} \mathcal{E}(\mathbf{G}^F, (\sigma^*)^{-1}(t)) = \mathcal{E}_\ell(\mathbf{G}^F, (\sigma^*)^{-1}(s))$$

due to [NTT08, Corollary 2.4]. Since $\sigma(e_\chi) = e_{{}^\sigma\chi}$ for any character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$, we conclude that

$$\sigma(e_s^{\mathbf{L}^F}) = \sum_{\chi\in\mathcal{E}_\ell(\mathbf{G}^F,s)} e_{{}^\sigma\chi} = \sum_{\chi\in{}^\sigma\mathcal{E}_\ell(\mathbf{G}^F,s)} e_\chi = e^{\mathbf{G}^F}_{(\sigma^*)^{-1}(s)}. \quad \square$$

Suppose now that $\mathbf{L}^*$ is an $F^*$-stable Levi subgroup of $\mathbf{G}^*$ with $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Assume that $\mathbf{L}$ is in duality with $\mathbf{L}^*$. Moreover, let $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ be an isogeny dual to $\sigma : \mathbf{G} \to \mathbf{G}$ such that $\sigma^*$ stabilizes $\mathbf{L}^*$ and such that $\sigma|_{\mathbf{L}}$ and $\sigma^*|_{\mathbf{L}^*}$ are in duality. Theorem 2.36 then implies the following useful consequence:

**Corollary 4.8.** *With the notation as above we have an isomorphism*

$$^{\sigma}(H_c^d(\mathbf{Y_U},\Lambda)e_s^{\mathbf{L}^F})^{\sigma} \cong H_c^d(\mathbf{Y_U},\Lambda)e^{\mathbf{L}^F}_{(\sigma^*)^{-1}(s)}$$

*of* $\Lambda\mathbf{G}^F$*-*$\Lambda\mathbf{L}^F$*-bimodules, where* $d = \dim(\mathbf{Y_U}) = \dim(\mathbf{Y}_{\sigma(\mathbf{U})})$*.*

*Proof.* By Lemma 4.5 we obtain an isomorphism

$$^{\sigma}(H_c^d(\mathbf{Y_U},\Lambda)e_s^{\mathbf{L}^F})^{\sigma} \cong H_c^d(\mathbf{Y}_{\sigma(\mathbf{U})},\Lambda)\sigma(e_s^{\mathbf{L}^F})$$

of $\Lambda\mathbf{G}^F$-$\Lambda\mathbf{L}^F$-bimodules. Consider the parabolic subgroups $\mathbf{P}_1 = \mathbf{P}$ and $\mathbf{P}_2 = \sigma(\mathbf{P})$ with unipotent radical $\mathbf{U}_1 = \mathbf{U}$ respectively $\mathbf{U}_2 = \sigma(\mathbf{U})$. Since $\mathbf{L}$ is $\sigma$-invariant by assumption $\mathbf{P}_1$ and $\mathbf{P}_2$ have $\mathbf{L}$ as common Levi complement. By Lemma 4.7 we have $\sigma(e_s^{\mathbf{L}^F}) = e^{\mathbf{L}^F}_{(\sigma^*)^{-1}(s)}$. We want to apply Theorem 2.36 to the semisimple element $(\sigma^*)^{-1}(s)$.

Recall that $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Since the Levi subgroup $\mathbf{L}^*$ is $\sigma^*$-stable it follows that

$$(\sigma^*)^{-1}(\mathrm{C}^\circ_{\mathbf{G}^*}(s)) = \mathrm{C}^\circ_{\mathbf{G}^*}((\sigma^*)^{-1}(s)) \subseteq (\sigma^*)^{-1}(\mathbf{L}^*) = \mathbf{L}^*.$$

Hence, Theorem 2.36 applies and we obtain an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\sigma(\mathbf{U})},\Lambda)e^{\mathbf{L}^F}_{(\sigma^*)^{-1}(s)} \cong H_c^{\dim}(\mathbf{Y_U},\Lambda)e^{\mathbf{L}^F}_{(\sigma^*)^{-1}(s)}.$$

This proves our corollary. □

## 4.3 Automorphisms and stabilizers of idempotents

Let $\mathbf{G}$ be a simple algebraic group of simply connected type not of type $D_4$ with Frobenius $F$. Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order. Recall that by Corollary 4.4 there exists a Frobenius endomorphism $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ with $F_0^r = F$ for some positive integer $r$ and a bijective morphism $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ commuting with $F_0$ such that $\tilde{\mathbf{G}}^F \rtimes \langle F_0, \sigma\rangle$ is the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$.

Let $\mathbf{L}^*$ be a minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$. Our first aim in this section is to show that we can assume that $F_0^*(s) = s$ and therefore $F_0^*(\mathbf{L}^*) = \mathbf{L}^*$.

To achieve this, we need to recall some general observations on conjugacy classes of semisimple elements. For this suppose that $\mathbf{H}$ is an $F^*$-stable connected reductive subgroup of $\mathbf{G}^*$ and let $t \in \mathbf{H}^{F^*}$. Then we denote $A_{\mathbf{H}}(t) = \mathrm{C}_{\mathbf{H}}(t)/\mathrm{C}^\circ_{\mathbf{H}}(t)$ and write $\mathcal{T}_{\mathbf{H},t}$ for the set of $\mathbf{H}^{F^*}$-conjugacy classes of elements of $\mathbf{H}^{F^*}$ which are $\mathbf{H}$-conjugate to $t$. Furthermore, we write $H^1(F^*, A_{\mathbf{H}}(t))$ for the set of $F^*$-conjugacy classes of $A_{\mathbf{H}}(t)$.

**Lemma 4.9.** *Under the notation as above the set $\mathcal{T}_{\mathbf{H},t}$ is in natural bijection with $H^1(F^*, A_{\mathbf{H}}(t))$.*

*Proof.* This is proved in [DM91, Proposition 3.21]. We recall the construction of this bijection. If $y \in \mathbf{H}^{F^*}$ is $\mathbf{H}$-conjugate to $t$ then there exists some $x \in \mathbf{H}$ such that ${}^x y = t$. Since both $y$ and $t$ are $F^*$-stable it follows that $x^{-1}F^*(x) \in \mathrm{C}_{\mathbf{H}}(t)$. Then one defines the map $\mathcal{T}_{\mathbf{H},t} \to H^1(F^*, A_{\mathbf{H}}(t))$ by sending the conjugacy class of $y$ to the $F^*$-conjugacy class of $x^{-1}F^*(x)$. $\square$

**Corollary 4.10.** *Assume additionally that $\mathbf{H}$ and $t$ are $F_0^*$-stable. Then the natural bijection $\mathcal{T}_{\mathbf{H},t} \to H^1(F^*, A_{\mathbf{H}}(t))$ is $F_0$-equivariant.*

*Proof.* We show that the bijection constructed in the proof of Lemma 4.9 is $F_0^*$-equivariant. Let $y \in \mathbf{H}^{F^*}$ be $\mathbf{H}$-conjugate to $t$ and $x \in \mathbf{H}$ such that ${}^x y = t$. Since $t$ is $F_0^*$-stable it follows that ${}^{F_0^*(x)}F_0^*(y) = t$. In particular, $F_0^*(y)$ is $\mathbf{H}$-conjugate to $t$. It follows that $F_0^*(x^{-1}F^*(x))$ is the image of the conjugacy class of $F_0^*(y)$ under the map $\mathcal{T}_{\mathbf{H},t} \to H^1(F^*, A_{\mathbf{H}}(t))$. The claim follows. $\square$

Recall that we assume that the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $s$ is $F_0^*$-stable. Using the equivariant bijection from Corollary 4.10 we can show the following:

**Lemma 4.11.** *Let $\mathbf{L}^*$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$. Then we may assume that $\mathbf{L}^*$ is $F_0$-stable and that the $(\mathbf{L}^*)^{F^*}$-conjugacy class of $s$ is $F_0^*$-stable.*

*Proof.* Since the $\mathbf{G}^*$-conjugacy class of $s$ is $F_0^*$-stable it follows that there exists some $t \in \mathbf{G}^*$ which is $F_0^*$-fixed and $\mathbf{G}^*$-conjugate to $s$. Let $\mathbf{K}^*$ be the unique minimal Levi subgroup containing $\mathrm{C}_{\mathbf{G}^*}(t)$. Since $F_0^*(t) = t$ it follows that $\mathbf{K}^*$ is $F_0^*$-stable. Moreover, we have $\mathrm{C}_{\mathbf{G}^*}(t) = \mathrm{C}_{\mathbf{K}^*}(t)$ and therefore $A_{\mathbf{K}^*}(t) = A_{\mathbf{G}^*}(t)$. Let $\mathcal{T}_{\mathbf{K}^*,s} \hookrightarrow \mathcal{T}_{\mathbf{G}^*,s}$ be the natural map. Then we have a commutative square:

$$\begin{array}{ccc} \mathcal{T}_{\mathbf{K}^*,t} & \xrightarrow{\;\cong\;} & H^1(F^*, A_{\mathbf{K}^*}(t)) \\ \big\downarrow & & = \big\downarrow \\ \mathcal{T}_{\mathbf{G}^*,t} & \xrightarrow{\;\cong\;} & H^1(F^*, A_{\mathbf{G}^*}(t)) \end{array}$$

From this we deduce that we have a bijection between the set of $(\mathbf{K}^*)^{F^*}$-conjugacy classes of elements which are $\mathbf{K}^*$-conjugate to $t$ and the set of $(\mathbf{G}^*)^{F^*}$-conjugacy classes of elements which are respectively $\mathbf{G}^*$-conjugate to $t$.

In fact, since this bijection is $F_0^*$-equivariant by Lemma 4.10 we deduce that it maps $F_0^*$-stable classes to $F_0^*$-stable classes. As the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $s$ is $F_0^*$-stable there exists some $x \in (\mathbf{K}^*)^{F^*}$ which is $(\mathbf{G}^*)^{F^*}$-conjugate to $s$. Moreover, the $(\mathbf{K}^*)^{F^*}$-conjugacy class of $x$ is $F_0^*$-stable. Thus, the assumptions of the lemma are satisfied if we replace $s$ by $x$ and $\mathbf{L}^*$ by $\mathbf{K}^*$. □

**Remark 4.12.** Let $\mathbf{T}_0$ be an $F_0$-stable maximal torus of $\mathbf{G}$. Suppose that the triple $(\mathbf{G}^*, \mathbf{T}_0^*, F_0^*)$ is in duality with $(\mathbf{G}, \mathbf{T}_0, F_0)$. We obtain a bijection between the $\mathbf{G}^{F_0}$-conjugacy classes of $F_0$-stable Levi subgroups of $\mathbf{G}$ and the $(\mathbf{G}^*)^{F_0^*}$-conjugacy classes of $F_0^*$-stable Levi subgroups of $\mathbf{G}^*$, see Lemma 2.15. Moreover, it follows that $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ is in duality with $(\mathbf{G}, \mathbf{T}_0, F)$, where $F^* := (F_0^*)^r$. This in turn gives a bijection between the $\mathbf{G}^F$-conjugacy classes of $F$-stable Levi subgroups of $\mathbf{G}$ and the $(\mathbf{G}^*)^{F^*}$-conjugacy classes of $F^*$-stable Levi subgroups of $\mathbf{G}^*$. This bijection is compatible with the aforementioned bijection, i.e. if $\mathbf{L}$ is an $F_0$-stable Levi subgroup of $\mathbf{G}$ in duality with the $F_0^*$-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ then $\mathbf{L}$ and $\mathbf{L}^*$ are $F$- respectively $F^*$-stable and correspond to each other under the bijection induced by the duality between $(\mathbf{G}, \mathbf{T}_0, F)$ and $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$. However, note that two $F_0$-stable Levi subgroups can be $\mathbf{G}^F$-conjugate but not $\mathbf{G}^{F_0}$-conjugate.

For the remainder of this section we may assume by Lemma 4.11 that $\mathbf{L}^*$ is $F_0$-stable and that the $\mathbf{L}^*$-conjugacy class of $s$ is $F_0$-stable. Hence, by Remark 4.12 there exists an $F_0$-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ which is in duality with $\mathbf{L}^*$ under the duality between $(\mathbf{G}, F_0)$ and $(\mathbf{G}^*, F_0^*)$.

Recall that we assume that $\sigma : \mathbf{G} \to \mathbf{G}$ is a bijective morphism with $F_0 \circ \sigma = \sigma \circ F_0$ which stabilizes the idempotent $e_s^{\mathbf{G}^F}$. By duality, we therefore obtain a bijective morphism $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ of algebraic groups with $\sigma^* \circ F_0^* = F_0^* \circ \sigma^*$. Recall that $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ is only unique up to inner automorphisms of $(\mathbf{G}^*)^{F_0^*}$, see Remark 2.18.

The following proposition compares the stabilizers of $e_s^{\mathbf{L}^F}$ and $e_s^{\mathbf{G}^F}$.

**Proposition 4.13.** *There exists some $x \in \mathbf{G}^{F_0}$ such that $x\sigma$ normalizes $\mathbf{L}$ and ${}^{x\sigma} e_s^{\mathbf{L}^F} = e_s^{\mathbf{L}^F}$.*

*Proof.* Since the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $s$ is $\sigma^*$-stable it follows that the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $\mathbf{L}^*$ is $\sigma^*$-stable. By Corollary 2.19 it follows that the $\mathbf{G}^F$-conjugacy class of $\mathbf{L}$ is $\sigma$-stable. We can therefore find $g \in \mathbf{G}^F$ and $h \in (\mathbf{G}^*)^{F^*}$ such that $\sigma_0 := g\sigma$ stabilizes $\mathbf{L}$ and $\sigma_0^* := h\sigma^*$ stabilizes $\mathbf{L}^*$. Moreover, we can choose $g$ and $h$ with the additional property that $\sigma_0|_{\mathbf{L}}$ and $\sigma_0^*|_{\mathbf{L}^*}$ are in duality with each other. Since the $(\mathbf{G}^*)^{F^*}$-conjugacy class of $s$ is $\sigma_0$-stable there exists some $n^* \in (\mathbf{G}^*)^{F^*}$ such that $\sigma_0^*(s) = {}^{n^*}s$. Since $\mathbf{L}^*$ is $\sigma_0^*$-stable it follows that $n^* \in \mathrm{N}_{(\mathbf{G}^*)^{F^*}}(\mathbf{L}^*)$. Let $n \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})$ be an element corresponding to $n^*$ under the canonical isomorphism

$$\mathrm{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F \cong \mathrm{N}_{(\mathbf{G}^*)^{F^*}}(\mathbf{L}^*)/(\mathbf{L}^*)^{F^*}$$

induced by duality. By applying Lemma 4.7 twice we obtain

$$\sigma_0(e_s^{\mathbf{L}^F}) = e_{\sigma_0^*(s)}^{\mathbf{L}^F} = e_{{}^{n^*}s}^{\mathbf{L}^F} = {}^n e_s^{\mathbf{L}^F}.$$

Therefore $y := g^{-1}n$ satisfies ${}^{y\sigma}\mathbf{L} = \mathbf{L}$ and ${}^{y\sigma}e_s^{\mathbf{L}^F} = e_s^{\mathbf{L}^F}$. Since $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$ are $F_0$-stable we conclude that $F_0(y)y^{-1} \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{L}, e_s^{\mathbf{L}^F}) = \mathbf{L}^F$. Therefore by applying Lang's theorem to $F_0 : \mathbf{L} \to \mathbf{L}$ there exists some $l \in \mathbf{L}$ such that $F_0(y)y^{-1} = F_0(l)l^{-1}$. This implies that $x := l^{-1}g \in \mathbf{G}^{F_0}$ and $x\sigma$ normalizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$. □

The next proposition describes the set of automorphisms stabilizing the idempotent $e_s^{\mathbf{G}^F}$ in a nice way:

**Proposition 4.14.** *Let $\mathbf{G}$ be a simple algebraic group of simply connected type not of type $D_4$ with Frobenius $F$. Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order. There exists a Frobenius endomorphism $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ with $F_0^r = F$ for some positive integer $r$ and a bijective morphism $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ such that $\mathcal{A} = \langle F_0, \sigma \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ satisfies:*

*(a) $F_0 \circ \sigma = \sigma \circ F_0$ as morphisms of $\tilde{\mathbf{G}}$.*

*(b) The image of $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$ in $\mathrm{Out}(\mathbf{G}^F)$ is the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$.*

*(c) There exists a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ in duality with $\mathbf{L}^*$ such that $\mathcal{A}$ stabilizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$.*

*Proof.* The existence of the bijective morphisms $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ satisfying properties (a) and (b) follows from Corollary 4.4. By construction, the Levi subgroup $\mathbf{L}$ is $F_0$-stable, see the remarks following Lemma 4.11. By Proposition 4.13 there exists $x \in \mathbf{G}^{F_0}$ such that $x\sigma$ stabilizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$. The result now follows by replacing $\sigma$ with $x\sigma$. □

For later reference in Remark 6.19 we observe the following.

**Remark 4.15.** Note that the natural map $\mathrm{Aut}(\mathbf{G}^F) \twoheadrightarrow \mathrm{Out}(\mathbf{G}^F)$ does not necessarily induce an isomorphism of $\mathcal{A}$ with its image in $\mathrm{Out}(\mathbf{G}^F)$. This is essentially the case since we need to replace the automorphism $\sigma$ by $\sigma x$ in the proof of Proposition 4.14.

## 4.4 Generalizations to disconnected reductive groups

In this section we suppose that $\hat{\mathbf{G}}$ is a reductive group with Frobenius $F$ and we let $\mathbf{G}$ be a closed normal connected $F$-stable subgroup of $\hat{\mathbf{G}}$. We let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order and $\mathbf{L}^*$ be a Levi subgroup of $\mathbf{G}^*$ such that $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}^*$.

As in Section 2.2 we suppose that $\mathbf{P} = \mathbf{L}\mathbf{U}$ and $\hat{\mathbf{P}} = \hat{\mathbf{L}}\mathbf{U}$ are two Levi decomposition of parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ and $\hat{\mathbf{P}}$ of $\hat{\mathbf{G}}$ such that $\hat{\mathbf{P}} \cap \mathbf{G} = \mathbf{P}$ and $\hat{\mathbf{L}} \cap \mathbf{G} = \mathbf{L}$.

Let $c \in \mathrm{Z}(\Lambda \mathbf{L}^F e_s^{\mathbf{L}^F})$ be the central idempotent corresponding to the central idempotent $b \in \mathrm{Z}(\Lambda \mathbf{G}^F e_s^{\mathbf{G}^F})$ under the Morita equivalence induced by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda) e_s^{\mathbf{L}^F}$.

**Lemma 4.16.** *We have* $\mathrm{N}_{\hat{\mathbf{L}}^F}(c) \subseteq \mathrm{N}_{\hat{\mathbf{G}}^F}(b)$. *In addition, we have* $\mathrm{N}_{\hat{\mathbf{L}}^F}(c)\mathbf{G}^F = \mathrm{N}_{\hat{\mathbf{G}}^F}(b)$ *whenever* $\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})\mathbf{G}^F = \mathrm{N}_{\hat{\mathbf{G}}^F}(e_s^{\mathbf{G}^F})$.

*Proof.* Let $\sigma : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ be a bijective endomorphism commuting with $F$ and stabilizing $\mathbf{L}^F$. By Theorem 2.36 and Lemma 4.5 it follows that $\sigma(c) \in \mathrm{Z}(\Lambda \mathbf{L}^F e_s^{\mathbf{L}^F})$ corresponds to the central idempotent $\sigma(b) \in \mathrm{Z}(\Lambda \mathbf{G}^F e_s^{\mathbf{G}^F})$ under the Morita equivalence induced by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda) e_s^{\mathbf{L}^F}$.

Applying this to an automorphism given by conjugation with an element of $\mathrm{N}_{\hat{\mathbf{L}}^F}(c)$ easily implies that $\mathrm{N}_{\hat{\mathbf{L}}^F}(c) \subseteq \mathrm{N}_{\hat{\mathbf{G}}^F}(b)$.

Suppose now that $\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})\mathbf{G}^F = \mathrm{N}_{\hat{\mathbf{G}}^F}(e_s^{\mathbf{G}^F})$. Then for $x \in \mathrm{N}_{\hat{\mathbf{G}}^F}(b)$ we find some $y \in \mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})$ such that $xy^{-1} \in \mathbf{G}^F$. In particular, both ${}^y c$ and $c$ map to the same central idempotent ${}^y b = {}^x b = b$ under the Morita equivalence given by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda) e_s^{\mathbf{L}^F}$. We conclude that $y \in \mathrm{N}_{\hat{\mathbf{L}}^F}(c)$. This shows that $\mathrm{N}_{\hat{\mathbf{L}}^F}(c)\mathbf{G}^F = \mathrm{N}_{\hat{\mathbf{G}}^F}(b)$. □

Let $\mathcal{D} := (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\hat{\mathbf{L}}^F)$ and $\mathcal{D}'$ be the stabilizer of the idempotent $e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}$ in $\hat{\mathbf{G}}^F \times (\hat{\mathbf{L}}^F)^{\mathrm{opp}}$. Note that we have $\mathcal{D}' = \mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\, \Delta(\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F}))$ by Lemma 4.16.

We generalize Theorem 2.36 to disconnected reductive groups.

**Lemma 4.17.** *Let* $\mathbf{Q} = \mathbf{L}\mathbf{V}$ *and* $\hat{\mathbf{Q}} = \hat{\mathbf{L}}\mathbf{V}$ *respectively be two Levi decomposition of parabolic subgroups* $\mathbf{Q}$ *of* $\mathbf{G}$ *and* $\hat{\mathbf{Q}}$ *of* $\hat{\mathbf{G}}$ *respectively which satisfy* $\hat{\mathbf{Q}} \cap \mathbf{G} = \mathbf{Q}$. *Then we have*

$$H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)\, \mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F}) \cong H_c^{\dim(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}}, \Lambda)\, \mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F})$$

*as* $\Lambda\mathcal{D}$*-bimodules.*

*Proof.* By Theorem 2.36 we have a quasi-isomorphism

$$\Theta_{\mathbf{U},\mathbf{V}} := \psi_{\mathbf{V},F(\mathbf{U}),s} \circ \mathrm{sh}^* \circ \psi^{-1}_{\mathbf{U},\mathbf{V},s} : \mathrm{R}\,\Gamma_c(\mathbf{Y}_\mathbf{U},\Lambda)e_s^{\mathbf{L}^F} \to \mathrm{R}\,\Gamma_c(\mathbf{Y}_\mathbf{V},\Lambda)e_s^{\mathbf{L}^F}.$$

The varieties involved in the construction of this map have a $\mathcal{D}$-structure (extending the usual $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-structure), see Lemma 2.4 and Lemma 2.5. The maps between them in the proof of Theorem 2.36 are easily seen to be $\mathcal{D}$-equivariant. Therefore, the map $\Theta_{\mathbf{V},\mathbf{U}}$ is a quasi-isomorphism of $\Lambda\mathcal{D}'$-complexes. By applying the functor $\mathrm{Ind}_{\mathcal{D}'}^{\mathcal{D}}$ we obtain an isomorphism

$$H_c^{\dim(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}})}(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F}) \cong H_c^{\dim(\mathbf{Y}_\mathbf{V}^{\hat{\mathbf{G}}})}(\mathbf{Y}_\mathbf{V}^{\hat{\mathbf{G}}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F})$$

of $\Lambda\mathcal{D}$-modules, see Lemma 1.31. □

**Corollary 4.18.** *Let $\sigma : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ be a bijective endomorphism of $\hat{\mathbf{G}}$ commuting with the action of $F$ and stabilizing $\hat{\mathbf{L}}$ and $\mathbf{G}$. Then we have an isomorphism*

$${}^{\sigma}(H_c^d(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F}))^{\sigma} \cong H_c^d(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(\sigma(e_s^{\mathbf{L}^F}))$$

*of $\Lambda\hat{\mathbf{G}}^F$-$\Lambda\hat{\mathbf{L}}^F$-bimodules, where $d = \dim(\mathbf{Y}_\mathbf{U}) = \dim(\mathbf{Y}_{\sigma(\mathbf{U})})$.*

*Proof.* This is a consequence of Lemma 4.17 and Lemma 4.5. □

**Lemma 4.19.** *Suppose that $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ and $\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})\mathbf{G}^F = \mathrm{N}_{\hat{\mathbf{G}}^F}(e_s^{\mathbf{G}^F})$. Then the bimodule $H_c^{\dim}(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}},\Lambda)e_\mathcal{X}$ induces a Morita equivalence between $\Lambda\hat{\mathbf{G}}^F\,\mathrm{Tr}^{\hat{\mathbf{G}}^F}_{\mathrm{N}_{\hat{\mathbf{G}}^F}(e_s^{\mathbf{G}^F})}(e_s^{\mathbf{G}^F})$ and $\Lambda\hat{\mathbf{L}}^F\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F})$.*

*Proof.* By Lemma 1.35 it follows that the bimodule

$$\mathrm{Ind}_{\mathcal{D}}^{\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}}}(H_c^{\dim}(\mathbf{Y}_\mathbf{U}^{\mathbf{G}},\Lambda))\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F})$$

induces a Morita equivalence between $\Lambda\hat{\mathbf{G}}^F\,\mathrm{Tr}^{\hat{\mathbf{G}}^F}_{\mathrm{N}_{\hat{\mathbf{G}}^F}(e_s^{\mathbf{G}^F})}(e_s^{\mathbf{G}^F})$ and $\Lambda\hat{\mathbf{L}}^F\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F})$. On the other hand, Lemma 2.8 implies that

$$\mathrm{Ind}_{\mathcal{D}}^{\hat{\mathbf{G}}^F\times(\hat{\mathbf{L}}^F)^{\mathrm{opp}}}\,H_c^{\dim}(\mathbf{Y}_\mathbf{U}^{\mathbf{G}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F}) \cong H_c^{\dim}(\mathbf{Y}_\mathbf{U}^{\hat{\mathbf{G}}},\Lambda)\,\mathrm{Tr}^{\hat{\mathbf{L}}^F}_{\mathrm{N}_{\hat{\mathbf{L}}^F}(e_s^{\mathbf{L}^F})}(e_s^{\mathbf{L}^F}).$$

□

## 4.5 Independence of Godement resolution

Let $G$ be a finite group and $L$ be a subgroup of $G$. Let $e$ be a central idempotent of $kG$ and $f$ be a central idempotent $kL$. In this section we consider two complexes $C_1$ and $C_2$ which both induce a splendid equivalence between $kGe$ and $kLf$ and we want to give a criterion when $C_1 \cong C_2$ in $\mathrm{Ho}^b(k[G \times L^{\mathrm{opp}}])$.

Let us recall the following statement, see [BDR17a, Lemma A.3].

**Lemma 4.20.** *Let $G$ be a finite group and $C$ be a bounded complex of $\ell$-permutation $kG$-modules. Suppose that $H^i(\mathrm{Br}_Q(C)) = 0$ for all $\ell$-subgroups $Q$ and all $i \neq 0$. Then $H^0(C) \cong C$ in $\mathrm{Ho}^b(kG)$.*

The following lemma should be compared to [BDR17a, Lemma A.5]. Note that we denote by $Q = 1$ the trivial subgroup of $G$.

**Lemma 4.21.** *Let $C_1$ and $C_2$ be two bounded complexes of $\ell$-permutation $kGe$-$kLf$-modules inducing a splendid Rickard equivalence between $kGe$ and $kLf$. Suppose that for all $\ell$-subgroups $Q$ of $L$ there exists an integer $d_Q$ such that the cohomology of $\mathrm{Br}_{\Delta Q}(C_1)$ and $\mathrm{Br}_{\Delta Q}(C_2)$ is concentrated in the same degree $d_Q$. In addition, assume that $H^{d_1}(C_1) \cong H^{d_1}(C_2)$. Then we have $C_1 \cong C_2$ in $\mathrm{Ho}^b(k[G \times L^{\mathrm{opp}}])$.*

*Proof.* By Theorem 1.20 the complex $C_1^\vee \otimes_{\Lambda G} C_2$ induces a splendid Rickard self-equivalence of $kLf$. Therefore, we have $\mathrm{Br}_R(C_1^\vee \otimes_{kG} C_2) \cong 0$ in $\mathrm{Ho}^b(k[L \times L^{\mathrm{opp}}])$ if $R$ is not conjugate to a subgroup of $\Delta L$. Moreover, by Lemma 1.14 we have

$$\mathrm{Br}_{\Delta Q}(C_1^\vee \otimes_{kG} C_2) \cong \mathrm{Br}_{\Delta Q}(C_1^\vee) \otimes_{k\,\mathrm{C}_G(Q)} \mathrm{Br}_{\Delta Q}(C_2)$$

for all $\ell$-subgroups $Q$ of $L$. Note that $\mathrm{Br}_{\Delta Q}(C_1)$, $\mathrm{Br}_{\Delta Q}(C_2)$ are complexes of finitely generated projective $k\mathrm{C}_G(Q)$-modules (see Lemma 1.13) and their cohomology is by assumption concentrated in the same degree $d_Q$. By [Ben98, Theorem 2.7.1] we thus have $H^i(\mathrm{Br}_{\Delta Q}(C_1^\vee \otimes_{kG} C_2)) = 0$ for $i \neq 0$. Therefore, we can apply Lemma 4.20 and obtain that

$$C_1^\vee \otimes_{kG} C_2 \cong H^0(C_1^\vee \otimes_{kG} C_2) \cong H^{d_1}(C_1)^\vee \otimes_{kG} H^{d_1}(C_2).$$

in $\mathrm{Ho}^b(k[G \times L^{\mathrm{opp}}])$. By assumption we have $H^{d_1}(C_1) \cong H^{d_1}(C_2)$. Moreover, the bimodule $H^{d_1}(C_1)$ induces a Morita equivalence between $kLf$ and $kGe$ by 1.23. From this we can conclude that $kLf \cong C_1^\vee \otimes_{kG} C_2$ in $\mathrm{Ho}^b(k[L \times L^{\mathrm{opp}}])$. Therefore, we have

$$C_1 \cong C_1 \otimes_{kL} kLf \cong C_1 \otimes_{kL} C_1^\vee \otimes_{kG} C_2 \cong C_2$$

in $\mathrm{Ho}^b(k[G \times L^{\mathrm{opp}}])$. $\square$

**Corollary 4.22.** *Let $\mathbf{G}$ be a connected reductive group, $s \in (\mathbf{G}^*)^{F^*}$ semisimple of $\ell'$-order and $\mathbf{L}^*$ the minimal Levi subgroup of $\mathbf{G}^*$ with $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Let $\mathbf{P} = \mathbf{LU}$ and $\mathbf{Q} = \mathbf{LV}$ be two parabolic subgroups of $\mathbf{G}$ with Levi subgroup $\mathbf{L}$. Then we have*

$$G\Gamma_c(\mathbf{Y_U}, k)e_s^{\mathbf{L}^F} \cong G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}[\dim(\mathbf{Y_V^G}) - \dim(\mathbf{Y_U^G})]$$

*in* $\mathrm{Ho}^b(k\mathbf{G}^F \otimes_k (k\mathbf{L}^F)^{\mathrm{opp}})$ *if*

$$\dim(\mathbf{Y_U^G}) - \dim(\mathbf{Y_V^G}) = \dim(\mathbf{Y}_{\mathrm{C}_\mathbf{U}(Q)}^{\mathrm{C}_\mathbf{G}(Q)}) - \dim(\mathbf{Y}_{\mathrm{C}_\mathbf{V}(Q)}^{\mathrm{C}_\mathbf{G}(Q)})$$

*for all $\ell$-subgroups $Q$ of $\mathbf{L}^F$.*

*Proof.* By Theorem 2.37 the complex $G\Gamma_c(\mathbf{Y_U}, k)e_s^{\mathbf{L}^F}$ induces a splendid Rickard equivalence between $k\mathbf{L}^F e_s^{\mathbf{L}^F}$ and $k\mathbf{G}^F e_s^{\mathbf{G}^F}$. Its cohomology is concentrated in degree $\dim(\mathbf{Y_U})$. Moreover, the cohomology of

$$\mathrm{Br}_{\Delta Q}(G\Gamma_c(\mathbf{Y_U^G}, k)e_s^{\mathbf{L}^F}) \cong G\Gamma_c(\mathbf{Y}_{\mathrm{C}_\mathbf{U}(Q)}^{\mathrm{C}_\mathbf{G}(Q)}, k)\, \mathrm{br}_Q(e_s^{\mathbf{L}^F})$$

is concentrated in degree $\dim(\mathbf{Y}_{\mathrm{C}_\mathbf{U}(Q)}^{\mathrm{C}_\mathbf{G}(Q)})$. The same holds for the variety $\mathbf{Y_V^G}$. By Theorem 2.36, $H_c^{\dim}(\mathbf{Y_U}, \Lambda)e_s^{\mathbf{L}^F} \cong H_c^{\dim}(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$. Hence the statement of the corollary is an immediate consequence of Lemma 4.21. □

We don't know when the condition of Corollary 4.22 holds in general. The following example is an application of Corollary 4.22.

**Example 4.23.** Suppose that $\sigma : \mathbf{G} \to \mathbf{G}$ is a bijective endomorphism with $\sigma \circ F = F \circ \sigma$ and stabilizing $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$. Suppose that a Sylow $\ell$-subgroup $D$ of $\mathbf{L}^F$ is cyclic. Up to changing $\sigma$ by inner automorphisms of $\mathbf{L}^F$ we may assume that $D$ is $\sigma$-stable. Hence, for any subgroup $Q$ of $D$ we have $\sigma(Q) = Q$. It follows that

$$\dim(\mathbf{Y}_{\mathrm{C}_{\sigma(\mathbf{U})}(Q)}^{\mathrm{C}_\mathbf{G}(Q)}) = \dim(\sigma(\mathbf{Y}_{\mathrm{C}_\mathbf{U}(Q)}^{\mathrm{C}_\mathbf{G}(Q)})) = \dim(\mathbf{Y}_{\mathrm{C}_\mathbf{U}(Q)}^{\mathrm{C}_\mathbf{G}(Q)}).$$

From this and Corollary 4.22 we conclude that

$$G\Gamma_c(\mathbf{Y}_{\sigma(\mathbf{U})}, k)e_s^{\mathbf{L}^F} \cong G\Gamma_c(\mathbf{Y_U}, k)e_s^{\mathbf{L}^F}$$

in $\mathrm{Ho}^b(k\mathbf{G}^F \otimes_k (k\mathbf{L}^F)^{\mathrm{opp}})$ Therefore, by Lemma 4.6 we have

$${}^\sigma(G\Gamma_c(\mathbf{Y_U}, k)e_s^{\mathbf{L}^F})^\sigma \cong G\Gamma_c(\mathbf{Y_U}, k)e_s^{\mathbf{L}^F}$$

in $\mathrm{Ho}^b(k\mathbf{G}^F \otimes_k (k\mathbf{L}^F)^{\mathrm{opp}})$.

## 4.6 Comparing Rickard and Morita equivalences

Let $\mathbf{G}$ be a not necessarily connected reductive group and $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Let $\mathcal{X}$ be a $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-regular series of $(\mathbf{L}, F)$. Denote by $\mathcal{Y}$ the unique series of $(\mathbf{G}, F)$ containing $\mathcal{X}$. We denote $d := \dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$.

We recall the following important result:

**Proposition 4.24.** *We have*

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}((G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, k)e_{\mathcal{X}})^{\mathrm{red}}) \cong \mathrm{End}_{D^b(k\mathbf{G}^F)}((G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, k)e_{\mathcal{X}})^{\mathrm{red}})$$

*in* $\mathrm{Ho}^b(k[\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$.

*Proof.* This is proved in Step 1 of the proof of [BDR17a, Theorem 7.6]. Note that the assumption $\mathbf{G}$ is connected is not needed in this step of the proof. □

**Proposition 4.25.** *Let $b$ be a block of $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ and $c$ be a block of $\Lambda\mathbf{L}^F e_{\mathcal{X}}$. Denote $C := bG\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^{\mathrm{red}}c$ and $d := \dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$. Then the complex $C$ induces a splendid Rickard equivalence between $\Lambda\mathbf{G}^F b$ and $\Lambda\mathbf{L}^F c$ if and only if $H^d(C)$ induces a Morita equivalence between $\Lambda\mathbf{G}^F b$ and $\Lambda\mathbf{L}^F c$.*

*Proof.* Let us first assume that $\Lambda = k$. By Lemma 2.12 the complex $C$ is splendid. Moreover, by Proposition 4.24 we have

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C) \cong \mathrm{End}_{D^b(k\mathbf{G}^F)}(C).$$

Since $C$ is a complex of projective $k\mathbf{G}^F$-modules we have $\mathrm{End}_{D^b(k\mathbf{G}^F)}(C) \cong H^0(\mathrm{End}^\bullet_{k\mathbf{G}^F}(C))$ and as the cohomology of $C$ is concentrated in degree $d$, we deduce that $H^0(\mathrm{End}^\bullet_{k\mathbf{G}^F}(C)) \cong \mathrm{End}_{k\mathbf{G}^F}(H^d(C))$. Therefore, $\mathrm{End}^\bullet_{k\mathbf{G}^F}(C) \cong \mathrm{End}_{k\mathbf{G}^F}(H^d(C))$ in $\mathrm{Ho}^b(k[\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$. By Theorem 1.22 it follows that $C$ induces a Rickard equivalence if and only if $H^d(C)$ induces a Morita equivalence.

Let us now assume that $\Lambda = \mathcal{O}$. If $H^d(C)$ induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F b$ and $\mathcal{O}\mathbf{L}^F c$ then $H^d(C \otimes_{\mathcal{O}} k) \cong H^d(C) \otimes_{\mathcal{O}} k$ induces a Morita equivalence between $k\mathbf{G}^F b$ and $k\mathbf{L}^F c$. Using the result for the case $\Lambda = k$ shows that the complex $C \otimes_{\mathcal{O}} k$ induces a splendid Rickard equivalence between $k\mathbf{G}^F b$ and $k\mathbf{L}^F c$. Thus, by Theorem 1.21 the complex $C$ induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F b$ and $\mathcal{O}\mathbf{L}^F c$. On the other hand, if the complex $C$ induces a Rickard equivalence then it follows by Lemma 1.23 that $H^d(C)$ induces a Morita equivalence. □

## 4.7 Morita equivalences for local subgroups

In this section we give some applications of Proposition 4.25. We keep the notation of the previous section and assume additionally that the rational series $\mathcal{X}$ of $(\mathbf{L}, F)$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-superregular.

**Corollary 4.26.** *Suppose that* $\mathrm{N}_{\mathbf{L}^F}(e_s^{(\mathbf{L}^\circ)^F})(\mathbf{G}^\circ)^F = \mathrm{N}_{\mathbf{G}^F}(e_s^{(\mathbf{G}^\circ)^F})$*. Then the complex* $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_{\mathcal{X}}$ *induces a splendid Rickard equivalence between* $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ *and* $\Lambda\mathbf{L}^F e_{\mathcal{X}}$*.*

*Proof.* By Lemma 4.19 the bimodule $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_{\mathcal{X}}$ induces a Morita equivalence between $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ and $\Lambda\mathbf{L}^F e_{\mathcal{X}}$. Write $e_{\mathcal{X}} = c_1 + \cdots + c_r$ as a sum of block idempotents. Then there exists a decomposition $e_{\mathcal{Y}} = b_1 + \cdots + b_r$ into block idempotents such that $H_c^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)c_i$ induces a Morita equivalence between $\Lambda\mathbf{G}^F b_i$ and $\Lambda\mathbf{L}^F c_i$. Set $C := G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^{\mathrm{red}} e_{\mathcal{X}}$. It follows from Proposition 4.25 that the complex $b_i C c_i$ induces a splendid Rickard equivalence between $\Lambda\mathbf{G}^F b_i$ and $\Lambda\mathbf{L}^F c_i$. Consequently, the complex $\bigoplus_{i=1}^r c_i C b_i$ induces a splendid Rickard equivalence between $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ and $\Lambda\mathbf{L}^F e_{\mathcal{X}}$.

For $j \neq i$ consider the complex $X := b_i C c_j$. By the proof of Lemma Proposition 4.25 we have

$$X^\vee \otimes_{k\mathbf{G}^F} X \cong \mathrm{End}^\bullet_{k\mathbf{G}^F}(X) \cong \mathrm{End}_{k\mathbf{G}^F}(H^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})}(X)) \cong 0$$

in $\mathrm{Ho}^b(k[\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$. By the proof of [Ric96, Theorem 2.1], the complex $X$ is a direct summand of $X \otimes_{k\mathbf{L}^F} X^\vee \otimes_{k\mathbf{G}^F} X$. This shows that $X = b_i C c_j \cong 0$ in $\mathrm{Ho}^b(k[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}])$ for $j \neq i$. Hence, the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_{\mathcal{X}}$ induces a splendid Rickard equivalence between $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ and $\Lambda\mathbf{L}^F e_{\mathcal{X}}$. □

Suppose that we are in the situation of Corollary 4.26. Let $b$ be a block of $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ corresponding to the block $c$ of $\Lambda\mathbf{L}^F e_{\mathcal{X}}$ under the splendid Rickard equivalence between $\Lambda\mathbf{G}^F e_{\mathcal{Y}}$ and $\Lambda\mathbf{L}^F e_{\mathcal{X}}$ given by $C := G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^{\mathrm{red}} e_{\mathcal{X}}$. Let $(Q, c_Q)$ be a $c$-Brauer pair and $(Q, b_Q)$ be the unique $b$-Brauer pair of $k\mathrm{C}_{\mathbf{G}^F}(Q)$ such that the complex $b_Q \mathrm{Br}_{\Delta Q}(C)c_Q \cong \mathrm{Br}_{\Delta Q}(C)c_Q$ induces a Rickard equivalence between $k\,\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$ and $k\,\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$, see Proposition 1.16.

The following proposition is yet another application of Proposition 4.25.

**Proposition 4.27.** *Suppose that* $\mathrm{N}_{\mathbf{L}^F}(e_s^{(\mathbf{L}^\circ)^F})(\mathbf{G}^\circ)^F = \mathrm{N}_{\mathbf{G}^F}(e_s^{(\mathbf{G}^\circ)^F})$*. Then the bimodule* $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)c_Q$ *induces a Morita equivalence between the blocks* $\Lambda\,\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$ *and* $\Lambda\,\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$*.*

*Proof.* Recall that $c$ is a block of $\Lambda\mathbf{L}^F e_{\mathcal{X}}$. Since $(Q, c_Q)$ is a $c$-subpair we have $\mathrm{br}_Q^{\mathbf{L}^F}(c)c_Q = c_Q$. Thus, there exists some rational series $\mathcal{X}' \in (i_Q^{\mathbf{L}})^{-1}(\mathcal{X})$ such that $c_Q$ is a block of $k\,\mathrm{C}_{\mathbf{L}^F}(Q)e_{\mathcal{X}'}$, see Lemma 2.28. Let $\mathcal{Y}'$ be the unique rational series of $(\mathrm{C}_{\mathbf{G}}(Q),F)$ containing $\mathcal{X}'$, see Lemma 2.29(a).

Since the complex $\mathrm{Br}_{\Delta Q}(C)c_Q \cong G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},k)c_Q$ induces a Rickard equivalence between $k\,\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$ and $k\,\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$ it follows by Lemma 2.29(c) that $b_Q$ is a block of $k\mathrm{C}_{\mathbf{G}^F}(Q)e_{\mathcal{Y}'}$. By Lemma 2.12 the complex $G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\mathcal{O})c_Q$ is a splendid complex of $\mathcal{O}\,\mathrm{C}_{\mathbf{G}^F}(Q)$-$\mathcal{O}\,\mathrm{C}_{\mathbf{L}^F}(Q)$-bimodules, which is a lift to $\mathcal{O}$ of $G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},k)c_Q$. It follows by Theorem 1.21 that $G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\mathcal{O})c_Q$ induces a Rickard equivalence between $\mathcal{O}\,\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$ and $\mathcal{O}\,\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$. It therefore follows by Proposition 4.25 that $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\Lambda)c_Q$ induces a Morita equivalence between $\Lambda\,\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$ and $\Lambda\,\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$. □

In the following we consider the subgroup

$$\mathcal{D} := \{(x,y)\in \mathrm{N}_{\mathbf{G}^F}(Q)\times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}} \mid x\,\mathrm{C}_{\mathbf{G}^F}(Q) = y^{-1}\,\mathrm{C}_{\mathbf{G}^F}(Q)\}$$

of $\mathrm{N}_{\mathbf{G}^F}(Q)\times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}$.

In addition, we let $B_Q := \mathrm{Tr}_{\mathrm{N}_{\mathbf{G}^F}(Q,b_Q)}^{\mathrm{N}_{\mathbf{G}^F}(Q)}(b_Q)$ and $C_Q := \mathrm{Tr}_{\mathrm{N}_{\mathbf{L}^F}(Q,c_Q)}^{\mathrm{N}_{\mathbf{L}^F}(Q)}(c_Q)$. The following can be seen as a geometric version of Proposition 1.36.

**Theorem 4.28.** *Suppose that* $\mathrm{N}_{\mathbf{L}^F}(e_s^{(\mathbf{L}^\circ)^F})(\mathbf{G}^\circ)^F = \mathrm{N}_{\mathbf{G}^F}(e_s^{(\mathbf{G}^\circ)^F})$. *Then the bimodule* $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda)C_Q$ *induces a Morita equivalence between* $\Lambda\mathrm{N}_{\mathbf{L}^F}(Q)C_Q$ *and* $\Lambda\mathrm{N}_{\mathbf{G}^F}(Q)B_Q$.

*Proof.* Corollary 1.18 shows that the factor groups $\mathrm{N}_{\mathbf{L}^F}(Q,c_Q)/\,\mathrm{C}_{\mathbf{L}^F}(Q)$ and $\mathrm{N}_{\mathbf{G}^F}(Q,b_Q)/\,\mathrm{C}_{\mathbf{G}^F}(Q)$ are isomorphic via the inclusion $\mathrm{N}_{\mathbf{L}^F}(Q)\subseteq \mathrm{N}_{\mathbf{G}^F}(Q)$. Moreover, by Proposition 1.16 we deduce ${}^{x}b_Q H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\Lambda)c_Q = 0$ for all $x\in \mathrm{N}_{\mathbf{G}^F}(Q)\setminus \mathrm{N}_{\mathbf{G}^F}(Q,b_Q)$. The bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\Lambda)c_Q$ induces by Proposition 4.27 a Morita equivalence between the blocks $\Lambda\mathrm{C}_{\mathbf{L}^F}(Q)c_Q$ and $\Lambda\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$.

Recall from Example 2.3 that $\mathrm{N}_{\mathbf{G}}(Q)$ is a reductive group. Moreover, $\mathrm{N}_{\mathbf{P}}(Q)$ is a parabolic subgroup of $\mathrm{N}_{\mathbf{G}}(Q)$ with Levi decomposition $\mathrm{N}_{\mathbf{P}}(Q) = \mathrm{N}_{\mathbf{L}}(Q)\ltimes \mathrm{C}_{\mathbf{U}}(Q)$. Note that $\mathrm{C}_{\mathbf{G}}(Q)$ is a normal subgroup of $\mathrm{N}_{\mathbf{G}}(Q)$ and we have a Levi decomposition $\mathrm{C}_{\mathbf{P}}(Q) = \mathrm{C}_{\mathbf{L}}(Q)\ltimes \mathrm{C}_{\mathbf{U}}(Q)$ in $\mathrm{C}_{\mathbf{G}}(Q)$, see Example 2.3. By Corollary 2.10 it follows that the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\Lambda)$ has a natural $\mathcal{D}$-action and we have an isomorphism

$$\mathrm{Ind}_{\mathcal{D}}^{\mathrm{N}_{\mathbf{G}^F}(Q)\times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}}\, H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)},\Lambda) \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda).$$

By Lemma 1.35 it follows that the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda)C_Q$ induces a Morita equivalence between $\Lambda\mathrm{N}_{\mathbf{L}^F}(Q)C_Q$ and $\Lambda\mathrm{N}_{\mathbf{G}^F}(Q)B_Q$. □

# Chapter 5

# Extending the Morita equivalence

Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order, $\mathbf{L}^*$ be the minimal $F^*$-stable Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$ and $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}^*$. Then by Theorem 2.37 the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ induces a Morita equivalence between $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$. In Proposition 4.14 we have constructed a group $\mathcal{A}$ such that $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$ generates the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$. The aim of this chapter is to show that the Morita equivalence induced by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^{\mathbf{L}^F}$ lifts (under mild assumptions on $\ell$) to a Morita equivalence between $\mathcal{O}\tilde{\mathbf{L}}^F \mathcal{A} e_s^{\mathbf{L}^F}$ and $\mathcal{O}\tilde{\mathbf{G}}^F \mathcal{A} e_s^{\mathbf{G}^F}$.

## 5.1 Disconnected reductive groups and Morita equivalences

Let $\mathbf{G}$ be a connected reductive group with Frobenius $F : \mathbf{G} \to \mathbf{G}$ and $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Consider an algebraic automorphism $\tau : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ satisfying $\tau \circ F = F \circ \tau$ and $\tau(\mathbf{G}) = \mathbf{G}$. By the discussion at the beginning of [CS13, Paragraph 2.4] it follows that the automorphism $\tau$ is uniquely determined by its restriction to $\tilde{\mathbf{G}}^F$. Consequently, the automorphisms $\tau$ and its restriction to $\tilde{\mathbf{G}}^F$ have the same order. As in Example 2.2 we consider the not necessarily connected reductive group $\tilde{\mathbf{G}} \rtimes \langle \tau \rangle$.

Let $\mathbf{G}^*$ be in duality with $\mathbf{G}$. Fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order and let $\mathbf{L}^*$ be a Levi subgroup with $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. We have a Levi decomposition $\tilde{\mathbf{P}} = \tilde{\mathbf{L}} \ltimes \mathbf{U}$ in $\tilde{\mathbf{G}}$, where $\tilde{\mathbf{P}} := \mathbf{P}\,\mathrm{Z}(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{L}} := \mathbf{L}\,\mathrm{Z}(\tilde{\mathbf{G}})$.

Suppose that the parabolic subgroup $\mathbf{P}$ is $\tau$-stable. Then the group $\hat{\mathbf{P}} := \tilde{\mathbf{P}}\langle \tau \rangle$ is a parabolic subgroup of $\hat{\mathbf{G}} := \tilde{\mathbf{G}} \rtimes \langle \tau \rangle$ with Levi decomposition

$\hat{\mathbf{P}} = \hat{\mathbf{L}} \ltimes \mathbf{U}$, where $\hat{\mathbf{L}} := \tilde{\mathbf{L}}\langle\tau\rangle$, see Example 2.2. The Frobenius endomorphism $F$ extends to a Frobenius endomorphism of $\tilde{\mathbf{G}} \rtimes \langle\tau\rangle$ by defining

$$F : \tilde{\mathbf{G}} \rtimes \langle\tau\rangle \to \tilde{\mathbf{G}} \rtimes \langle\tau\rangle,\ g\tau \mapsto F(g)\tau.$$

Since $\tau$ and its restriction to $\tilde{\mathbf{G}}^F$ have the same order we have an isomorphism

$$(\tilde{\mathbf{G}} \rtimes \langle\tau\rangle)^F \cong \tilde{\mathbf{G}}^F \rtimes \langle\tau|_{\tilde{\mathbf{G}}^F}\rangle.$$

In the following, we will as in Notation 4.3 use the same letter $\tau$ for the automorphism $\tau : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and its restriction to $\tilde{\mathbf{G}}^F$.

We let $\mathcal{Y}$ and $\mathcal{X}$ be the rational series of $(\mathbf{G}\langle\tau\rangle, F)$ and $(\mathbf{L}\langle\tau\rangle, F)$ which contain the rational series associated to the semisimple element $s$ of $\mathbf{G}$ and $\mathbf{L}$ respectively.

Let $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ be a bijective morphism of algebraic groups commuting with the action of $\tau$ and $F$. Then $\sigma$ extends to a bijective morphism

$$\sigma : \tilde{\mathbf{G}} \rtimes \langle\tau\rangle \to \tilde{\mathbf{G}} \rtimes \langle\tau\rangle,\ g\tau \mapsto \sigma(g)\tau.$$

With this notation we have the following:

**Lemma 5.1.** *The bimodule* $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_{\mathcal{X}}$ *is endowed with a natural* $(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\tilde{\mathbf{L}}^F\langle\tau\rangle)$*-action. If* $\mathbf{L}$ *is* $\sigma$*-stable then we have*

$$H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_{\mathcal{X}} \cong {}^{\sigma}H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^{\sigma}\sigma(e_{\mathcal{X}})$$

*as* $\Lambda[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\tilde{\mathbf{L}}^F\langle\tau\rangle)]$*-bimodules.*

*Proof.* This is a direct consequence of Lemma 4.18 applied to our situation. □

## 5.2 Local equivalences

We keep the assumptions of the previous section and consider the local situation. Suppose that $b$ is a block of $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ corresponding to a block $c$ of $\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}$ under the Morita equivalence induced by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$.

Let $(Q, c_Q)$ be a $c$-Brauer pair and $(Q, b_Q)$ the corresponding $b$-Brauer pair such that $b_Q H_c^{\dim}(\mathbf{Y}_{C_{\mathbf{U}}(Q)}^{C_{\mathbf{G}}(Q)}, k) = H_c^{\dim}(\mathbf{Y}_{C_{\mathbf{U}}(Q)}^{C_{\mathbf{G}}(Q)}, k)c_Q$. As usually, we define $B_Q := \mathrm{Tr}_{N_{\mathbf{G}^F}(Q,b_Q)}^{N_{\mathbf{G}^F}(Q)}(b_Q)$ and $C_Q := \mathrm{Tr}_{N_{\mathbf{L}^F}(Q,c_Q)}^{N_{\mathbf{L}^F}(Q)}(c_Q)$. We will now provide a local version of Lemma 5.1. The technical difficulty is to keep track of the diagonal actions.

**Theorem 5.2.** *Assume that* $\hat{\mathbf{Q}} = \hat{\mathbf{L}} \ltimes \mathbf{V}$ *is a parabolic subgroup of* $\hat{\mathbf{G}}$ *with Levi subgroup* $\hat{\mathbf{L}}$. *Then we have*

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{V}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q$$

*as* $\Lambda[(\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, C_Q))]$*-modules.*

*Proof.* Firstly, recall that $\mathrm{N}_{\hat{\mathbf{G}}}(Q)$ is a reductive group with closed connected normal subgroup $\mathrm{C}^\circ_{\hat{\mathbf{G}}}(Q)$, see Example 2.3. We have a Levi decomposition $\mathrm{N}_{\hat{\mathbf{P}}}(Q) = \mathrm{N}_{\hat{\mathbf{L}}}(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$ in $\mathrm{N}_{\hat{\mathbf{G}}}(Q)$. Furthermore, $\mathrm{C}^\circ_{\mathbf{G}}(Q)$ is a closed normal subgroup of $\mathrm{N}_{\hat{\mathbf{G}}}(Q)$ and we have a Levi decomposition $\mathrm{C}^\circ_{\mathbf{P}}(Q) = \mathrm{C}^\circ_{\mathbf{L}}(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$ in the connected reductive group $\mathrm{C}^\circ_{\mathbf{G}}(Q)$, see also Example 2.3. In addition, we have

$$\mathrm{N}_{\hat{\mathbf{L}}}(Q) \cap \mathrm{C}^\circ_{\mathbf{G}}(Q) = \mathrm{C}_{\hat{\mathbf{L}}\cap\mathbf{G}}(Q) \cap \mathrm{C}^\circ_{\mathbf{G}}(Q) = \mathrm{C}_{\mathbf{L}}(Q) \cap \mathrm{C}^\circ_{\mathbf{G}}(Q) = \mathrm{C}^\circ_{\mathbf{L}}(Q)$$

and similarly $\mathrm{N}_{\hat{\mathbf{P}}}(Q) \cap \mathrm{C}^\circ_{\mathbf{G}}(Q) = \mathrm{C}^\circ_{\mathbf{P}}(Q)$. This shows that we are in the situation of Section 2.3.

Recall that since $(Q, c_Q)$ is a $c$-subpair we have $\mathrm{br}_Q^{\mathbf{L}^F}(c)c_Q = c_Q$. Thus, there exists some rational series $\mathcal{X}' \in (i_Q^{\mathbf{L}})^{-1}(\mathcal{X})$ such that $c_Q$ is a block of $k\,\mathrm{C}_{\mathbf{L}^F}(Q)e_{\mathcal{X}'}$.

Let $\mathcal{Z}$ be a rational series of $\mathrm{C}^\circ_{\mathbf{L}}(Q)$ contained in $\mathcal{X}'$. By Lemma 2.28 we obtain that the rational series $\mathcal{Z}$ is $(\mathrm{C}^\circ_{\mathbf{G}}(Q), \mathrm{C}^\circ_{\mathbf{L}}(Q))$-superregular. By the proof of Lemma 4.17 we thus obtain an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}^\circ_{\mathbf{G}}(Q)}, \Lambda)e_{\mathcal{Z}} \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{V}}(Q)}^{\mathrm{C}^\circ_{\mathbf{G}}(Q)}, \Lambda)e_{\mathcal{Z}}$$

of $\Lambda[(\mathrm{C}^\circ_{\mathbf{G}}(Q)^F \times (\mathrm{C}^\circ_{\mathbf{L}}(Q)^F)^{\mathrm{opp}})\Delta\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, e_{\mathcal{Z}})]$-modules. Moreover we have $e_{\mathcal{X}'} = \mathrm{Tr}^{\mathrm{C}_{\mathbf{L}^F}(Q)}_{(\mathrm{C}^\circ_{\mathbf{L}}(Q))^F}(e_{\mathcal{Z}})$ by Lemma 2.26, which implies that $\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, e_{\mathcal{X}'}) = \mathrm{C}_{\mathbf{L}^F}(Q)\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, e_{\mathcal{Z}})$. By Remark 2.9 we obtain an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)e_{\mathcal{X}'} \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{V}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)e_{\mathcal{X}'}$$

of $\Lambda[(\mathrm{C}_{\mathbf{G}^F}(Q) \times (\mathrm{C}_{\mathbf{L}^F}(Q))^{\mathrm{opp}})\Delta\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, e_{\mathcal{X}'})]$-modules. Since $c_Q$ is a block of $k\mathrm{C}_{\mathbf{L}^F}(Q)e_{\mathcal{X}'}$ we obtain, by truncating to $c_Q$, an isomorphism $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)c_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{V}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)c_Q$ of $\Lambda[\mathrm{C}_{\mathbf{G}^F}(Q) \times (\mathrm{C}_{\mathbf{L}^F}(Q))^{\mathrm{opp}}\Delta\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, c_Q)]$-modules. Applying Lemma 2.8 yields an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{V}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q$$

of $\Lambda[(\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta\mathrm{N}_{\hat{\mathbf{L}}^F}(Q, C_Q)]$-modules. □

The methods of this section rely on the parabolic subgroup $\mathbf{P}$ being $\tau$-stable. In the upcoming sections, we will use an idea from [Dig99] to reduce to this situation.

## 5.3 Restriction of scalars for Deligne–Lusztig varieties

Let $\mathbf{G}$ be a reductive group with Frobenius endomorphism $F_0 : \mathbf{G} \to \mathbf{G}$. For an integer $r$ we let $F := F_0^r : \mathbf{G} \to \mathbf{G}$. We consider the reductive group $\underline{\mathbf{G}} = \mathbf{G}^r$ with Frobenius endomorphism $F_0 \times \cdots \times F_0 : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$ which we also denote by $F_0$. We consider the permutation

$$\tau : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$$

given by $\tau(g_1, \dots, g_r) = (g_2, \dots, g_r, g_1)$. Consider the projection onto the first component

$$\mathrm{pr} : \underline{\mathbf{G}} \to \mathbf{G},\ (g_1, \dots, g_r) \mapsto g_1.$$

The restriction of pr to $\underline{\mathbf{G}}^{F_0\tau}$ induces an isomorphism

$$\mathrm{pr} : \underline{\mathbf{G}}^{F_0\tau} \to \mathbf{G}^F$$

of finite groups with inverse map given by $\mathrm{pr}^{-1}(g) = (g, F_0^{r-1}(g), \dots, F_0(g))$ for $g \in \mathbf{G}^F$.

For any subset $\mathbf{H}$ of $\mathbf{G}$ we set

$$\underline{\mathbf{H}} := \mathbf{H} \times F_0^{r-1}(\mathbf{H}) \times \cdots \times F_0(\mathbf{H}).$$

Note that if $\mathbf{H}$ is $F$-stable then $\underline{\mathbf{H}}$ is $\tau F_0$-stable and the projection map $\mathrm{pr} : \underline{\mathbf{H}} \to \mathbf{H}$ induces an isomorphism $\underline{\mathbf{H}}^{\tau F_0} \cong \mathbf{H}^F$. Conversely, one easily sees that any $\tau F_0$-stable subset of $\underline{\mathbf{G}}$ is of the form $\underline{\mathbf{H}}$ for some $F$-stable subset $\mathbf{H}$ of $\mathbf{G}$.

Let $\mathbf{L}$ be an $F$-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Then $\underline{\mathbf{P}}$ is a parabolic subgroup of $\underline{\mathbf{G}}$ with Levi decomposition $\underline{\mathbf{P}} = \underline{\mathbf{L}} \ltimes \underline{\mathbf{U}}$ such that $\tau F_0(\underline{\mathbf{L}}) = \underline{\mathbf{L}}$. We can therefore consider the Deligne–Lusztig variety $\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, F_0\tau}$ which is a $\underline{\mathbf{G}}^{F_0\tau} \times (\underline{\mathbf{L}}^{F_0\tau})^{\mathrm{opp}}$-variety. Under the isomorphism $\mathbf{G}^F \cong \underline{\mathbf{G}}^{F_0\tau}$ we will in the following regard it as a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-variety.

The following proposition is proved in [Dig99, Proposition 3.1] under the additional assumptions that $\mathbf{G}$ is connected and that the Levi subgroup $\mathbf{L}$ is $F_0$-stable. Here, we give a complete proof of this proposition and thereby show that these assumptions are superfluous.

**Proposition 5.3.** *Let* $\mathbf{L}$ *be an* $F$*-stable Levi subgroup of* $\mathbf{G}$ *and* $\mathbf{P}$ *a parabolic subgroup of* $\mathbf{G}$ *with Levi decomposition* $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$*. Then the projection* $\mathrm{pr} : \underline{\mathbf{G}} \to \mathbf{G}$ *onto the first coordinate defines an isomorphism*

$$\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0} \cong \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}$$

*of varieties which is* $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$*-equivariant.*

*Proof.* Let $\underline{g} = (g_1, \ldots, g_r) \in \underline{\mathbf{G}}$. Then $\underline{g}\underline{\mathbf{U}} \in \mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}$ if and only if

$$\underline{g}^{-1}(\tau F_0)(\underline{g}) \in \underline{\mathbf{U}}(\tau F_0)(\underline{\mathbf{U}}) = \mathbf{U}F(\mathbf{U}) \times F_0^{r-1}(\mathbf{U}) \times \cdots \times F_0(\mathbf{U}).$$

This is equivalent to $g_1^{-1}F_0(g_2) \in \mathbf{U}F(\mathbf{U})$ and $g_i^{-1}F_0(g_{i+1}) \in F_0^{r+1-i}(\mathbf{U})$ for all $i = 2, \ldots, r$ (where $g_{r+1} := g_1$). Therefore, $\underline{g}\underline{\mathbf{U}} \in \mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}$ if and only if

$$\underline{g}\underline{\mathbf{U}} = (g_1, F_0^{r-1}(g_1), \ldots, F_0(g_1))\underline{\mathbf{U}} \text{ and } g_1^{-1}F(g_1) \in \mathbf{U}F(\mathbf{U}).$$

Hence, an element $\underline{g}\underline{\mathbf{U}} \in \mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}$ is uniquely determined by its first component $g_1\mathbf{U} \in \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$ and each element of $\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}}$ arises from an element $g_1\mathbf{U} \in \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. This shows that $\mathrm{pr} : \underline{\mathbf{G}} \to \mathbf{G}$ induces an isomorphism

$$\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F} \cong \mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0},$$

which is clearly $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-equivariant. $\square$

For any $F$-stable closed subgroup $\mathbf{H}$ of $\mathbf{G}$, the projection map $\mathrm{pr} : \underline{\mathbf{H}}^{\tau F_0} \to \mathbf{H}^F$ induces an isomorphism

$$\mathrm{pr}^\vee : \Lambda\mathbf{H}^F\text{-mod} \to \Lambda\underline{\mathbf{H}}^{\tau F_0}\text{-mod}.$$

The isomorphism of the previous lemma therefore shows that the following diagram is commutative.

$$\begin{array}{ccc} G_0(\Lambda\mathbf{L}^F) & \xrightarrow{R_{\mathbf{L}}^{\mathbf{G}, F}} & G_0(\Lambda\mathbf{G}^F) \\ \Big\downarrow \mathrm{pr}^\vee & & \Big\downarrow \mathrm{pr}^\vee \\ G_0(\Lambda\underline{\mathbf{L}}^{\tau F_0}) & \xrightarrow{R_{\underline{\mathbf{L}}}^{\underline{\mathbf{G}}, \tau F_0}} & G_0(\Lambda\underline{\mathbf{G}}^{\tau F_0}) \end{array}$$

We will now provide a local version of Proposition 5.3. Let $Q$ be a finite solvable $p'$-subgroup of $\mathbf{L}$. Recall from Example 2.3 that the normalizer $\mathrm{N}_{\mathbf{G}}(Q)$ is a reductive group and $\mathrm{N}_{\mathbf{P}}(Q)$ is a parabolic subgroup of $\mathrm{N}_{\mathbf{G}}(Q)$ with Levi decomposition $\mathrm{N}_{\mathbf{P}}(Q) = \mathrm{N}_{\mathbf{L}}(Q) \ltimes \mathrm{C}_{\mathbf{U}}(Q)$. We denote

$$\underline{Q} := Q \times F_0^{r-1}(Q) \times \cdots \times F_0(Q)$$

and observe that $\underline{Q}$ is a finite solvable $p'$-subgroup of $\underline{\mathbf{L}}$. By the same argument as before, we see that $\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q})$ is a reductive group with parabolic subgroup $\mathrm{N}_{\underline{\mathbf{P}}}(\underline{Q})$ and Levi decomposition $\mathrm{N}_{\underline{\mathbf{P}}}(\underline{Q}) = \mathrm{N}_{\underline{\mathbf{L}}}(\underline{Q}) \ltimes \mathrm{C}_{\underline{\mathbf{U}}}(\underline{Q})$.

We can therefore consider the Deligne–Lusztig variety $\mathbf{Y}_{C_{\underline{\mathbf{U}}}(\underline{Q})}^{N_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}$ which is a $N_{\underline{\mathbf{G}}^{F_0\tau}}(\underline{Q}) \times N_{\underline{\mathbf{L}}^{F_0\tau}}(\underline{Q})^{\mathrm{opp}}$-variety. Under the isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{F_0\tau} \to \mathbf{G}^F$ we may consider it as a $N_{\mathbf{G}^F}(Q) \times N_{\mathbf{L}^F}(Q)^{\mathrm{opp}}$-variety.

Thus, we can apply Proposition 5.3 in this situation and obtain the following corollary:

**Corollary 5.4.** *Suppose that we are in the situation of Proposition 5.3 and assume that $Q$ is a finite solvable $p'$-group of $\mathbf{L}$. Then the projection map* $\mathrm{pr} : N_{\underline{\mathbf{G}}}(\underline{Q}) \to N_{\mathbf{G}}(Q)$ *induces an isomorphism*

$$\mathbf{Y}_{C_{\underline{\mathbf{U}}}(\underline{Q})}^{N_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0} \cong \mathbf{Y}_{C_{\mathbf{U}}(Q)}^{N_{\mathbf{G}}(Q),F}$$

*of varieties which is* $N_{\mathbf{G}^F}(Q) \times N_{\mathbf{L}^F}(Q)^{\mathrm{opp}}$*-equivariant.*

## 5.4 Duality in the context of restriction of scalars

Recall from the previous section we denote $\underline{\mathbf{G}} = \mathbf{G}^r$ and we consider the automorphism

$$\tau : \underline{\mathbf{G}} \to \underline{\mathbf{G}},\ (g_1, \dots, g_r) \mapsto (g_2, \dots, g_r, g_1).$$

Our aim in this and the subsequent sections is to study the representation theory of the finite group $\underline{\mathbf{G}}^{\tau F_0}$. In order to understand the Lusztig series of the group $\underline{\mathbf{G}}^{\tau F_0}$ we will need to explicitly construct the dual group of $\underline{\mathbf{G}}^{\tau F_0}$. Note that in the following we will therefore heavily use the notation introduced in Section 2.6.

Suppose that the triple $(\mathbf{G}^*, \mathbf{T}_0^*, F_0^*)$ is in duality with $(\mathbf{G}, \mathbf{T}_0, F_0)$ under a duality isomorphism $\delta : X(\mathbf{T}_0) \to Y(\mathbf{T}_0^*)$. We consider the $r$-fold product $\underline{\mathbf{G}^*} := (\mathbf{G}^*)^r$ of the dual group $\mathbf{G}^*$ endowed with the Frobenius endomorphism $F_0^* := F_0^* \times \cdots \times F_0^* : \underline{\mathbf{G}^*} \to \underline{\mathbf{G}^*}$. Moreover, let

$$\tau^* : \underline{\mathbf{G}^*} \to \underline{\mathbf{G}^*},\ (g_1, \dots, g_r) \mapsto (g_r, g_1 \dots, g_{r-1}).$$

We denote by $\mathrm{pr} : \underline{\mathbf{G}^*} \to \mathbf{G}^*$ the projection onto the first coordinate. For any $F^*$-stable closed subgroup $\mathbf{H}$ of $\mathbf{G}^*$ we set

$$\underline{\mathbf{H}} := \mathbf{H} \times F_0^*(\mathbf{H}) \times \cdots \times (F_0^*)^{r-1}(\mathbf{H}).$$

Let $\pi_i : \underline{\mathbf{T}_0} \to F_0^{r-i+1}(\mathbf{T}_0)$ the projection onto the $i$th coordinate. For any character $\underline{\chi} \in X(\underline{\mathbf{T}_0})$ we let $\chi_i \in X(F_0^{r-i+1}(\mathbf{T}_0))$ be the unique character such that $\chi_i \circ \pi_i = \chi$ and we write $\underline{\chi} = (\chi_1, \dots, \chi_r)$.

Similarly, for $\underline{\gamma} \in Y(\underline{\mathbf{T}_0^*})$ we consider the projection $\pi_i : \underline{\mathbf{T}_0^*} \to (F_0^*)^i(\mathbf{T}_0^*)$ onto the $i$th coordinate. Let $\gamma_i \in Y((F_0^*)^i(\mathbf{T}_0^*))$ be the cocharacter defined by $\gamma_i = \pi_i \circ \underline{\gamma}$ and write $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$. We then define

$$\underline{\delta} : X(\underline{\mathbf{T}_0}) \to Y(\underline{\mathbf{T}_0^*}),\ (\chi_1, \dots, \chi_r) \mapsto (\delta(\chi_1), \dots, \delta(\chi_r)).$$

Recall that the torus $\underline{\mathbf{T}_0}$ is $\tau F_0$-stable and the torus $\underline{\mathbf{T}_0^*}$ is $\tau^* F_0^*$-stable.

**Lemma 5.5.** *The triple $(\underline{\mathbf{G}}, \underline{\mathbf{T}_0}, F_0\tau)$ is in duality with $(\underline{\mathbf{G}^*}, \underline{\mathbf{T}_0^*}, \tau^* F_0^*)$.*

*Proof.* For $\underline{\chi} = (\chi_1, \dots, \chi_r) \in X(\underline{\mathbf{T}_0})$ we have

$$(F_0\tau)(\underline{\chi}) = (F_0(\chi_r), F_0(\chi_1), \dots, F_0(\chi_{r-1})).$$

On the other hand, for $\underline{\gamma} = (\gamma_1, \dots, \gamma_r) \in Y(\underline{\mathbf{T}_0^*})$ we have $(\tau^* F_0^*)^\vee(\underline{\gamma}) = ((F_0^*)^\vee(\gamma_r), (F_0^*)^\vee(\gamma_1), \dots, (F_0^*)^\vee(\gamma_{r-1}))$. Therefore, we have

$$\underline{\delta}((\tau F_0)(\chi_1, \dots, \chi_r)) = (\tau^* F_0^*)^\vee(\underline{\delta}(\chi_1, \dots, \chi_r)).$$

We conclude that $(\underline{\mathbf{G}}, \underline{\mathbf{T}_0}, F_0\tau)$ is in duality with $(\underline{\mathbf{G}^*}, \underline{\mathbf{T}_0^*}, \tau^* F_0^*)$. ☐

**Lemma 5.6.** *The following diagram is commutative:*

$$\begin{array}{ccc} \operatorname{Irr}(\underline{\mathbf{T}_0}^{\tau F_0}) & \xleftarrow{\ \operatorname{pr}^\vee\ } & \operatorname{Irr}(\mathbf{T}_0^F) \\ \downarrow{\scriptstyle \underline{\delta}_1} & & \downarrow{\scriptstyle \delta_1} \\ (\underline{\mathbf{T}_0^*})^{\tau^* F_0^*} & \xrightarrow{\ \operatorname{pr}\ } & ((\mathbf{T}_0^*)^{F^*}) \end{array}$$

*More concretely, if $(\mathbf{T}_0, \theta) \in \nabla(\mathbf{G}, F)$ is in duality with $(\mathbf{T}_0^*, \operatorname{pr}(\underline{s})) \in \mathcal{S}(\mathbf{G}^*, F^*)$ then $(\underline{\mathbf{T}_0}, \theta \circ \operatorname{pr}) \in \nabla(\underline{\mathbf{G}}, F_0\tau)$ is in duality with $(\underline{\mathbf{T}_0^*}, \underline{s}) \in \mathcal{S}(\underline{\mathbf{G}^*}, \tau^* F_0^*)$.*

*Proof.* Let $\theta \in \operatorname{Irr}(\mathbf{T}_0^F)$ be a character and suppose that $\chi \in X(\mathbf{T}_0)$ satisfies $\theta = \operatorname{Res}^{\mathbf{T}_0}_{\mathbf{T}_0^F}(\kappa \circ \chi)$, where $\kappa : \overline{\mathbb{F}_p}^\times \hookrightarrow \overline{\mathbb{Q}_\ell}^\times$ is the ring homomorphism from Section 2.6. Then we have $\operatorname{Res}^{\underline{\mathbf{T}_0}}_{\underline{\mathbf{T}_0}^{\tau F_0}}(\kappa \circ (\chi, 1, \dots, 1)) = \theta \circ \operatorname{pr}$.

Recall that in Section 2.6 we fixed an injective morphism $\iota : (\mathbb{Q}/\mathbb{Z})_{p'} \to \overline{\mathbb{F}_q}^\times$ and defined $\zeta \in \overline{\mathbb{F}_q}^\times$ to be $\iota(\frac{1}{q^n - 1}) \in \overline{\mathbb{F}_q}^\times$.

Let $\gamma := \delta(\chi)$ and $s := N_{F^n/F}(\gamma(\zeta))$ where $n$ is chosen such that $\mathbf{T}_0^*$ is split over $\mathbb{F}_{q^n}$. Then by definition we have $\delta_1(\theta) = s$, where $\delta_1 : \operatorname{Irr}(\mathbf{T}_0^F) \to (\mathbf{T}_0^*)^{F^*}$ is the isomorphism induced by duality. We have $(\gamma, 1, \dots, 1) = \underline{\delta}(\chi, 1, \dots, 1)$. For any $t \in \underline{\mathbf{T}_0^*}$ we have

$$N_{(F_0^*\tau^*)^r/F_0^*}(t, 1, \dots, 1) = (t, F_0^*(t), \dots, (F_0^*)^{r-1}(t)).$$

We conclude that

$$N_{(F_0^*\tau^*)^{rn}/F_0^*\tau^*}((\gamma,1,\dots,1)(\zeta)) = (N_{F^n/F}(\gamma(\zeta)), N_{F^n/F}(F_0(\gamma(\zeta)),\dots,N_{F^n/F}(F_0^{r-1}(\gamma(\zeta)))$$

and therefore

$$N_{(F_0^*\tau^*)^{rn}/F_0^*\tau^*}((\gamma,1,\dots,1)(\zeta)) = (s, F_0(s),\dots,F_0^{r-1}(s)) = \underline{s}.$$

Note that $\underline{\mathbf{T}_0}$ is split over $\mathbb{F}_{q^{rn}}$ since $\underline{\mathbf{T}_0}^{(\tau F_0)^{rn}} \cong \mathbf{T}_0^{F^n}$. Let $\underline{\delta}_1 : \mathrm{Irr}(\underline{\mathbf{T}_0}^{\tau F_0}) \to (\underline{\mathbf{T}_0^*})^{F_0\tau^*}$ be the isomorphism induced by duality between $(\underline{\mathbf{G}}, \underline{\mathbf{T}_0}, F_0\tau)$ and $(\underline{\mathbf{G}^*}, \underline{\mathbf{T}_0^*}, \tau^*F_0^*)$. Then by construction we have $\underline{\delta}_1(\theta\circ \mathrm{pr}) = \underline{s}$, as claimed in the lemma. □

## 5.5 Comparing Weyl groups

For the following lemma note that

$$W(\underline{\mathbf{T}_0}) = W(\mathbf{T}_0)\times W(F_0^{r-1}(\mathbf{T}_0))\times\cdots\times W(F_0(\mathbf{T}_0)).$$

The map $F_0 : W(\mathbf{T}_0) \to W(F_0(\mathbf{T}_0))$ is an isomorphism of finite groups and we denote by $F_0^{-1} : W(F_0(\mathbf{T}_0)) \to W(\mathbf{T}_0)$ its inverse.

**Lemma 5.7.** *The product map*

$$\mathrm{prod}_F : W(\underline{\mathbf{T}_0}) \to W(\mathbf{T}_0),\ (w_1,\dots,w_r) \to w_1\cdot F_0(w_2)\cdot\dots\cdot F_0^{r-1}(w_r)$$

*induces a bijection between the $F_0\tau$-conjugacy classes of $W(\underline{\mathbf{T}_0})$ and the $F$-conjugacy classes of $W(\mathbf{T}_0)$. In particular, any element $\underline{w}\in W(\underline{\mathbf{T}_0})$ is $F_0\tau$-conjugate to $(\mathrm{prod}_F(\underline{w}),1,\dots,1)$.*

*Proof.* Let $\underline{x} = (x_1,\dots,x_r), \underline{g} = (g_1,\dots,g_r) \in W(\underline{\mathbf{T}_0})$ be arbitrary. Then

$$\underline{g}\underline{x}(F_0\tau(\underline{g}))^{-1} = (g_1x_1F_0(g_2^{-1}), g_2x_2F_0(g_3^{-1}),\dots,g_rx_rF_0(g_1^{-1})),$$

which implies

$$\mathrm{prod}_F(\underline{g}\underline{x}(F_0\tau(\underline{g}))^{-1}) = g_1\mathrm{prod}_F(x_1,\dots,x_r)F(g_1)^{-1} = g_1\mathrm{prod}_F(\underline{x})F(g_1)^{-1}.$$

This shows that $\mathrm{prod}_F$ induces a map from the $F_0\tau$-conjugacy classes of $W(\underline{\mathbf{T}_0})$ to $F$-conjugacy classes of $W(\mathbf{T}_0)$. The map $\mathrm{prod}_F : W(\underline{\mathbf{T}_0}) \to W(\mathbf{T}_0)$ is clearly surjective since for $w\in W(\mathbf{T}_0)$ we have $\mathrm{prod}_F(w,1,\dots,1) = w$. Therefore, the induced map on conjugacy classes is surjective as well. It remains to show that this map is injective.

Let $\underline{x} = (x_1, \ldots, x_r) \in W(\underline{\mathbf{T}_0})$ and $\underline{y} = (y_1, \ldots, y_r) \in W(\underline{\mathbf{T}_0})$ such that

$$g_1 \mathrm{prod}_F(\underline{x}) F(g_1)^{-1} = \mathrm{prod}_F(\underline{y})$$

for some $g_1 \in W(\mathbf{T}_0)$. This is equivalent to

$$\mathrm{prod}_F(\underline{y})^{-1} g_1 \mathrm{prod}_F(\underline{x}) = F(g_1).$$

We want to show that there exist $g_i \in W(F_0^{r-i+1}(\mathbf{T}_0))$, $i = 2, \ldots, r$, such that $\underline{g} := (g_1, \ldots, g_r) \in W(\underline{\mathbf{T}_0})$ satisfies $\underline{g}\underline{x}(F_0\tau(\underline{g}))^{-1} = \underline{y}$ which is equivalent to

$$(g_1 x_1 F_0(g_2^{-1}), g_2 x_2 F_0(g_3^{-1}), \ldots, g_r x_r F_0(g_1^{-1})) = (y_1, \ldots, y_r).$$

This is tantamount to

$$F_0(g_i) = y_{i-1}^{-1} g_{i-1} x_{i-1} \text{ for all } i = 2, \ldots, r \text{ and } F_0(g_1) = y_r^{-1} g_r x_r.$$

Hence we can inductively define $g_i := F_0^{-1}(y_{i-1}^{-1} g_{i-1} x_{i-1})$ for $i = 2, \ldots, r$. It remains to show that the equality $F_0(g_1) = y_r^{-1} g_r x_r$ holds. By definition of $g_r$ we have

$$y_r^{-1} g_r x_r = y_r^{-1} F_0^{-1}(y_{r-1}^{-1}) F_0^{-1}(g_{r-1}) F_0^{-1}(x_{r-1}) x_r$$

Iterating gives

$$y_r^{-1} g_r x_r = \prod_{i=1}^{r} F_0^{1-i}(y_{r-(1-i)}^{-1}) F_0^{-r+1}(g_1) \prod_{i=1}^{r} F_0^{-r+i}(x_i).$$

Therefore, we have

$$y_r^{r-1} g_r x_r = F_0^{r-1}(\mathrm{prod}_F(\underline{y})^{-1} g_1 \mathrm{prod}_F(\underline{x})) = F_0^{r-1}(F(g_1)) = F_0(g_1).$$

This shows that $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ are $F_0\tau$-conjugate. Thus, we have shown that the map $\mathrm{prod}_F : W(\underline{\mathbf{T}_0}) \to W(\mathbf{T}_0)$ induces a bijection between the $F_0\tau$-conjugacy classes of $W(\underline{\mathbf{T}_0})$ and the $F$-conjugacy classes of $W(\mathbf{T}_0)$. □

For the dual group we define

$$\mathrm{prod}_F^* : W(\underline{\mathbf{T}_0^*}) \to W(\mathbf{T}_0^*),\ (w_1, \ldots, w_r) \mapsto (F_0^*)^{-r+1}(w_r) \ldots (F_0^*)^{-1}(w_2) \cdot w_1.$$

**Lemma 5.8.** *For* $\underline{w} \in W(\underline{\mathbf{T}_0})$ *we have* $\mathrm{prod}_F(\underline{w})^* = \mathrm{prod}_F^*(\underline{w}^*)$.

*Proof.* We have

$$\mathrm{prod}_F(\underline{w})^* = (F_0^{r-1}(w_r))^* \cdot \dots \cdot (F_0(w_2))^* \cdot w_1^*.$$

As $((F_0^i(w))^*) = (F_0^*)^{-i}(w^*)$ for all $i$ and $w \in W(\mathbf{T}_0)$ we conclude that

$$\mathrm{prod}_F(\underline{w})^* = (F_0^*)^{-r+1}(w_r^*) \dots (F_0^*)^{-1}(w_2^*) \cdot w_1^* = \mathrm{prod}_F^*(\underline{w}^*).$$

This proves the result. □

**Corollary 5.9.** *The dual product map* $\mathrm{prod}_F^* : W(\underline{\mathbf{T}_0^*}) \to W(\mathbf{T}_0^*)$ *induces a bijection between the* $F_0^*\tau^*$*-conjugacy classes of* $W(\underline{\mathbf{T}_0^*})$ *and the* $F^*$*-conjugacy classes of* $W(\mathbf{T}_0^*)$.

*Proof.* The map $^* : W(\mathbf{T}_0) \to W(\mathbf{T}_0^*)$ induces a bijection between $F$- and $F^*$-conjugacy classes, see Section 2.6. On the other hand, the map $^*$ : $W(\underline{\mathbf{T}_0}) \to W(\underline{\mathbf{T}_0^*})$ induces a bijection between $\tau F_0$-conjugacy classes and $F_0^*\tau^*$-conjugacy classes, see loc. cit.. By Lemma 5.7 the map $\mathrm{prod}_F$ : $W(\underline{\mathbf{T}_0}) \to W(\mathbf{T}_0)$ induces a bijection between $F$- and $\tau F_0$-conjugacy classes. The statement follows from Lemma 5.8. □

Let $\underline{w} \in W(\underline{\mathbf{T}_0})$ and $w := \mathrm{prod}_F(\underline{w}) \in W(\mathbf{T}_0)$. We consider the projection map $\mathrm{pr} : \underline{\mathbf{T}_0}^{\underline{w}\tau F_0} \to \mathbf{T}_0^{wF}$ onto the first coordinate. Let us show that this is well-defined, i.e., it maps $\underline{w}\tau F_0$-stable elements to $wF$-stable elements. Let $\underline{t} = (t_1, \dots, t_r) \in \underline{\mathbf{T}_0}$ be $\underline{w}\tau F_0$-stable. Then $(t_1, \dots, t_r) = ({}^{w_1 F_0}t_2, \dots, {}^{w_{r-1}F_0}t_r, {}^{w_r F_0}t_1)$ from which we deduce that

$$t_1 = {}^{w_1 F_0}t_2 = {}^{w_1 F_0(w_2) F_0^2} = \dots = {}^{\mathrm{prod}_F(\underline{w})F}t_1.$$

This shows that $\mathrm{pr}(\underline{t}) = t_1$ is $wF$-stable.

Now a similar calculation shows that $\mathrm{pr} : (\underline{\mathbf{T}_0^*})^{\tau^* F_0^* \underline{w}^*} \to (\mathbf{T}_0^*)^{F^* w^*}$ is well-defined. We are ready to state the next lemma, which is a generalization of Lemma 5.6.

**Lemma 5.10.** *Let* $\underline{w} \in W(\underline{\mathbf{T}_0})$ *and set* $w := \mathrm{prod}_F(\underline{w}) \in W(\mathbf{T}_0)$*. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{Irr}(\underline{\mathbf{T}_0}^{\underline{w}\tau F_0}) & \xleftarrow{\mathrm{pr}^\vee} & \mathrm{Irr}(\mathbf{T}_0^{wF}) \\
\Big\downarrow{\scriptstyle \delta_{\underline{w}}} & & \Big\downarrow{\scriptstyle \delta_w} \\
(\underline{\mathbf{T}_0^*})^{\tau^* F_0^* \underline{w}^*} & \xrightarrow{\mathrm{pr}} & (\mathbf{T}_0^*)^{F^* w^*}
\end{array}$$

*Proof.* To simplify the calculations we observe the following: By Lemma 5.7 any element $\underline{w} \in W(\underline{\mathbf{T}_0})$ is $F_0\tau$-conjugate to $(\mathrm{prod}_F(\underline{w}), 1, \ldots, 1)$. On the other hand, the statement of the lemma only depends on the $\tau F_0$-conjugacy class respectively $F$-conjugacy class of the element $\underline{w}$, respectively $\mathrm{prod}_F(\underline{w})$. We may thus assume that $\underline{w} = (w, 1, \ldots, 1)$.

The rest of the calculation is as in Lemma 5.6. Let $\theta \in \mathrm{Irr}(\mathbf{T}_0^{wF})$ be a character and suppose that $\chi \in X(\mathbf{T}_0)$ satisfies $\theta = \mathrm{Res}^{\mathbf{T}_0}_{\mathbf{T}_0^{wF}}(\kappa \circ \chi)$. Then we have $\mathrm{Res}^{\underline{\mathbf{T}_0}}_{\underline{\mathbf{T}_0}^{\underline{w}\tau F_0}}(\kappa \circ (\chi, 1, \ldots, 1)) = \theta \circ \mathrm{pr}$. Let $\gamma := \delta(\chi)$ and $s := N_{(F^* w^*)^n / F^* w^*}(\gamma(\zeta))$ where $n$ is chosen such that $\mathbf{T}_0^*$ is split over $\mathbb{F}_{q^n}$. Then by definition we have $\delta_w(\theta) = s$, where $\delta_w : \mathrm{Irr}(\mathbf{T}_0^{wF}) \to (\mathbf{T}_0^*)^{F^* w^*}$ is the isomorphism induced by duality. We have $(\gamma, 1, \ldots, 1) = \underline{\delta}(\chi, 1, \ldots, 1)$. For any $t \in \underline{\mathbf{T}_0^*}$ we have

$$N_{(F_0^* \tau^* \underline{w}^*)^r / F_0^* \tau^* \underline{w}^*}(t, 1, \ldots, 1) = (t, {}^{F_0^* w^*}t, \ldots, {}^{(F_0^*)^{r-1} w^*}t).$$

Observe that the inverse of the projection map $\mathrm{pr} : (\underline{\mathbf{T}_0^*})^{\tau^* F_0^* \underline{w}^*} \to (\mathbf{T}_0^*)^{F^* w^*}$ is given by $(\mathbf{T}_0^*)^{F^* w^*} \to (\underline{\mathbf{T}_0^*})^{\tau^* F_0^* \underline{w}^*}$, $t \mapsto (t, {}^{F_0^* w^*}t, \ldots, {}^{(F_0^*)^{r-1} w^*}t)$. We conclude that

$$N_{(F_0 \tau^* \underline{w}^*)^{rn} / F_0 \tau \underline{w}^*}(\gamma, 1, \ldots, 1)(\zeta) = (s, {}^{F_0^* w^*}s, \ldots, {}^{(F_0^*)^{r-1} w^*}s) = \underline{s}.$$

Note that $\underline{\mathbf{T}_0}$ is split over $\mathbb{F}_{q^{rn}}$ since $\underline{\mathbf{T}_0}^{(\tau F_0)^{rn}} \cong \mathbf{T}_0^{F^n}$. Let $\underline{\delta}_{\underline{w}} : \mathrm{Irr}(\underline{\mathbf{T}_0}^{\tau F_0}) \to (\underline{\mathbf{T}_0^*})^{F_0 \tau^*}$ be the isomorphism induced by duality between the triples $(\underline{\mathbf{G}}, \underline{\mathbf{T}}_0, \underline{w} F_0 \tau)$ and $(\underline{\mathbf{G}}^*, \underline{\mathbf{T}_0^*}, \tau^* F_0^* \underline{w}^*)$. Then by construction we have $\underline{\delta}_{\underline{w}}(\theta \circ \mathrm{pr}) = \underline{s}$, as claimed in the statement. $\square$

## 5.6 Restriction of scalars and Lusztig series

From now on we identify the groups $\underline{\mathbf{G}}^{\tau F_0}$ and $\mathbf{G}^F$ under the fixed isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$.

The following proposition is probably known. A very similiar result in a different language can be found in [Tay19, Corollary 8.8].

**Proposition 5.11.** *For any semisimple $\underline{x} \in (\underline{\mathbf{G}}^*)^{F_0^* \tau^*}$ the sets $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{x}))$ and $\mathcal{E}(\underline{\mathbf{G}}^{F_0 \tau}, \underline{x})$ coincide via the isomorphism $\mathbf{G}^F \cong \underline{\mathbf{G}}^{\tau F_0}$ given by* pr.

*Proof.* Note that the isomorphism $\mathrm{pr} : (\underline{\mathbf{G}}^*)^{F_0^* \tau^*} \to (\mathbf{G}^*)^{F^*}$ induces a bijection between semisimple conjugacy classes of $(\underline{\mathbf{G}}^*)^{F_0^* \tau^*}$ and semisimple conjugacy classes of $(\mathbf{G}^*)^{F^*}$. Moreover, note that we have two partitions of irreducible characters into rational Lusztig series:

$$\mathrm{Irr}(\underline{\mathbf{G}}^{F_0 \tau}) = \coprod_{\underline{x}} \mathcal{E}(\underline{\mathbf{G}}^{F_0 \tau}, \underline{x})$$

and

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_{\underline{x}} \mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{x})),$$

where in both cases $\underline{x}$ runs over a set of representatives for the $(\underline{\mathbf{G}}^*)^{F_0^*\tau^*}$-conjugacy classes of semisimple elements.

Thus, if we can show that $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{x})) \subseteq \mathcal{E}(\underline{\mathbf{G}}^{F_0\tau}, \underline{x})$ (via the projection map $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$) for all semisimple $\underline{x} \in \underline{\mathbf{G}}^{*F_0\tau}$ then we therefore automatically have $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{x})) = \mathcal{E}(\underline{\mathbf{G}}^{F_0\tau}, \underline{x})$.

We fix an $F$-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ and suppose that the triple $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ is in duality with $(\mathbf{G}, \mathbf{T}_0, F)$.

Let $\mathbf{T}^*$ be an $F^*$-stable maximal torus of $\mathbf{G}^*$ with $x \in (\mathbf{T}^*)^{F^*}$. There exists $h \in \mathbf{G}^*$ such that $\mathbf{T}^* = {}^h\mathbf{T}_0^*$. Define $w^* \in W(\mathbf{T}_0^*)$ by the property that $F(w^*) := h^{-1}F^*(h)\mathbf{T}_0^* \in W(\mathbf{T}_0^*)$. Then $(w^*, {}^{h^{-1}}x)$ maps to $(\mathbf{T}^*, x)$ under the bijection $\mathcal{S}(\mathbf{T}_0^*, W(\mathbf{T}_0^*), F^*)/W(\mathbf{T}_0^*) \to \mathcal{S}(\mathbf{G}^*, F^*)/(\mathbf{G}^*)^{F^*}$. Let $\theta \in \mathrm{Irr}(\mathbf{T}_0^{wF})$ with $\delta_w(\theta) = {}^{h^{-1}}x$. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ in duality with $\mathbf{T}^*$ and let $g \in \mathbf{G}$ such that $\mathbf{T} = {}^g\mathbf{T}_0$. Note that $g^{-1}F(g) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ with image $w \in W(\mathbf{T}_0)$. Observe that $\underline{\mathbf{T}} = \mathbf{T} \times F_0^{r-1}(\mathbf{T}) \times \cdots \times F_0(\mathbf{T}) = {}^{\underline{g}}\underline{\mathbf{T}_0}$, where $\underline{g} := (g, F_0^{r-1}(g), \dots, F_0(g)) \in \underline{\mathbf{G}}$. Moreover,

$$\underline{g}^{-1}(\tau F_0)(\underline{g}) = (g^{-1}F(g), 1 \dots, 1) \in \mathrm{N}_{\underline{\mathbf{G}}}(\underline{\mathbf{T}_0}).$$

We denote by $\underline{w} := (w, 1, \dots, 1) \in W(\underline{\mathbf{T}_0})$ its image in the Weyl group. Using the isomorphism in Proposition 5.3 we obtain

$$R_{\mathbf{T}}^{\mathbf{G},F}({}^g\theta) = R_{\underline{\mathbf{T}}}^{\underline{\mathbf{G}},\tau F_0}(({}^g\theta) \circ \mathrm{pr}) = R_{\underline{\mathbf{T}}}^{\underline{\mathbf{G}},\tau F_0}({}^{\underline{g}}(\theta \circ \mathrm{pr})).$$

Set $\underline{y} := \delta_{\underline{w}}(\theta \circ \mathrm{pr}) \in (\underline{\mathbf{T}_0^*})^{\tau^* F_0^* \underline{w^*}}$. Since $\delta_w(\theta) = {}^{h^{-1}}x$ it follows that $\mathrm{pr}(\underline{y}) = {}^{h^{-1}}x$ by Lemma 5.10.

We denote $\underline{h} := (h, F_0^*(h), \dots, (F_0^*)^{r-1}(h))$ and observe that $\underline{\mathbf{T}}^* = \mathbf{T}^* \times F_0^*(\mathbf{T}^*) \times \cdots \times (F_0^*)^{r-1}(\mathbf{T}^*) = {}^{\underline{h}}\underline{\mathbf{T}_0^*}$. Moreover, we have

$$\underline{h}^{-1}(\tau^* F_0^*)(\underline{h}) = (h^{-1}F^*(h), 1, \dots, 1) \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*).$$

From this we deduce that $(\underline{w}^*, \underline{y})$ maps to $(\underline{\mathbf{T}^*}, {}^{\underline{h}}\underline{y}) \in \mathcal{S}(\underline{\mathbf{G}}^*, F_0^*\tau^*)$ under the bijection $\mathcal{S}(\underline{\mathbf{T}_0^*}, W(\underline{\mathbf{T}_0^*}), F_0^*\tau^*)/W(\underline{\mathbf{T}_0^*}) \to \mathcal{S}(\underline{\mathbf{G}}^*, F_0^*\tau^*)/\underline{\mathbf{G}}^{*F_0^*\tau^*}$.

Since $\mathrm{pr}({}^{\underline{h}}\underline{y}) = {}^{\mathrm{pr}(\underline{h})}\mathrm{pr}(\underline{y}) = {}^h({}^{h^{-1}}x) = x = \mathrm{pr}(\underline{x})$ and $\mathrm{pr} : (\underline{\mathbf{G}}^*)^{F_0^*\tau^*} \to (\mathbf{G}^*)^{F^*}$ is bijective we deduce that ${}^{\underline{h}}\underline{y} = \underline{x}$. In particular, the constituents of $R_{\underline{\mathbf{T}}}^{\underline{\mathbf{G}},\tau F_0}(\theta \circ \mathrm{pr})$ lie in the Lusztig series $\mathcal{E}(\underline{\mathbf{G}}^{\tau F_0}, \underline{x})$. This shows the inclusion $\mathcal{E}(\mathbf{G}^F, x) \subseteq \mathcal{E}(\underline{\mathbf{G}}^{\tau F_0}, \underline{x})$. □

**Corollary 5.12.** *For any semisimple $\ell'$-element $\underline{s} \in (\underline{\mathbf{G}}^*)^{\tau^* F_0^*}$ we have $e^{\mathbf{G}^F}_{\mathrm{pr}(\underline{s})} = e^{\underline{\mathbf{G}}^{F_0\tau}}_{\underline{s}}$ considered as idempotents of $\Lambda\mathbf{G}^F$ under the isomorphism $\Lambda\underline{\mathbf{G}}^{\tau F_0} \cong \Lambda\mathbf{G}^F$ given by* pr*.*

*Proof.* Note that $e^{\mathbf{G}^F}_{\mathrm{pr}(\underline{s})}$ is the idempotent associated to $\mathcal{E}_\ell(\mathbf{G}^F, \mathrm{pr}(\underline{s}))$ and $e^{\underline{\mathbf{G}}^{F_0\tau}}_{\underline{s}}$ is the idempotent associated to $\mathcal{E}_\ell(\underline{\mathbf{G}}^{\tau F_0}, \underline{s})$. Thus it is clearly sufficient to show that $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{x})) = \mathcal{E}(\underline{\mathbf{G}}^{F_0\tau}, \underline{x})$ for any semisimple $\underline{x} \in (\underline{\mathbf{G}^*})^{\tau^* F_0^*}$. This was however proved in Proposition 5.11. □

## 5.7 Restriction of scalars and Jordan decomposition of characters

In the following section we use ideas from [Dig99, Corollary 3.5] and apply them to our set-up.

The following notation will be in force until the end of this chapter. We let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order and $\mathbf{L}^*$ be an $F_0^*$-stable Levi subgroup of $\mathbf{G}^*$ with $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$. Suppose that $\mathbf{L}$ is an $F_0$-stable Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}^*$. Let

$$\underline{s} := (s, F_0(s), \dots, F_0^{r-1}(s)) \in (\underline{\mathbf{G}}^*)^{F_0^* \tau^*}.$$

In addition, we let $\sigma : \mathbf{G} \to \mathbf{G}$ be a bijective morphism with $F_0 \circ \sigma = \sigma \circ F_0$ and $\sigma(\mathbf{L}) = \mathbf{L}$. We denote by

$$\underline{\sigma} = \sigma \times \cdots \times \sigma : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$$

the induced map on $\underline{\mathbf{G}}$ which commutes with the action of $\tau F_0$ and its restriction

$$\underline{\sigma} : \underline{\mathbf{G}}^{F_0\tau} \to \underline{\mathbf{G}}^{F_0\tau}.$$

Observe that if the isogeny $\sigma^*$ is dual to $\sigma$ then the isogeny $\underline{\sigma}^*\tau^*$ is dual to $\tau\underline{\sigma}$. We note that $\sigma \in \mathrm{Aut}(\mathbf{G}^F)$ corresponds to $\underline{\sigma} \in \mathrm{Aut}(\underline{\mathbf{G}}^{\tau F_0})$ under the isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$.

**Lemma 5.13.** *The automorphism $F_0 : \mathbf{G}^F \to \mathbf{G}^F$ corresponds under the identification of $\mathbf{G}^F$ with $\underline{\mathbf{G}}^{F_0\tau}$ via the projection map* pr *to the automorphism $\tau^{-1} : \underline{\mathbf{G}}^{F_0\tau} \to \underline{\mathbf{G}}^{F_0\tau}$.*

*Proof.* This follows from the fact that any element of $\underline{g} \in \underline{\mathbf{G}}^{\tau F_0}$ satisfies $\tau F_0(\underline{g}) = \underline{g}$ or in other words $\tau^{-1}(\underline{g}) = F_0(\underline{g})$. □

Let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Moreover, assume that $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ are extensions of $\sigma$ and $F_0$, which still commute with each other. As always, we let $\tilde{\mathbf{L}} := \mathrm{Z}(\tilde{\mathbf{G}})\mathbf{L}$ and $\tilde{\mathbf{P}} := \mathrm{Z}(\tilde{\mathbf{G}})\mathbf{P}$ such that we have a Levi decomposition $\tilde{\mathbf{P}} = \tilde{\mathbf{L}} \ltimes \mathbf{U}$ in $\tilde{\mathbf{G}}$.

We consider the unipotent radical $\underline{\mathbf{U}}' := \mathbf{U}^r$ of the parabolic subgroup $\underline{\mathbf{P}}' = \mathbf{P}^r$ of $\underline{\mathbf{G}}$. Note that we have a Levi decomposition $\underline{\mathbf{P}}' = \underline{\mathbf{L}} \ltimes \underline{\mathbf{U}}'$ in $\underline{\mathbf{G}}$ and the parabolic subgroup $\underline{\mathbf{P}}'$ is $\tau$-stable. The following is an application of Lemma 5.1:

**Lemma 5.14.** *Suppose that the idempotent $e_s^{\mathbf{L}^F}$ is $\langle F_0, \sigma\rangle$-stable. Then $H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is endowed with a natural $\Lambda[(\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\underline{\tilde{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle)]$-structure. Moreover, $H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is $(\underline{\sigma}, \underline{\sigma}^{-1})$-invariant as $\Lambda[(\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\underline{\tilde{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle)]$-module.*

*Proof.* The pair $(\underline{\mathbf{L}}, \underline{\mathbf{P}}')$ is $\tau$-stable and $\underline{\mathbf{L}}$ is $\underline{\sigma}$-stable. We have $\mathrm{pr}(\underline{s}) = s$. Since $e_s^{\mathbf{L}^F}$ is $\langle F_0, \sigma\rangle$-stable it therefore follows from Lemma 5.13 that $e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is $\langle \tau, \underline{\sigma}\rangle$-stable. Moreover, $\mathbf{L}^*$ is $F_0^*$-stable by assumption, so we obtain

$$\mathrm{C}^\circ_{\underline{\mathbf{G}}^*}(\underline{s}) = \mathrm{C}^\circ_{\mathbf{G}^*}(s) \times \cdots \times \mathrm{C}^\circ_{\mathbf{G}^*}(F_0^{r-1}(s)) \subseteq \underline{\mathbf{L}}^*.$$

We conclude that Lemma 5.1 applies which gives the claim of the lemma. □

Combining Lemma 5.14 and Lemma 5.13 yields the following important observation.

**Proposition 5.15.** *Suppose that $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$ are $\langle F_0, \sigma\rangle$-stable. Then the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, \Lambda)e_s^{\mathbf{L}^F}$ can be equipped with a $\Lambda[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\tilde{\mathbf{L}}^F\langle F_0\rangle)]$-module structure with which it is $(\sigma, \sigma^{-1})$-stable.*

*Proof.* By Theorem 2.36, we have an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}} \cong H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$$

of $\Lambda[(\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\underline{\tilde{\mathbf{L}}}^F)]$-modules.

It follows by Lemma 5.14 that the bimodule $H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ has a $\Lambda[(\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\underline{\tilde{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle)]$-structure with which it is $(\underline{\sigma}, \underline{\sigma}^{-1})$-stable. By Proposition 5.3 and Corollary 5.12 the bimodule $H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}},\tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is isomorphic to $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ as $\Lambda[(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\tilde{\mathbf{L}}^F)]$-modules. As noted above, the group isomorphism $\sigma \in \mathrm{Aut}(\mathbf{G}^F)$ corresponds to $\underline{\sigma} \in \mathrm{Aut}(\underline{\mathbf{G}}^{\tau F_0})$ under the isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$. Moreover, by Lemma 5.13 the automorphism $\tau \in \mathrm{Aut}(\underline{\mathbf{G}}^{\tau F_0})$ corresponds to $F_0^{-1} \in \mathrm{Aut}(\mathbf{G}^F)$. From this we can, by transport of structure, endow the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, \Lambda)e_s^{\mathbf{L}^F}$ with a $\Lambda[(\mathbf{G}^F \times \mathbf{L}^{F^{\mathrm{opp}}})\Delta(\tilde{\mathbf{L}}^F\langle F_0\rangle)]$-module structure with which it is $(\sigma, \sigma^{-1})$-stable. □

In the following, we denote $\mathcal{A} = \langle \sigma, F_0 \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ and $\mathcal{D} = (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\Delta(\tilde{\mathbf{L}}^F \mathcal{A})$. Furthermore, let $A \in \{K, \mathcal{O}, k\}$.

**Theorem 5.16.** *Suppose that* $\mathbf{L}$ *and* $e_s^{\mathbf{L}^F}$ *are* $\mathcal{A}$*-stable. Assume that* $\mathrm{C}_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ *and the order of* $\sigma : \tilde{\mathbf{G}}^F \to \tilde{\mathbf{G}}^F$ *is invertible in* $A$*. Then* $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, A)e_s^{\mathbf{L}^F}$ *extends to an* $A\mathcal{D}$*-module* $M$*. Moreover, the bimodule* $\mathrm{Ind}_{\mathcal{D}}^{\tilde{\mathbf{G}}^F \mathcal{A} \times (\tilde{\mathbf{L}}^F \mathcal{A})^{\mathrm{opp}}}(M)$ *induces a Morita equivalence between* $A\tilde{\mathbf{L}}^F \mathcal{A} e_s^{\mathbf{L}^F}$ *and* $A\tilde{\mathbf{G}}^F \mathcal{A} e_s^{\mathbf{G}^F}$*.*

*Proof.* The existence of the extension $M$ follows from Proposition 5.15 and Lemma 1.32. The bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, A)e_s^{\mathbf{L}^F}$ induces a Morita equivalence between $A\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $A\mathbf{L}^F e_s^{\mathbf{L}^F}$. Since $e_s^{\mathbf{L}^F}$ is $\mathcal{A}$-invariant we conclude that the assumptions of Theorem 1.24 are satisfied. From this it follows that $\mathrm{Ind}_{\mathcal{D}}^{\tilde{\mathbf{G}}^F \mathcal{A} \times (\tilde{\mathbf{L}}^F \mathcal{A})^{\mathrm{opp}}}(M)$ gives a Morita equivalence between $A\tilde{\mathbf{L}}^F \mathcal{A} e_s^{\mathbf{L}^F}$ and $A\tilde{\mathbf{G}}^F \mathcal{A} e_s^{\mathbf{G}^F}$. □

We remark the following consequence of Theorem 5.16 which will become important in Section 6.

**Corollary 5.17.** *In the situation of Theorem 5.16 we have the following commutative square:*

$$\begin{array}{ccc} G_0(A\tilde{\mathbf{L}}^F \langle F_0, \sigma \rangle e_s^{\mathbf{L}^F}) & \xrightarrow{[M \otimes -]} & G_0(A\tilde{\mathbf{G}}^F \langle F_0, \sigma \rangle e_s^{\mathbf{G}^F}) \\ \Big\downarrow \mathrm{Res} & & \Big\downarrow \mathrm{Res} \\ G_0(A\mathbf{L}^F e_s^{\mathbf{L}^F}) & \xrightarrow{(-1)^{\dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})} R_{\mathbf{L}}^{\mathbf{G}}} & G_0(A\mathbf{G}^F e_s^{\mathbf{G}^F}) \end{array}$$

*Proof.* This has been discussed in Remark 1.26(a). □

## 5.8 Reduction to isolated series

We keep the assumptions of the previous section. Furthermore, as in Section 2.12 we assume that $\mathbf{L}^* \, \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} = \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \mathbf{L}^*$ and define $\mathbf{N}^* := \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \mathbf{L}^*$. Recall that we denote by $\mathbf{N}$ the subgroup of $\mathrm{N}_{\mathbf{G}}(\mathbf{L})$ which corresponds to the subgroup $\mathbf{N}^*$ of $\mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)$ under the isomorphism $\mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \cong \mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ given by duality.

Suppose now that $F_0(e_s^{\mathbf{L}^F}) = e_s^{\mathbf{L}^F}$. Then the $(\mathbf{L}^*)^{F^*}$-conjugacy class of $s$ is $F_0$-stable. It follows that $\mathbf{N}^*$ is $F_0$-stable. Since $\mathbf{L}$ is in duality with $\mathbf{L}^*$ under the duality between $(\mathbf{G}, F_0)$ and $(\mathbf{G}^*, F_0^*)$ we can conclude that $\mathbf{N}$ is $F_0$-stable, see the remarks following Remark 2.16.

In this section we give a partial answer to the question whether the equivalence constructed by Bonnafé–Dat–Rouquier is automorphism-equivariant. The results of this section will not be used in the remainder of this thesis. We work with the following assumption:

**Assumption 5.18.** *Assume that the idempotent $e_s^{\mathbf{L}^F}$ is $F_0$-stable. Moreover, suppose that the quotient group $\mathbf{N}^F/\mathbf{L}^F$ is cyclic and that $\mathbf{N}^{F_0}/\mathbf{L}^{F_0} \cong \mathbf{N}^F/\mathbf{L}^F$.*

**Theorem 5.19.** *Suppose that Assumption 5.18 is satisfied. Then there exists a $\Lambda[\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta(\langle F_0\rangle)]$-module $M$ extending $H_c^{\dim}(\mathbf{Y}_\mathbf{U},\Lambda)e_s^{\mathbf{L}^F}$ which induces a Morita equivalence between $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$. Consequently, the bimodule $\tilde{M} := \mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\mathbf{N}^F\langle F_0\rangle}^{\mathbf{G}^F\langle F_0\rangle\times(\mathbf{N}^F\langle F_0\rangle)^{\mathrm{opp}}}(M)$ induces a Morita equivalence between $\Lambda\mathbf{N}^F\langle F_0\rangle e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F\langle F_0\rangle e_s^{\mathbf{G}^F}$.*

*Proof.* It follows from Assumption 5.18 that there exists some $n \in \mathbf{N}^F$ generating the quotient group $\mathbf{N}^F/\mathbf{L}^F$ and such that $F_0(n)n^{-1} \in \mathbf{L}$. By Lang's theorem there exists some $l \in \mathbf{L}^F$ such that $ln$ is $F_0$-invariant. We may thus assume that $n$ is $F_0$-fixed.

By applying Proposition 5.15 to the automorphism $\sigma : \mathbf{G} \to \mathbf{G}$, $x \mapsto {}^n x$ it follows that the bimodule $H_c^{\dim}(\mathbf{Y}_\mathbf{U},\Lambda)e_s^{\mathbf{L}^F}$ can be equipped with a $\Lambda[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\langle F_0\rangle)]$-module structure with which it is $\mathbf{N}^F$-invariant. By Assumption 5.18 the quotient group $\mathbf{N}^F/\mathbf{L}^F$ is cyclic and $\mathbf{N}^F$ normalizes $\mathbf{L}^F\langle F_0\rangle$. Thus, we can apply Lemma 1.32 and it follows that $H_c^{\dim}(\mathbf{Y}_\mathbf{U},\Lambda)e_s^{\mathbf{L}^F}$ extends to a $\Lambda[\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta(\langle F_0\rangle)]$-module $M$.

Using Theorem 2.35 we conclude that $M$ induces a Morita equivalence between $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$. Consequently, by Theorem 1.24 the bimodule $\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta(\langle F_0\rangle)}^{\mathbf{G}^F\langle F_0\rangle\times(\mathbf{N}^F\langle F_0\rangle)^{\mathrm{opp}}}(M)$ induces a Morita equivalence between $\Lambda\mathbf{N}^F\langle F_0\rangle e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F\langle F_0\rangle e_s^{\mathbf{G}^F}$. □

Using the full strength of the proof of [BDR17a, Theorem 7.6] we can prove an even stronger statement:

**Theorem 5.20.** *Suppose that we are in the situation of Theorem 5.19. Then there exists a bounded complex $\tilde{C}$ of $\Lambda[\mathbf{G}^F\langle F_0\rangle \times (\mathbf{N}^F\langle F_0\rangle)^{\mathrm{opp}}]$-modules with cohomology concentrated in one degree and isomorphic to $\tilde{M}$ such that $\tilde{C}$ induces a splendid Rickard equivalence between $\Lambda\mathbf{N}^F\langle F_0\rangle e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F\langle F_0\rangle e_s^{\mathbf{G}^F}$.*

*Proof.* We have a Levi decomposition $\underline{\mathbf{P}} = \underline{\mathbf{L}} \ltimes \underline{\mathbf{U}}'$ in $\underline{\mathbf{G}}$. Denote $\hat{\mathbf{P}} := \underline{\mathbf{P}}\langle\tau\rangle$, $\hat{\mathbf{L}} := \underline{\mathbf{L}}\langle\tau\rangle$ and $\hat{\mathbf{U}} := \underline{\mathbf{U}}'$. Then as in Section 5.1 we have a Levi decomposition $\hat{\mathbf{P}} = \hat{\mathbf{L}} \ltimes \hat{\mathbf{U}}$ in the reductive group $\hat{\mathbf{G}} := \underline{\mathbf{G}}\langle\tau\rangle$. We define $\hat{\mathbf{N}} := \underline{\mathbf{N}}\langle\tau\rangle$, which is a closed subgroup of $\hat{\mathbf{G}}$ since $\mathbf{N}$ is $F_0$-stable. We first prove the following fundamental observation:

**Lemma 5.21.** *We have* $\hat{\mathbf{N}}^{\tau F_0} = \mathrm{N}_{\hat{\mathbf{G}}^{\tau F_0}}(\hat{\mathbf{L}}, e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}})$.

*Proof.* By Assumption 5.18 we have $nF_0(n^{-1}) \in \mathbf{L}$ for every $n \in \mathbf{N}^F$. From this and Lemma 5.13 it follows that $\hat{\mathbf{N}}^{\tau F_0}$ stabilizes the Levi subgroup $\hat{\mathbf{L}} = \underline{\mathbf{L}}\langle\tau\rangle$. Lemma 5.13 implies that the map

$$\psi : \hat{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F\langle F_0\rangle,\ \underline{x}\tau \mapsto \mathrm{pr}(\underline{x})F_0^{-1},$$

is an isomorphism of groups. Moreover, the image of $\hat{\mathbf{N}}^{\tau F_0}$ under $\psi$ is $\mathbf{N}^F\langle F_0\rangle$. We have $\psi(e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}) = e_s^{\mathbf{L}^F}$ and $e_s^{\mathbf{L}^F}$ is $\mathbf{N}^F\langle F_0\rangle$-fixed. It follows that $\hat{\mathbf{N}}^{\tau F_0} \subseteq \mathrm{N}_{\hat{\mathbf{G}}^{\tau F_0}}(\hat{\mathbf{L}}, e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}})$. On the other hand, let $z \in \mathrm{N}_{\hat{\mathbf{G}}^{\tau F_0}}(\hat{\mathbf{L}}, e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}})$. Then we have ${}^{\psi(z)}e_s^{\mathbf{L}^F} = e_s^{\mathbf{L}^F}$. Furthermore, $z$ normalizes $\underline{\mathbf{L}}$ and it follows that $\psi(z)$ normalizes $\mathbf{L}$. From this we deduce that $\psi(z) \in \mathbf{N}^F\langle F_0\rangle$ and therefore $z \in \hat{\mathbf{N}}^{\tau F_0}$. It follows that $\hat{\mathbf{N}}^{\tau F_0} = \mathrm{N}_{\hat{\mathbf{G}}^{\tau F_0}}(\hat{\mathbf{L}}, e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}})$. □

We can now continue the proof of our theorem. Due to Theorem 1.21 we may assume that $\Lambda = k$. By the construction in the proof of Theorem 5.19 the $k[\hat{\mathbf{G}}^{\tau F_0} \times (\hat{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}]$-module $H_c^{\dim}(\mathbf{Y}_{\hat{\mathbf{U}}}^{\hat{\mathbf{G}},\tau F_0}, k)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ extends to a $k[\hat{\mathbf{G}}^{\tau F_0} \times (\hat{\mathbf{N}}^{\tau F_0})^{\mathrm{opp}}]$-module $\tilde{M}$ which induces a Morita equivalence between $k\hat{\mathbf{N}}^{\tau F_0}e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ and $k\hat{\mathbf{G}}^{\tau F_0}e_{\underline{s}}^{\underline{\mathbf{G}}^{\tau F_0}}$.

Consider the complex $C := G\Gamma_c(\mathbf{Y}_{\hat{\mathbf{U}}}^{\hat{\mathbf{G}},\tau F_0}, k)^{\mathrm{red}}e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ of $\hat{\mathbf{G}}^{\tau F_0} \times (\hat{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}$-modules. Note that $e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is a $(\underline{\mathbf{G}}, \underline{\mathbf{L}})$-regular rational series of $(\underline{\mathbf{L}}, \tau F_0)$, see proof of Lemma 5.14. Furthermore, $e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is $\tau$-stable and is therefore by Lemma 2.26 a $(\hat{\mathbf{G}}, \hat{\mathbf{L}})$-regular rational series of $(\hat{\mathbf{L}}, \tau F_0)$.

Using Steps 1-3 in the proof of [BDR17a, Theorem 7.6], which directly apply to our set-up, we can conclude that there exists a direct summand $\tilde{C}$ of the complex $\mathrm{Ind}_{\hat{\mathbf{G}}^{\tau F_0}\times(\hat{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}}^{\hat{\mathbf{G}}^{\tau F_0}\times(\hat{\mathbf{N}}^{\tau F_0})^{\mathrm{opp}}}(C)$ which is quasi-isomorphic to the bimodule $\tilde{M}$.

Since $\tilde{M}$ is a Morita bimodule it is in particular a direct sum of indecomposable pairwise non-isomorphic bimodules. By Assumption 5.18, the quotient group $(\hat{\mathbf{N}}/\hat{\mathbf{L}})^{\tau F_0} \cong \mathbf{N}^F/\mathbf{L}^F$ is cyclic of $\ell'$ order. It follows therefore by Lemma 1.32 that the $k[\hat{\mathbf{G}}^{\tau F_0} \times (\hat{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}]$-module $H_c^{\dim}(\mathbf{Y}_{\hat{\mathbf{U}}}^{\hat{\mathbf{G}},\tau F_0}, k)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is a direct sum of indecomposable pairwise non-isomorphic modules. From this we conclude that Step 4 of the proof of [BDR17a, Theorem 7.6] applies and we obtain

$$\mathrm{Res}_{\hat{\mathbf{G}}^{\tau F_0}\times(\hat{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}}^{\hat{\mathbf{G}}^{\tau F_0}\times(\hat{\mathbf{N}}^{\tau F_0})^{\mathrm{opp}}}(\tilde{C}) \cong C.$$

By Step 5 of the proof of [BDR17a, Theorem 7.6] we conclude that $\tilde{C}$ induces a splendid Rickard equivalence between $k\hat{\mathbf{N}}^{\tau F_0}e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ and $k\hat{\mathbf{G}}^{\tau F_0}e_{\underline{s}}^{\underline{\mathbf{G}}^{\tau F_0}}$. Hence, we obtain a splendid Rickard equivalence between $k\mathbf{N}^F\langle F_0\rangle e_s^{\mathbf{L}^F}$ and $k\mathbf{G}^F\langle F_0\rangle e_s^{\mathbf{G}^F}$. □

## 5.9 Jordan decomposition for local subgroups

We keep the assumptions of Section 5.7. The aim of this section is to obtain a local version of Theorem 5.16. We will essentially use the same strategy of Section 5.7 to prove this local version. However, we need to adapt some of the arguments.

Recall that the projection map $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$ onto the first coordinate induces an isomorphism of groups, which extends to an isomorphism $\mathrm{pr} : \Lambda\underline{\mathbf{G}}^{\tau F_0} \to \Lambda\mathbf{G}^F$ of $\Lambda$-algebras. Hence, under the isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$ the notions of blocks, Brauer subpairs and defect groups translate.

From now on we will use the following notation: If $H$ is a subgroup of $\mathbf{G}^F$ we let $\underline{H} := \mathrm{pr}^{-1}(H)$ and if $x \in \Lambda H$ then we let $\underline{x} := \mathrm{pr}^{-1}(x) \in \Lambda\underline{H}$.

Let $b \in \mathrm{Z}(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$ and $c \in \mathrm{Z}(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F})$ be blocks which correspond to each other under the splendid Rickard equivalence given by $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$. By Proposition 5.3 and Corollary 5.12 the projection map pr yields an isomorphism between $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ and $G\Gamma_c(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$.

Hence, the blocks $\underline{b} \in \mathrm{Z}(\Lambda\underline{\mathbf{G}}^{\tau F_0} e_{\underline{s}}^{\underline{\mathbf{G}}^{\tau F_0}})$ and $\underline{c} \in \mathrm{Z}(\Lambda\underline{\mathbf{L}}^{\tau F_0} e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}})$ correspond to each other under the splendid Rickard equivalence induced by $G\Gamma_c(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$. We fix a maximal $c$-Brauer pair $(D, c_D)$ and let $(D, b_D)$ be the $b$-Brauer pair corresponding to it under the splendid Rickard equivalence induced by $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)c$ in the sense of Proposition 1.16. Consequently, the $\underline{c}$-subpair $(\underline{D}, \underline{c}_D)$ corresponds to the $\underline{b}$-subpair $(\underline{D}, \underline{b}_D)$ under the Rickard equivalence induced by $G\Gamma_c(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$.

Furthermore, we let $Q$ be a subgroup of $D$ and let $(Q, c_Q) \leq (D, c_D)$ and $(Q, b_Q) \leq (D, b_D)$. We denote $B_Q = \mathrm{Tr}_{\mathrm{N}_{\mathbf{G}^F}(Q, b_Q)}^{\mathrm{N}_{\mathbf{G}^F}(Q)}(b_Q)$ and $C_Q = \mathrm{Tr}_{\mathrm{N}_{\mathbf{L}^F}(Q, c_Q)}^{\mathrm{N}_{\mathbf{L}^F}(Q)}(c_Q)$.

**Proposition 5.22.** *The bimodule* $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q$ *can be equipped with a* $\Lambda[\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}} \Delta \mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q, C_Q)]$*-module structure.*

*Proof.* By Theorem 5.2 (set $\hat{\mathbf{G}} := \underline{\mathbf{G}}$), we have

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}), \tau F_0})\underline{C}_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}), \tau F_0})\underline{C}_Q$$

as $\Lambda[(\mathrm{N}_{\underline{\mathbf{G}}^{\tau F_0}}(\underline{Q}) \times \mathrm{N}_{\underline{\mathbf{L}}^{\tau F_0}}(\underline{Q})^{\mathrm{opp}})\Delta(\mathrm{N}_{\underline{\tilde{\mathbf{L}}}^{\tau F_0}}(\underline{Q}, \underline{C}_Q))]$-modules. Moreover, Corollary 5.4 shows that $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}), \tau F_0})\underline{C}_Q$ is isomorphic to $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)C_Q$ as $\Lambda[(\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q, c_Q))]$-modules.

Since $\tau(\underline{\mathbf{U}}') = \underline{\mathbf{U}}'$ we obtain a Levi decomposition $\underline{\tilde{\mathbf{P}}}\langle\tau\rangle = \underline{\tilde{\mathbf{L}}}\langle\tau\rangle \ltimes \underline{\mathbf{U}}'$ in the reductive group $\underline{\tilde{\mathbf{G}}} \rtimes \langle\tau\rangle$. Hence we obtain a Levi decomposition $\mathrm{N}_{\underline{\tilde{\mathbf{P}}}\langle\tau\rangle}(\underline{Q}) = \mathrm{N}_{\underline{\tilde{\mathbf{L}}}\langle\tau\rangle}(\underline{Q}) \ltimes \mathrm{C}_{\underline{\mathbf{U}}'}(\underline{Q})$ in the reductive group $\mathrm{N}_{\underline{\tilde{\mathbf{G}}}\langle\tau\rangle}(\underline{Q})$, see Example 2.3. From this we conclude (see Lemma 2.4) that the bimodule

$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}}'}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0})\underline{C}_Q$ has a natural $\Delta(\mathrm{N}_{\underline{\tilde{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle}(\underline{Q},\underline{C}_Q))$-action. By Lemma 5.13 it follows that the Morita bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda)C_Q$ can be equipped with a $\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q,C_Q))$-action. □

From now on we will assume that $Q$ is a characteristic subgroup of the defect group $D$. Recall that $\mathcal{A} = \langle F_0, \sigma\rangle$.

**Lemma 5.23.** *Let $Q$ be a characteristic subgroup of $D$. Then we have $\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q)\mathrm{N}_{\mathbf{G}^F}(Q) = \mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q,B_Q)$.*

*Proof.* Let $\hat{L}$ denote the stabilizer of $c$ in $\tilde{\mathbf{L}}^F\mathcal{A}$ and $\hat{G}$ the stabilizer of $b$ in $\tilde{\mathbf{G}}^F\mathcal{A}$. We abbreviate $L := \mathbf{L}^F$ and $G := \mathbf{G}^F$. By Lemma 1.45, we have

$$\mathrm{N}_{\tilde{\mathbf{L}}^F\rtimes\mathcal{A}}(Q,C_Q) = \mathrm{N}_{\hat{L}}(Q,C_Q) = \mathrm{N}_{\hat{L}}(Q)$$

and $\mathrm{N}_{\hat{L}}(Q)/\mathrm{N}_L(Q) = \hat{L}/L$. Similarly, we have $\mathrm{N}_{\tilde{\mathbf{G}}^F\rtimes\mathcal{A}}(Q,B_Q) = \mathrm{N}_{\hat{G}}(Q)$ and $\mathrm{N}_{\hat{G}}(Q)/\mathrm{N}_G(Q) = \hat{G}/G$. On the other hand, we have $\hat{L}/L \cong \hat{G}/G$ by Lemma 4.16. This yields $\mathrm{N}_{\hat{L}}(Q)/\mathrm{N}_L(Q) \cong \mathrm{N}_{\hat{G}}(Q)/\mathrm{N}_G(Q)$ and the claim of the lemma follows easily from this. □

Let us denote $B_Q' = \mathrm{Tr}_{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q,B_Q)}^{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q)}(B_Q)$ and $C_Q' = \mathrm{Tr}_{\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q)}^{\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)}(C_Q)$. Recall that $A \in \{K, \mathcal{O}, k\}$.

**Theorem 5.24.** *Suppose that the assumptions of Theorem 5.16 are satisfied. Let $Q$ be a characteristic subgroup of $D$. Then $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},A)C_Q$ extends to an $A[(\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q))]$-module $M_Q$. In particular, the bimodule*

$$\mathrm{Ind}_{(\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q))}^{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q)\times\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)^{\mathrm{opp}}}(M_Q)$$

*induces a Morita equivalence between $A\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q)B_Q'$ and $A\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)C_Q'$.*

*Proof.* In Proposition 5.22 we have proved that $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}}'}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0})\underline{C}_Q$ is isomorphic to $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},A)C_Q$. This allowed us to endow the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},A)C_Q$ with an $A[(\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q,C_Q))]$-module structure.

Since $Q$ is a characteristic subgroup of the defect group $D$ of $c$, it follows that the quotient group $\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q)/\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q,C_Q)$ is cyclic and of order divisible by the order of $\sigma \in \mathrm{Aut}(\tilde{\mathbf{G}}^F)$. Hence, there exist $x \in \tilde{\mathbf{L}}^F$ and a bijective morphism $\phi_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ such that $x\phi_0|_{\tilde{\mathbf{G}}^F}$ generates the quotient group $\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q)/\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q,C_Q)$. Let $\underline{x} := (x, F_0^{r-1}(x),\dots,F_0(x)) \in \underline{\mathbf{G}}^{\tau F_0}$ such that $\mathrm{pr}(\underline{x}) = x$. Denote

$$\underline{\phi_0} : \underline{\tilde{\mathbf{G}}}\langle\tau\rangle \to \underline{\tilde{\mathbf{G}}}\langle\tau\rangle,\ (g_1,\dots,g_r)\tau \mapsto (\phi_0(g_1),\dots,\phi_0(g_r))\tau$$

and consider the bijective morphism

$$\underline{\phi} := \underline{x}\,\underline{\phi_0} : \tilde{\underline{\mathbf{G}}}\langle\tau\rangle \to \tilde{\underline{\mathbf{G}}}\langle\tau\rangle, z \mapsto {}^{\underline{x}}\underline{\phi_0}(z),$$

of the reductive group $\tilde{\underline{\mathbf{G}}}\langle\tau\rangle$. Note that $\underline{\phi}$ stabilizes $\tilde{\underline{\mathbf{G}}}$ and commutes with the Frobenius endomorphism $\tau F_0$ of $\tilde{\underline{\mathbf{G}}} \rtimes \langle\tau\rangle$. Moreover, ${}^{\underline{x}}(\tilde{\underline{\mathbf{L}}}\langle\tau\rangle) = \tilde{\underline{\mathbf{L}}}\langle\tau\rangle$ and $\phi_0(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}}$. Therefore, the bijective morphism $\underline{\phi}$ also stabilizes the Levi subgroup $\tilde{\underline{\mathbf{L}}}\langle\tau\rangle$ of $\tilde{\underline{\mathbf{G}}} \rtimes \langle\tau\rangle$. Since $\underline{\phi}|_{\tilde{\underline{\mathbf{G}}}^{\tau F_0}} \in \mathrm{Aut}(\tilde{\underline{\mathbf{G}}}^{\tau F_0})$ corresponds to the automorphism $x\phi_0 \in \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ under the isomorphism $\mathrm{pr} : \tilde{\underline{\mathbf{G}}}^{\tau F_0} \to \tilde{\mathbf{G}}^F$ we deduce that $\underline{\phi}(\underline{Q}, \underline{C}_Q) = (\underline{Q}, \underline{C}_Q)$. Hence, Lemma 4.5 applies and we obtain an isomorphism

$${}^{\underline{\phi}}(H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}, \Lambda)\underline{C}_Q)^{\underline{\phi}} \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\phi}(\underline{\mathbf{U}'})}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}, \Lambda)\underline{C}_Q$$

of $\Lambda[(\mathrm{N}_{\underline{\mathbf{G}}^{\tau F_0}}(\underline{Q}) \times \mathrm{N}_{\underline{\mathbf{L}}^{\tau F_0}}(\underline{Q})^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\underline{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle}(\underline{Q}, \underline{C}_Q))]$-modules. We have two Levi decompositions

$$\tilde{\underline{\mathbf{P}}}\langle\tau\rangle = \tilde{\underline{\mathbf{L}}}\langle\tau\rangle \ltimes \underline{\mathbf{U}} \text{ and } \underline{\phi}(\tilde{\underline{\mathbf{P}}}\langle\tau\rangle) = \tilde{\underline{\mathbf{L}}}\langle\tau\rangle \ltimes \underline{\phi}(\underline{\mathbf{U}})$$

with the same Levi subgroup $\tilde{\underline{\mathbf{L}}}\langle\tau\rangle$ of $\tilde{\mathbf{G}}\langle\tau\rangle$. Therefore, Theorem 5.2 yields

$$H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\phi}(\underline{\mathbf{U}'})}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}, \Lambda)\underline{C}_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}, \Lambda)\underline{C}_Q.$$

It follows from this that $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}, \Lambda)\underline{C}_Q$ is $(\underline{\phi}, \underline{\phi}^{-1})$-invariant. Hence, the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, A)C_Q$ is by transport of structure $(x\phi_0, x\phi_0^{-1})$-invariant as $A[(\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q, C_Q))]$-module. Lemma 1.32 therefore shows that there exists an $A[(\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q, C_Q))]$-module $M_Q$ extending $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, A)C_Q$. By Theorem 4.28 the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, A)C_Q$ induces a Morita equivalence between the blocks $A\mathrm{N}_{\mathbf{G}^F}(Q)B_Q$ and $A\mathrm{N}_{\mathbf{L}^F}(Q)C_Q$. Moreover, Lemma 5.23 shows that

$$\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q, C_Q)\mathrm{N}_{\mathbf{G}^F}(Q) = \mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q, B_Q).$$

Hence, Lemma 1.33 implies that the bimodule

$$\mathrm{Ind}_{(\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q,C_Q))}^{\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q)\times\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)^{\mathrm{opp}}}(M_Q)$$

induces a Morita equivalence between $A\mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(Q)B'_Q$ and $A\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)C'_Q$. $\square$

**Remark 5.25.** If one could prove a version of Theorem 5.16 with Morita equivalence replaced by splendid Rickard equivalence then Theorem 5.24 would be obtained as a consequence of that theorem, see Theorem 1.36. However this seems to be difficult since we would have to show that the Rickard–Rouquier complex $G\Gamma_c(\mathbf{Y_U}, \Lambda)e_s^{\mathbf{L}^F}$ is independent of the choice of the unipotent radical $\mathbf{U}$ used in its definition. In the case where the Sylow $\ell$-subgroups of $\mathbf{G}^F$ are cyclic we obtained such an independence result in Example 4.23.

# Chapter 6

# Application to the inductive Alperin–McKay condition

In this chapter we show how the results from the previous chapters can be used in the verification of the Alperin–McKay conjecture. More precisely, we show that in order to prove the inductive Alperin-McKay condition for all blocks of groups of Lie type it is sufficient to consider their quasi-isolated blocks.

## 6.1 The inductive Alperin–McKay condition

The aim of this section is to recall the inductive Alperin–McKay condition as introduced in [Spä13, Definition 7.2].

Recall that a *character triple* $(G, N, \theta)$ consists of a finite group $G$ with normal subgroup $N$ and a $G$-invariant character $\theta \in \mathrm{Irr}(N)$. A *projective representation* is a set-theoretic map $\mathcal{P} : G \to \mathrm{GL}_n(K)$ such that for all $g, g' \in G$ there exists a scalar $\alpha(g, g') \in K$ with $\mathcal{P}(gg') = \alpha(g, g')\mathcal{P}(g)\mathcal{P}(g')$. The projective representation $\mathcal{P} : G \to \mathrm{GL}_n(K)$ is said to be *associated to* $(G, N, \theta)$ if the restriction $\mathcal{P}|_N$ affords the character $\theta$ and for all $n \in N$ and $g \in G$ we have $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ and $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$.

We recall the following order relation on character triples, see [Spä18, Definition 2.1]:

**Definition 6.1.** Let $(G, N, \theta)$ and $(H, M, \theta')$ be two character triples. We write

$$(G, N, \theta) \geq (H, M, \theta')$$

if the following conditions are satisfied:

(i) $G = NH$, $M = N \cap H$ and $\mathrm{C}_G(N) \leq H$.

(ii) There exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated with $(G, N, \theta)$ and $(H, M, \theta')$ such that their factor sets $\alpha$ and $\alpha'$ satisfy $\alpha|_{H\times H} = \alpha'$.

Let $\mathcal{P} : G \to \mathrm{GL}_n(K)$ be a projective representation associated to the character triple $(G, N, \theta)$. Then for $c \in \mathrm{C}_G(N)$ we have $\mathcal{P}(c)\mathcal{P}(n) = \mathcal{P}(n)\mathcal{P}(c)$ for every $n \in N$. Since $\mathcal{P}|_N$ affords the irreducible character $\theta$ it follows by Schur's lemma that $\mathcal{P}(c)$ is a scalar matrix.

For the inductive Alperin–McKay we need a refinement of the order relation "$\geq$" on character triples. More precisely, we also require that the scalars of the projective representations on elements of $\mathrm{C}_G(N)$ coincide:

**Definition 6.2.** Let $(G, N, \theta)$ and $(H, M, \theta')$ be two character triples such that $(G, N, \theta) \geq (H, M, \theta')$ via the projective representations $\mathcal{P}$ and $\mathcal{P}'$. Then we write

$$(G, N, \theta) \geq_c (H, M, \theta')$$

if for every $c \in \mathrm{C}_G(N)$ the scalars associated to $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ coincide.

Let $N$ be a normal subgroup of a finite group $G$ and $\chi \in \mathrm{Irr}(G)$. Then we write $\mathrm{Irr}(N \mid \chi)$ for the set of irreducible constituents of $\mathrm{Res}^G_N(\chi)$. Moreover for $\theta \in \mathrm{Irr}(N)$ we write $\mathrm{Irr}(G \mid \theta)$ for the set of all irreducible constituents $\chi$ of the induced character $\mathrm{Ind}^G_N(\theta)$. We then say that the character $\chi$ *covers* the character $\theta$.

**Theorem 6.3.** *Let $(G, N, \theta)$ and $(H, M, \theta')$ be two character triples such that $(G, N, \theta) \geq (H, M, \theta')$ with respect to the projective representations $\mathcal{P}$ and $\mathcal{P}'$. Then for every intermediate subgroup $N \leq J \leq G$ there exists a bijection*

$$\sigma_J : \mathbb{N}\,\mathrm{Irr}(J \mid \theta) \to \mathbb{N}\,\mathrm{Irr}(J \cap H \mid \theta')$$

*such that $\sigma_J(\mathrm{Irr}(J \mid \theta)) = \mathrm{Irr}(J \cap H \mid \theta')$.*

*Proof.* This is [Spä18, Theorem 2.2]. □

The following properties which are collected in the next lemma are a consequence of the fact that "$\geq$" induces a strong isomorphism of character triples in the sense of [Isa06, Problem 11.13].

**Lemma 6.4.** *Let $(G, N, \theta)$ and $(H, M, \theta')$ be two character triples satisfying $(G, N, \theta) \geq (H, M, \theta')$. Then for any $N \leq J_1 \leq J_2 \leq G$ and $\chi \in \mathbb{N}\,\mathrm{Irr}(J_2 \mid \theta)$ the following holds:*

*(a)* $\mathrm{Res}^{J_2\cap H}_{J_1\cap H}(\sigma_{J_2}(\chi)) = \sigma_{J_1}(\mathrm{Res}^{J_2}_{J_1}(\chi))$.

*(b)* $(\sigma_{J_2}(\chi\beta)) = \sigma_{J_2}(\chi)\mathrm{Res}^{J_2}_{J_2\cap H}(\beta)$ *for every $\beta \in \mathbb{N}\,\mathrm{Irr}(J_2/N)$.*

*(c)* $(\sigma_{J_2}(\chi))^h = \sigma_{J_2}(\chi^h)$ *for every* $h \in H$.

*Proof.* This is [Spä18, Corollary 2.4]. □

**Lemma 6.5.** *Let* $(G, N, \theta)$ *and* $(H, M, \theta')$ *be two character triples with* $(G, N, \theta) \geq_c (H, M, \theta')$. *Then for every* $N \leq J \leq G$ *and* $\kappa \in \mathrm{Irr}(\mathrm{C}_J(G))$ *we have*

$$\sigma_J(\mathrm{Irr}(J \mid \theta) \cap \mathrm{Irr}(J \mid \kappa)) \subseteq \mathrm{Irr}(J \cap H \mid \kappa).$$

*Proof.* See [Spä18, Lemma 2.10]. □

Let $G$ be a finite group and $\chi \in \mathrm{Irr}(G)$. Then we write $\mathrm{bl}(\chi)$ for the $\ell$-block of $G$ containing $\chi$.

The following definition is in [Spä18, Definition 4.2].

**Definition 6.6.** Let $(G, N, \theta)$ and $(H, M, \theta')$ be two character triples with $(G, N, \theta) \geq_c (H, M, \theta')$. Then we write

$$(G, N, \theta) \geq_b (H, M, \theta')$$

if the following hold:

(i) A defect group $D$ of $\mathrm{bl}(\theta')$ satisfies $\mathrm{C}_G(D) \leq H$.

(ii) The maps $\sigma_J$ induced by $(\mathcal{P}, \mathcal{P}')$ satisfy

$$\mathrm{bl}(\psi) = \mathrm{bl}(\sigma_J(\psi))^J$$

for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$.

**Remark 6.7.** Let $G$ be a finite group and $b$ a block of $G$ with defect group $D$. Let $M$ be a subgroup of $G$ containing $\mathrm{N}_G(D)$. By Brauer correspondence there exists a unique block $B_D$ of $\mathrm{N}_G(D)$ such that $(B_D)^G = b$. On the other hand, since $\mathrm{N}_M(D) = \mathrm{N}_G(D)$ Brauer correspondence yields a bijection $\mathrm{Bl}(\mathrm{N}_G(D) \mid D) \to \mathrm{Bl}(M \mid D)$. It follows that $B := (B_D)^M$ is the unique block of $M$ with defect group $D$ satisfying $B^G = b$, see [Nav98, Problem 4.2].

If $G$ is a finite group with normal subgroup $N$ and $\chi \in \mathrm{Irr}(N)$ an irreducible character, then we write $G_\chi$ for the inertia group of the character $\chi$ in $G$. For $b$ a block of $G$ with defect group $D$ we denote by

$$\mathrm{Irr}_0(G, b) := \{\chi \in \mathrm{Irr}(G, b) \mid \chi(1)_\ell = [G : D]_\ell\}$$

the set of *$\ell$-height zero characters* of the block $b$.

**Definition 6.8.** Let $G$ be a finite group and $b$ a block of $G$ with non-central defect group $D$. Assume that for $\Gamma := \mathrm{N}_{\mathrm{Aut}(G)}(D, b)$ there exist

(i) a $\Gamma$-stable subgroup $M$ with $\mathrm{N}_G(D) \leq M \lneq G$;

(ii) a $\Gamma$-equivariant bijection $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(M, B)$ where $B \in \mathrm{Bl}(M \mid D)$ is the unique block with $B^G = b$;

(iii) $\Psi(\mathrm{Irr}_0(b \mid \nu)) \subseteq \mathrm{Irr}_0(B \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ and

$$(G \rtimes \Gamma_\chi, G, \chi) \geq_b (M \rtimes \Gamma_\chi, M, \Psi(\chi)),$$

for every $\chi \in \mathrm{Irr}_0(G, b)$.

Then we say that $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(M, B)$ is an *iAM-bijection for the block $b$* with respect to the subgroup $M$.

We will usually work with iAM-bijections in the following. However, to formulate the inductive iAM-condition we need a slightly stronger version of Definition 6.8:

**Definition 6.9.** We say that $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(M, B)$ is a *strong iAM-bijection for the block $b$* if it is an iAM-bijection which additionally satisfies

$$(G/Z \rtimes \Gamma_\chi, G/Z, \overline{\chi}) \geq_b (M/Z \rtimes \Gamma_\chi, M/Z, \overline{\Psi(\chi)}),$$

for every $\chi \in \mathrm{Irr}_0(G, b)$ and $Z = \mathrm{Ker}(\chi) \cap \mathrm{Z}(G)$, where $\overline{\chi}$ and $\overline{\Psi(\chi)}$ lift to $\chi$ and $\Psi(\chi)$, respectively.

**Remark 6.10.** Note that if $(G/Z \rtimes \Gamma_\chi, G/Z, \overline{\chi}) \geq_b (M/Z \rtimes \Gamma_\chi, M/Z, \overline{\Psi(\chi)})$ then we automatically have $(G \rtimes \Gamma_\chi, G, \chi) \geq_b (M \rtimes \Gamma_\chi, M, \Psi(\chi))$ by [NS14, Lemma 3.12]. However, the converse is not known to hold.

Condition (iii) in Defintion 6.8 is made accessible through the following theorem:

**Theorem 6.11** (Butterfly theorem)**.** *Let $G_2$ be a finite group with normal subgroup $N$. Let $(G_1, N, \theta)$ and $(H_1, M, \theta')$ be two character triples with $(G_1, N, \theta) \geq_b (H_1, M, \theta')$. Assume that via the canonical morphism $\varepsilon_i : G_i \to \mathrm{Aut}(N)$, $i = 1, 2$, we have $\varepsilon_1(G_1) = \varepsilon_2(G_2)$. Then for $H_2 := \varepsilon_2^{-1}\varepsilon_1(H_1)$ we have*

$$(G_2, N, \theta) \geq_b (H_2, M, \varphi).$$

*Proof.* See [Spä18, Theorem 2.16] and [Spä18, Theorem 4.6]. □

The inductive Alperin–McKay condition was introduced by Späth in [Spä13, Definition 7.12]. We will in the following use its reformulation in the language of character triples, see [Spä18, Definition 4.12].

**Definition 6.12.** Let $S$ be a non-abelian simple group with universal covering group $G$ and $b$ a block of $G$ with non-central defect group $D$. If there exists a strong iAM-bijection $\Psi : \mathrm{Irr}_0(G,b) \to \mathrm{Irr}_0(M,B)$ for the block $b$ with respect to a subgroup $M$ then we say that the *block $b$ is AM-good for $\ell$.*

## 6.2 A criterion for block isomorphic character triples

In this section we establish a slightly more general version of [CS15, Lemma 3.2]. This will be required to obtain Lemma 6.15 below in the case where $\mathbf{G}^F$ is of type $D_4$. Thus, the reader who is not interested in the specifics of this case may skip this section entirely.

Let $A$ be a group acting on a finite group $H$. Then we denote by $\mathrm{Lin}_A(H)$ the subset of $A$-invariant linear characters of $\mathrm{Irr}(H)$. Moreover, if $N$ is a normal subgroup of $A$ and $b$ is a block of $N$, then we denote by $A[b]$ *the ramification of the block $b$* in $A$, which was introduced by Dade, see [CS15, Definition 3.1].

**Lemma 6.13.** *Let $A$ be a finite group. Suppose that $N \trianglelefteq A$ and $N \leq J \trianglelefteq A$ such that $J/N$ is abelian and $A/N$ is solvable. Assume that $\mathrm{Lin}_A(H) = \{1_H\}$ for every subgroup $H$ of the quotient group $([A,A]J)/J$. Let $b$ be a block of $N$ and $\chi, \phi \in \mathrm{Irr}(N,b)$. Let $\tilde{\chi} \in \mathrm{Irr}(A)$ and $\tilde{\phi} \in \mathrm{Irr}(A[b])$ be extensions of $\chi$ and $\phi$ respectively with $\mathrm{bl}(\mathrm{Res}^A_{J_1}(\tilde{\chi})) = \mathrm{bl}(\mathrm{Res}^{A[b]}_{J_1}(\tilde{\phi}))$ for every $J_1$ with $N \leq J_1 \leq J[b]$. Then there exists an extension $\tilde{\chi}_1 \in \mathrm{Irr}(A)$ of $\mathrm{Res}^A_J(\tilde{\chi})$ with*

$$\mathrm{bl}(\mathrm{Res}^A_{J_2}(\tilde{\chi}_1)) = \mathrm{bl}\big(\mathrm{Res}^{A[b]}_{J_2}(\tilde{\phi})\big)$$

*for every $J_2$ with $N \leq J_2 \leq A[b]$.*

*Proof.* We copy the first part of the proof of [CS15, Lemma 3.2]. Since $A/N$ is solvable, there exists some group $I$ with $N \leq I \leq A[b]$ such that $I/N$ is a Hall $\ell'$-subgroup of $A[b]/N$ and $(I \cap J)/N$ is a Hall $\ell'$-subgroup of $J[b]/N$, see [Asc00, Theorem 18.5]. According to [KS15, Theorem C(b)(1)] there exists an extension $\tilde{\chi}_2 \in \mathrm{Irr}(I)$ of $\chi$ to $I$ with $\mathrm{bl}(\tilde{\chi}_2) = \mathrm{bl}(\mathrm{Res}^{A[b]}_I(\tilde{\phi}))$. This extension also satisfies $\mathrm{bl}(\mathrm{Res}^I_{I\cap J}(\tilde{\chi}_2)) = \mathrm{bl}(\mathrm{Res}^{A[b]}_{I\cap J}(\tilde{\phi}))$ according to [KS15, Lemma 2.4] and [KS15, Lemma 2.5]. By [KS15, Lemma 3.7] there is a unique character in $\mathrm{Irr}(I \cap J \mid \chi)$ with this property, hence $\mathrm{Res}^A_{I\cap J}(\tilde{\chi}) = \mathrm{Res}^{A[b]}_{I\cap J}(\tilde{\chi}_2)$

by the assumptions on $\tilde{\chi}$. By [CS17, Lemma 5.8(a)] there exists an extension $\eta \in \mathrm{Irr}(IJ)$ of $\mathrm{Res}^A_J(\tilde{\chi})$ such that $\mathrm{Res}^{IJ}_I(\eta) = \tilde{\chi}_2$.

Since $\eta$ and $\mathrm{Res}^A_{IJ}(\tilde{\chi})$ are both extensions of the character $\mathrm{Res}^A_J(\tilde{\chi})$ there exists by Gallagher's theorem a unique linear character $\mu \in \mathrm{Irr}(IJ/J)$ such that $\eta = \mathrm{Res}^A_{IJ}(\tilde{\chi})\mu$. Moreover, by the uniqueness of $\mu$ it follows that $\mu$ is $A$-invariant. Now, $([A,A]J \cap IJ)/J \leq ([A,A]J)/J$. By assumption every $A$-invariant linear character of a subgroup of $([A,A]J)/J$ is trivial. Hence we obtain $\mathrm{Res}^{IJ}_{[A,A]J\cap IJ}(\mu) = 1_{[A,A]J\cap IJ}$. In other words, $[A,A]J \cap IJ$ is contained in the kernel of the linear character $\mu$. Hence we can consider $\mu$ as a character of $IJ/([A,A]J\cap IJ)$. Since $IJ/([A,A]J\cap IJ) \hookrightarrow A/[A,A]$ there exists a linear character $\tilde{\mu} \in \mathrm{Irr}(A)$ extending $\mu$. We define $\tilde{\chi}_1 := \tilde{\mu}\tilde{\chi}$ and by definition we have

$$\mathrm{Res}^A_{IJ}(\tilde{\chi}_1) = \mathrm{Res}^A_{IJ}(\tilde{\chi})\mu.$$

After having constructed the extension $\tilde{\chi}_1$ the same arguments from [CS15, Lemma 3.2] show the result. For the convenience of the reader we will recall the arguments here.

According to [KS15, Lemma 2.4] (which also holds for ordinary characters instead of Brauer characters) the character $\tilde{\chi}_1$ satisfies

$$\mathrm{bl}(\mathrm{Res}^A_{\langle N,x\rangle}(\tilde{\chi}_1)) = \mathrm{bl}\big(\mathrm{Res}^{A[b]}_{\langle N,x\rangle}(\tilde{\phi})) \text{ for every } x \in I \text{ of } \ell'\text{-order.}$$

Since $I/N$ is a Hall $\ell'$-subgroup of $A[b]/N$ it follows that every element $x \in A[b]$ of $\ell'$-order is conjugate to some element in $I$. Consequently, the above equality holds for every element $x \in A[b]$ of $\ell'$-order. By [KS15, Lemma 2.5] this implies

$$\mathrm{bl}\big(\mathrm{Res}^A_{J_2}(\tilde{\chi}_1)) = \mathrm{bl}(\mathrm{Res}^{A[b]}_{J_2}(\tilde{\phi})) \text{ for every group } J_2 \text{ with } N \leq J_2 \leq A[b]. \square$$

**Remark 6.14.** Note that if $A$ is abelian the assumption that $\mathrm{Irr}_A(H) = \{1_H\}$ for every subgroup $H$ of the quotient group $([A,A]J)/J$ is trivially satisfied. Hence, we obtain as a special case the original statement of [CS15, Lemma 3.2].

## 6.3 A condition on the stabilizer and the inductive conditions

In this section we introduce one of the most important results which is used in practice to verify the inductive Alperin–McKay condition for simple groups of Lie type.

Henceforth, $\mathbf{G}$ denotes a simple algebraic group of simply connected type with Frobenius $F : \mathbf{G} \to \mathbf{G}$ and such that the finite group $\mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F)$ is non-abelian and simple with universal covering group $\mathbf{G}^F$. We let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding as in Lemma 4.1. For every closed $F$-stable subgroup $\mathbf{H}$ of $\tilde{\mathbf{G}}$ we write $H := \mathbf{H}^F$.

In the following we denote by $\mathcal{B} := \langle \mathcal{G}, \phi_0 \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ the subgroup generated by the group $\mathcal{G} \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ of graph automorphisms and the field automorphism $\phi_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ from Lemma 4.1. By the description of automorphisms of simple simply connected algebraic groups in [GLS18, Theorem 2.5.1] it follows that $\mathrm{C}_{\tilde{G}\mathcal{B}}(G) = \mathrm{Z}(\tilde{G})$.

Let $s \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of $\ell'$-order and $b$ be a block of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ with defect group $D$. We let $Q$ be a characteristic subgroup of $D$. In the following we abbreviate $M := \mathrm{N}_G(Q)$ and $\tilde{M} := \mathrm{N}_{\tilde{G}}(Q)$. Moreover, $B_Q \in \mathrm{Bl}(M \mid Q)$ denotes the unique block with $(B_Q)^M = b$.

The following theorem is essentially due to Cabanes–Späth [CS15]. In previous work this theorem has turned out to be useful in the verification of the inductive Alperin–McKay condition for simple groups of Lie type.

**Theorem 6.15.** *Let $\chi \in \mathrm{Irr}(G, b)$ and $\chi' \in \mathrm{Irr}(M, B_Q)$ such that the following holds:*

*(i) We have $(\tilde{G}\mathcal{B})_\chi = \tilde{G}_\chi \mathcal{B}_\chi$ and $\chi$ extends to $(G\mathcal{B})_\chi$.*

*(ii) We have $(\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'} = \tilde{M}_{\chi'}\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$ and $\chi'$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$.*

*(iii) $(\tilde{G}\mathcal{B})_\chi = G(\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}$.*

*(iv) There exists $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}_\chi \mid \chi)$ and $\tilde{\chi}' \in \mathrm{Irr}(\tilde{M}_{\chi'} \mid \chi')$ such that the following holds:*

- *For all $m \in \mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$ there exists $\nu \in \mathrm{Irr}(\tilde{G}_\chi/G)$ with $\tilde{\chi}^m = \nu\tilde{\chi}$ and $\tilde{\chi}'^m = \mathrm{Res}^{\tilde{G}_\chi}_{\tilde{M}_{\chi'}}(\nu)\tilde{\chi}'$.*
- *The characters $\tilde{\chi}$ and $\tilde{\chi}'$ cover the same underlying central character of $\mathrm{Z}(\tilde{G})$.*

*(v) For all $G \leq J \leq \tilde{G}_\chi$ we have $\mathrm{bl}(\mathrm{Res}^{\tilde{G}_\chi}_J(\tilde{\chi})) = \mathrm{bl}(\mathrm{Res}^{\mathrm{N}_{\tilde{G}_{\chi'}}(Q)}_{\mathrm{N}_J(Q)}(\tilde{\chi}'))^J$.*

*Let $\mathrm{Z} := \mathrm{Ker}(\chi) \cap \mathrm{Z}(G)$. Then*

$$((\tilde{G}\mathcal{B})_\chi/Z, G/Z, \overline{\chi}) \geq_b ((\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}/Z, M/Z, \overline{\chi'}),$$

*where $\overline{\chi}$ and $\overline{\chi'}$ are the characters which inflate to $\chi$, respectively $\chi'$.*

*Proof.* If assumptions (i)-(iv) hold then we have

$$((\tilde{G}\mathcal{B})_\chi/Z, G/Z, \overline{\chi}) \geq_c ((\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}/Z, M/Z, \overline{\chi'}),$$

by [Spä12, Lemma 2.7]. It therefore remains to show that the additional property in Definition 6.6(ii) holds in order to show that the relation $\geq_b$ holds as well. For this, we go through the proof of [CS15, Proposition 4.2] which is still applicable under our assumptions since we can replace [CS15, Equation (4.2)] in the proof of [CS15, Proposition 4.2] by our assumption (v).

Now we want to apply the proof of [CS15, Theorem 4.1]. In the notation of [CS15, Proposition 4.2], Cabanes–Späth construct a group $A$ together with a central extension $\varepsilon : A \to \mathrm{Aut}(G)_\chi$ of $\mathrm{Aut}(G)_\chi$. Denote by $\mathrm{Aut}_{\tilde{G}_\chi}(G)$ the subgroup of automorphisms of $\mathrm{Aut}(G)$ induced by the conjugation action of $\tilde{G}_\chi$ and set $J := \varepsilon^{-1}(\mathrm{Aut}_{\tilde{G}_\chi}(G))$. By construction, $A/J \cong \mathcal{B}_\chi$, which is abelian unless possibly if $G$ is of type $D_4$.

If $A/J$ is abelian then the group-theoretic assumptions of [CS15, Lemma 3.2] are satisfied. Then we can apply the proof of [CS15, Theorem 4.1] without any change and we deduce that the characters $\chi$ and $\chi'$ satisfy the conditions in [CS15, Definition 2.1(c)].

If $A/J$ is non-abelian then as argued above $G$ is of type $D$ and it follows that $A/J \cong S_3 \times C_m$ for some integer $m$. We have $([A,A]J)/J = [A/J, A/J] \cong C_3$. No non-trivial character of $C_3$ is fixed by $S_3$. Hence, $\mathrm{Lin}_A(H) = \{1_H\}$ for every subgroup $H$ of $[A,A]J/J$. Thus, we can apply the proof of [CS15, Theorem 4.1] and instead of applying [CS15, Lemma 3.2] we use our Lemma 6.13. This shows that also in this case the characters $\chi$ and $\chi'$ satisfy the conditions in [CS15, Definition 2.1(c)].

However, since the characters $\chi$ and $\chi'$ satisfy the conditions in [CS15, Definition 2.1(c)] it follows by Theorem 6.11 that the additional property in Definition 6.6(ii) holds. This finishes the proof. □

We will check condition (v) in Theorem 6.15 using the following:

**Lemma 6.16.** *Let $\chi \in \mathrm{Irr}(G, b)$ and $\chi' \in \mathrm{Irr}(\mathrm{N}_G(Q), B_Q)$. Let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}_\chi \mid \chi)$ be an extension of $\chi$ and $\tilde{\chi}' \in \mathrm{Irr}(\tilde{M}_{\chi'} \mid \chi')$ be an extension of $\chi'$. Assume that $\mathrm{bl}(\tilde{\chi}) = \mathrm{bl}(\tilde{\chi}')^{\tilde{G}_\chi}$. Then we have*

$$\mathrm{bl}(\mathrm{Res}^{\tilde{G}_\chi}_J(\tilde{\chi})) = \mathrm{bl}(\mathrm{Res}^{\mathrm{N}_{\tilde{G}_\chi}(Q)}_{\mathrm{N}_J(Q)}(\tilde{\chi}'))^J$$

*for all $G \leq J \leq \tilde{G}_\chi$.*

*Proof.* Since $\tilde{G}/G$ is abelian, it follows that $J$ is a normal subgroup of $\tilde{G}_\chi$. Hence, $\mathrm{bl}(\mathrm{Res}^{\tilde{G}_\chi}_J(\tilde{\chi}))$ is the unique block which is covered by $\mathrm{bl}(\tilde{\chi})$.

On the other hand, $\mathrm{bl}(\mathrm{Res}^{\mathrm{N}_{\tilde{G}_\chi}(Q)}_{\mathrm{N}_J(Q)}(\tilde{\chi}'))$ is the unique block of $\mathrm{N}_J(Q)$ which is covered by $\mathrm{bl}(\tilde{\chi}')$. Moreover, we have $\mathrm{bl}(\chi')^G = \mathrm{bl}(\chi)$ and $\mathrm{bl}(\tilde{\chi}')^{\tilde{G}_\chi} = \mathrm{bl}(\tilde{\chi})$. By Corollary 1.44 we can therefore deduce that $\mathrm{bl}(\mathrm{Res}^{\tilde{G}_\chi}_J(\tilde{\chi})) = \mathrm{bl}(\mathrm{Res}^{\mathrm{N}_{\tilde{G}_\chi}(Q)}_{\mathrm{N}_J(Q)}(\tilde{\chi}'))^J$. □

## 6.4 Extension of characters

Suppose that $\mathbf{G}^F$ is not of untwisted type $D_4$. Fix a conjugacy class $(s)$ of a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of $\ell'$-order. Let $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ be the automorphisms constructed in the proof of Proposition 4.14 and denote by $\mathcal{A} \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ the subgroup generated by these automorphisms. Recall that there exists a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ in duality with the Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$, the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$, such that $\mathcal{A}$ stabilizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$.

The next lemma shows that extendibility to $G\mathcal{A}$ can be compared with extendibility to $G\mathcal{B}$. In the following, we denote by $\mathrm{ad}(x) : \tilde{G} \to \tilde{G}$ the inner automorphism of $\tilde{G}$ given by conjugation with $x \in \tilde{G}$.

**Lemma 6.17.** *Let $\chi \in \mathrm{Irr}(\mathbf{G}^F, e_s^{\mathbf{G}^F})$. Then the character $\chi$ extends to $G\mathcal{A}_\chi$ if and only if it extends to $G\mathcal{B}_\chi$.*

*Proof.* By Proposition 4.14 the image of $\tilde{G} \rtimes \mathcal{A}$ in $\mathrm{Out}(G)$ is the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(G)$. From this it follows that $\mathcal{A}_\chi$ and $\mathcal{B}_\chi$ generate the same group in $\mathrm{Out}(G)$. Thus, if $\mathcal{A}_\chi$ is cyclic then so is $\mathcal{B}_\chi$ and $\chi$ extends in both cases. Now assume that $\mathcal{A}_\chi$ is non-cyclic. Then by the proof of Proposition 4.14 there exists some $x \in \mathbf{G}^{F_0}$ such that $\mathcal{A} = \langle \mathrm{ad}(x)\gamma, F_0 \rangle$, where $F_0 \in \langle \phi_0 \rangle$. Since $\mathcal{A}_\chi$ is non-cyclic it follows that $\mathcal{A}_\chi = \langle \mathrm{ad}(x)\gamma, F_0^i \rangle$. Therefore, $\mathcal{B}_\chi = \langle \gamma, F_0^i \rangle$. By Clifford theory it follows that the character $\chi$ extends to $\mathcal{A}_\chi$ if and only if $\chi$ extends to an $\mathrm{ad}(x)\gamma$-invariant character of $G\langle F_0^i \rangle$. On the other hand, the character $\chi$ extends to $\mathcal{B}_\chi$ if and only if $\chi$ extends to a $\gamma$-invariant character of $G\langle F_0^i \rangle$. We conclude that $\chi$ extends to $G\mathcal{A}_\chi$ if and only if it extends to $G\mathcal{B}_\chi$. □

We have a local version of the previous lemma. Recall that $Q$ is assumed to be a characteristic subgroup of the defect group $D$ of $b$.

**Lemma 6.18.** *Let $\chi \in \mathrm{Irr}(\mathrm{N}_G(Q), B_Q)$. Then the character $\chi$ extends to its inertia group in $\mathrm{N}_{G\mathcal{A}}(Q, B_Q)$ if and only if it extends to its inertia group in $\mathrm{N}_{G\mathcal{B}}(Q, B_Q)$.*

*Proof.* A short calculation shows that $\mathrm{N}_{G\mathcal{A}}(Q, B_Q)/\mathrm{N}_G(Q) \cong \mathrm{N}_{\mathcal{A}}(b)$ and similarly $\mathrm{N}_{G\mathcal{B}}(Q, B_Q)/\mathrm{N}_G(Q) \cong \mathrm{N}_{G\mathcal{B}}(b)$. By the same argument as in Lemma 6.17 we may assume that the stabilizer $\mathrm{N}_{G\mathcal{A}}(Q)_\chi/\mathrm{N}_G(Q)$ is non-cyclic. Therefore, $\mathcal{A} = \langle \mathrm{ad}(x)\gamma, F_0\rangle$, where $F_0 \in \langle \phi_0 \rangle$. We conclude that there exist $y, z \in G$ such that $\mathrm{N}_{G\mathcal{A}}(Q)_\chi = \langle y\,\mathrm{ad}(x)\gamma, zF_0^i\rangle$. It follows that $\mathrm{N}_{G\mathcal{B}}(Q)_\chi = \langle yx\gamma, zF_0^i\rangle$. By Clifford theory it follows that the character $\chi$ extends to $\mathrm{N}_{G\mathcal{A}}(Q)_\chi$ if and only if $\chi$ extends to a $y\,\mathrm{ad}(x)\gamma$-invariant character of $G\langle zF_0^i\rangle$. On the other hand, the character $\chi$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if $\chi$ extends to a $yx\gamma$-invariant character of $G\langle F_0^i\rangle$. Therefore, both statements are equivalent. □

**Remark 6.19.** In Theorem 6.15 one could try to replace $\mathcal{B}$ by the group $\mathcal{A}$. However, $\mathrm{C}_{\tilde{G}\mathcal{A}}(G)$ could be larger than $\mathrm{Z}(\tilde{G})$, see Remark 4.15. We do not know however how to compute the values of the involved projective representations on this larger group.

## 6.5 The case $D_4$

In the last section we assumed that $\mathbf{G}^F$ is not of type $D_4$. The reason for this is that $\mathbf{G}^F$ admits in this case an additional graph automorphism. Thus, many of our considerations have to be altered in order to work in this case. The aim of this section is provide a certain criterion for the extendibility of characters which is tailored to the situation of Theorem 6.27.

Suppose in this section only that $\mathbf{G}$ is a simple, simply connected algebraic group of type $D_4$. We let $\phi_0 : \mathbf{G} \to \mathbf{G}$ be the field automorphism defined in Section 4.1 and we consider for any fixed prime power $q = p^f$ the Frobenius endomorphism $F = \phi_0^f : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure such that $\mathbf{G}^F$ is a finite quasi-simple group of untwisted type $D_4$.

**Corollary 6.20.** *There exists a subgroup $\mathcal{C}$ of $\mathcal{B}$ such that the image of $\tilde{G} \rtimes \mathcal{C}$ in* $\mathrm{Out}(G)$ *is the stabilizer of $e_s^{\mathbf{G}^F}$ in* $\mathrm{Out}(G)$.

*Proof.* Let $\mathrm{Diag}_{\mathbf{G}^F}$ be the image of the group of diagonal automorphisms in $\mathrm{Out}(\mathbf{G}^F)$. The stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$ contains $\mathrm{Diag}_{\mathbf{G}^F}$ by Lemma 2.32. Since $\tilde{G} \rtimes \mathcal{B}$ generates all automorphisms of $\mathbf{G}^F$ up to inner automorphisms, see Section 4.1, there exists a subgroup $\mathcal{C} \leq \mathcal{B}$ such that the image of $\tilde{G} \rtimes \mathcal{C}$ in $\mathrm{Out}(G)$ is the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(G)$. □

Recall that $b$ is a block of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ with defect group $D$ and characteristic subgroup $Q$. For every prime $r$ fix a Sylow $r$-subgroup $\mathcal{C}_r$ of $\mathcal{C}$. Note that $\mathcal{C}_r$ is contained in a Sylow $r$-subgroup of $\mathcal{B}$. Hence, there exists a graph automorphism $\gamma_r : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ of order dividing $r$ and a Frobenius $F_r = \phi_0^{i_r} : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$, with $i_r \mid f$, such that $\mathcal{C}_r = \langle \gamma_r, F_r\rangle$. We define $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ to be the Frobenius endomorphism which is the product of the $F_r$ over all primes $r$ diving the order of $\mathcal{B}$. In the following, we let $J \subseteq \{2,3\}$ be the set such that $\mathcal{C}_j$ is non-cyclic for $j \in J$.

**Lemma 6.21.** *Let $\chi \in \mathrm{Irr}(\mathrm{N}_{G\mathcal{B}}(Q)_\chi \mid B_Q)$. The character $\chi$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if for every $r \in J$ it extends to $\mathrm{N}_{G\mathcal{R}}(Q)_\chi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$.*

*Proof.* By Corollary 6.20, the image of $\tilde{G} \rtimes \mathcal{C}$ in $\mathrm{Out}(\mathbf{G}^F)$ is the stabilizer of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$. Consequently, we have $\mathrm{N}_{G\mathcal{B}}(Q)_\chi = \mathrm{N}_{G\mathcal{C}}(Q)_\chi$. Denote $H := \mathrm{N}_{G\mathcal{C}}(Q)_\chi/\mathrm{N}_G(Q)$. There exists a Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$ such that $\mathrm{N}_{G\mathcal{R}}(Q)_\chi/\mathrm{N}_G(Q)$ is a Sylow $r$-subgroup of $H$. (Note that this property is not necessarily true for all Sylow $r$-subgroups of $\mathcal{C}$.) Moreover, for $r \notin J$ all Sylow $r$-subgroups of $H$ are cyclic. By [Isa06, Corollary 11.31] the character $\chi$ extends to $\mathrm{N}_{G\mathcal{C}}(Q)_\chi$ if and only if $\chi$ extends to the preimage of a Sylow $r$-subgroup of $H$ for every prime $r$. Since all Sylow $r$-subgroups of $H$ for $r \notin J$ are cyclic it follows that $\chi$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if $\chi$ extends to $\mathrm{N}_{G\mathcal{R}}(Q)_\chi$ for all $r \in J$ and every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$. □

Let $\mathbf{L}^*$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$. By Lemma 4.11 there exists a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ in duality with $\mathbf{L}^*$ such that $\mathbf{L}$ is $F_0$-stable. Recall that $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$ are splendid Rickard equivalent, see Theorem 2.37. Hence by Theorem 1.15 we can and we will assume that the defect group $D$ of $b$ is contained in $\mathbf{L}^F$.

Let $j \in J$. By the proof of Proposition 4.14 there exist $x_j \in \mathbf{G}^{F_0}$ such that $\sigma_j := \mathrm{ad}(x_j)\gamma_j$ stabilizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$. If $j \in \{2,3\} \setminus J$ there exists some bijective morphism $\pi_j : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ with $\mathcal{C}_j = \langle \pi_j \rangle$ and again by the proof of Proposition 4.14 there exist $x_j \in \mathbf{G}^{F_0}$ such that $\sigma_j := \mathrm{ad}(x_j)\pi_j$ stabilizes $\mathbf{L}$ and $e_s^{\mathbf{L}^F}$. We then define $\mathcal{A} := \langle \sigma_2, \sigma_3, F_0 \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$.

For $r \in J$ consider an arbitrary Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$. Then we have $\mathcal{R} = \langle \gamma, F_r \rangle$ for some graph automorphism $\gamma \in \mathcal{G}$. Hence, there exists some $x \in \mathbf{G}^{F_0}$ such that $\sigma := \mathrm{ad}(x)\gamma \in \mathcal{A}$. We then denote $\mathcal{R}_{\mathcal{A}} := \langle \sigma, F_r \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$.

**Lemma 6.22.** *A character $\chi \in \mathrm{Irr}(\mathrm{N}_{G\mathcal{B}}(Q)_\chi \mid B_Q)$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if for every $r \in J$ it extends to $\mathrm{N}_{G\mathcal{R}_{\mathcal{A}}}(Q)_\chi$ for all Sylow $r$-subgroups $\mathcal{R}$ of $\mathcal{C}$.*

*Proof.* By Lemma 6.21, the character $\chi$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if $\chi$ extends to $\mathrm{N}_{G\mathcal{R}}(Q)_\chi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$ with $r \in J$. Hence it suffices to show that for every such $\mathcal{R}$ the character $\chi$ extends to $\mathrm{N}_{G\mathcal{R}}(Q)_\chi$ if and only if $\chi$ extends to $\mathrm{N}_{G\mathcal{R}_{\mathcal{A}}}(Q)_\chi$. The proof of the latter is now however exactly the same as in Lemma 6.18. □

**Proposition 6.23.** *A character $\chi \in \mathrm{Irr}(\mathrm{N}_{G\mathcal{B}}(Q)_\chi \mid B_Q)$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if for all $r \in J$ the character $\psi := {}^*R_{\mathrm{N}_L(Q)}^{\mathrm{N}_G(Q)}(\chi)$ extends to $\mathrm{N}_{L\mathcal{R}_\mathcal{A}}(Q)_\psi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$.*

*Proof.* By Lemma 6.22 the character $\chi$ extends to $\mathrm{N}_{G\mathcal{B}}(Q)_\chi$ if and only if $\chi$ extends for every $r \in J$ to $\mathrm{N}_{G\mathcal{R}_\mathcal{A}}(Q)_\chi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$.

By Remark 1.26(b) extendibility of characters in the situation of Theorem 1.24 is preserved. Now for $r \in J$ recall that $\mathcal{R}_\mathcal{A} = \langle \sigma, F_r \rangle$ for some bijective morphism $\sigma$ of $\tilde{\mathbf{G}}$. Thus, we can apply Theorem 5.24 to the commuting automorphisms $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ and $F_r : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$. This implies that the character $\chi$ extends to $\mathrm{N}_{G\mathcal{R}_\mathcal{A}}(Q)_\chi$ if and only if $\psi$ extends to $\mathrm{N}_{L\mathcal{R}_\mathcal{A}}(Q)_\psi$. □

## 6.6 A first reduction of the iAM-condition

In this section we describe the first step to reducing the verification of the iAM-condition. Before stating the main theorem of this section we need two lemmas:

**Lemma 6.24.** *Let $\chi \in \mathrm{Irr}(\tilde{L})$, $\lambda \in \mathrm{Irr}(\tilde{G}/G)$ and $\psi \in \mathrm{Irr}(\mathrm{N}_{\tilde{L}}(Q))$. Then we have*

*(a)* $\lambda R_{\tilde{L}}^{\tilde{G}}(\chi) = R_{\tilde{L}}^{\tilde{G}}(\mathrm{Res}_{\tilde{L}}^{\tilde{G}}(\lambda)\chi)$,

*(b)* $\mathrm{Res}_{\mathrm{N}_{\tilde{G}}(Q)}^{\tilde{G}}(\lambda) R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\psi) = R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\mathrm{Res}_{\mathrm{N}_{\tilde{L}}(Q)}^{\tilde{G}}(\lambda)\psi)$.

*Proof.* Part (a) is classical. It is proved using the character formula for Deligne–Lusztig characters, see proof of [DM91, Proposition 13.30(ii)].

The character formula for Deligne–Lusztig characters has been generalized to disconnected reductive groups, see [DM94, Proposition 2.6(i)]. (Note that our definition of Levi subgroups and parabolic subgroups is more general than the one in [DM94], but the same proof applies to our set-up.) Using the explicit character formula for $R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\psi)$ gives the result in (b). □

**Lemma 6.25.** *Let $\mathbf{G}$ be a reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ and parabolic subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$, where $F(\mathbf{L}) = \mathbf{L}$. Then for $\chi \in \mathrm{Irr}(\mathbf{L}^F)$ the characters $\chi$ and $R_\mathbf{L}^\mathbf{G}(\chi)$ restrict to multiples of the same central character on $Z := (\mathrm{Z}(\mathbf{G}) \cap \mathbf{L})^F$.*

*Proof.* The diagonal action of $Z$ fixes the variety $\mathbf{Y}_\mathbf{U}^\mathbf{G}$ pointwise. Hence the diagonal action of $Z$ on the bimodule $H_c^i(\mathbf{Y}_\mathbf{U}^\mathbf{G}, K)$ is trivial. The claim of the lemma follows from this. □

For the following theorem we need to suppose that the following assumption holds.

**Assumption 6.26.** *In every $\tilde{G}$-orbit of* $\mathrm{Irr}(G, e_s^G)$ *there exists a character* $\chi \in \mathrm{Irr}(G, e_s^G)$ *satisfying assumption (i) of Theorem 6.15.*

In the following we abbreviate $M_L := \mathrm{N}_L(Q)$, $\hat{M} := \mathrm{N}_{G\mathcal{A}}(Q)$, $\hat{M}_L := \mathrm{N}_{L\mathcal{A}}(Q)$ and $\tilde{M}_L := \mathrm{N}_{\tilde{L}}(Q)$.

**Theorem 6.27.** *Let $b$ be a block idempotent of* $\mathrm{Z}(\mathcal{O}Ge_s^G)$ *and* $c \in \mathrm{Z}(\mathcal{O}Le_s^L)$ *the block idempotent corresponding to $b$ under the Morita equivalence between* $\mathcal{O}Le_s^L$ *and* $\mathcal{O}Ge_s^G$ *given by* $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^L$.

*Suppose that Assumption 6.26 is satisfied. Assume additionally that the following hold.*

*(i) There exists an* $\mathrm{Irr}(\tilde{M}_L/M_L) \rtimes \hat{M}_L$*-equivariant bijection* $\tilde{\varphi} : \mathrm{Irr}(\tilde{L} \mid \mathrm{Irr}_0(c)) \to \mathrm{Irr}(\tilde{M}_L \mid \mathrm{Irr}_0(C_Q))$ *such that it maps characters covering the character* $\nu \in \mathrm{Irr}(\mathrm{Z}(\tilde{G}))$ *to a character covering* $\nu$.

*(ii) There exists an* $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q)$*-equivariant bijection* $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(M_L, C_Q)$ *which satisfies the following two conditions:*

- *If* $\chi \in \mathrm{Irr}_0(L, c)$ *extends to a subgroup $H$ of $L\mathcal{A}$ then $\varphi(\chi)$ extends to* $\mathrm{N}_H(Q)$.
- $\tilde{\varphi}(\mathrm{Irr}(\tilde{L} \mid \chi)) = \mathrm{Irr}(\tilde{M} \mid \varphi(\chi))$ *for all* $\chi \in \mathrm{Irr}_0(c)$.

*(iii) For every* $\theta \in \mathrm{Irr}_0(c)$ *and* $\tilde{\theta} \in \mathrm{Irr}(\tilde{L} \mid \theta)$ *the following holds: If* $\theta_0 \in \mathrm{Irr}(\tilde{L}_\theta \mid \theta)$ *is the Clifford correspondent of* $\tilde{\theta} \in \mathrm{Irr}(\tilde{L})$ *then* $\mathrm{bl}(\theta_0) = \mathrm{bl}(\theta_0')^{\tilde{L}_\theta}$, *where* $\theta_0' \in \mathrm{Irr}(\tilde{M}_{\varphi(\theta)} \mid \varphi(\theta))$ *is the Clifford correspondent of* $\tilde{\varphi}(\tilde{\theta})$.

*Then the block $b$ is iAM-good.*

*Proof.* By [Bro90, Theorem 1.5(2)] and [Bro90, Theorem 3.1] it follows that derived equivalences between blocks of group algebras induce character bijections which preserve the height of corresponding characters. Hence, by Theorem 4.28 we obtain bijections

$$R_L^G : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(G, b) \text{ and } R_{\mathrm{N}_L(Q)}^{\mathrm{N}_G(Q)} : \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q).$$

We define $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$ to be the bijection which makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Irr}_0(L,c) & \xrightarrow{\quad R_L^G \quad} & \mathrm{Irr}_0(G,b) \\ \varphi \big\downarrow & & \big\downarrow \Psi \\ \mathrm{Irr}_0(M_L,C_Q) & \xrightarrow{R_{\mathrm{N}_L(Q)}^{\mathrm{N}_G(Q)}} & \mathrm{Irr}_0(M,B_Q) \end{array}$$

Note that the bijection $R_L^G : \mathrm{Irr}_0(L,c) \to \mathrm{Irr}_0(G,b)$ is $\mathrm{N}_{\tilde{L}\mathcal{A}}(c)$-equivariant and the bijection $R_{\mathrm{N}_L(Q)}^{\mathrm{N}_G(Q)} : \mathrm{Irr}_0(\mathrm{N}_L(Q),C_Q) \to \mathrm{Irr}_0(\mathrm{N}_G(Q),B_Q)$ is $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q,C_Q)$-equivariant by Lemma 4.18. As in Lemma 5.23, we see that $\mathrm{N}_G(Q)\mathrm{N}_{\tilde{L}\mathcal{A}}(Q,C_Q) = \mathrm{N}_{\tilde{G}\mathcal{A}}(Q,B_Q)$. It follows that the bijection $\Psi : \mathrm{Irr}_0(G,b) \to \mathrm{Irr}_0(M,B_Q)$ is $\mathrm{N}_{\tilde{G}\mathcal{A}}(Q,B_Q)$-equivariant.

Fix a character $\chi \in \mathrm{Irr}_0(G,b)$. For $l \in \tilde{L}$ we note that the block ${}^l b$ of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ also satisfies the assumptions of the theorem with the map $\varphi$ replaced by $\varphi' : \mathrm{Irr}_0(L,{}^l c) \to \mathrm{Irr}_0(\mathrm{N}_L({}^l Q),{}^l C_Q)$ given by $\varphi'(\theta) = {}^l\varphi({}^{l^{-1}}\theta)$ for $\theta \in \mathrm{Irr}_0(L,{}^l c)$. Using Assumption 6.26 we can, by possibly taking a $\tilde{G}$-conjugate of $b$, assume that the character $\chi$ satisfies assumption (i) of Theorem 6.15. We denote $\chi' := \Psi(\chi)$ and show that the characters $\chi$ and $\chi'$ satisfy the conditions of Theorem 6.15.

Since the bijection $\Psi : \mathrm{Irr}_0(G,b) \to \mathrm{Irr}_0(M,B_Q)$ is $\mathrm{N}_{\tilde{G}\mathcal{A}}(Q,B_Q)$-equivariant we deduce that condition (iii) in Theorem 6.15 is satisfied and we have

$$(\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'} = \tilde{M}_{\chi'}\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}.$$

To show condition (ii) in Theorem 6.15 let us first assume that $\mathbf{G}^F$ is not of type $D_4$. Since Assumption 6.26 holds for the character $\chi$ it follows by Lemma 6.17 that the character $\chi$ extends to its inertia group in $G\mathcal{A}$. Note that by Remark 1.26(b) extendibility of characters in the situation of Theorem 1.24 is preserved. It therefore follows by Theorem 3.19 that ${}^*R_L^G(\chi)$ extends to its inertia group in $L\mathcal{A}$. By assumption (ii), the character $\varphi({}^*R_L^G(\chi))$ extends to its inertia group in $\hat{M}_L$. Hence, by Theorem 5.24 the character $\chi'$ extends to $\hat{M}_{\chi'}$. Now Lemma 6.18 shows that condition (ii) in Theorem 6.15 is satisfied.

Now assume that $\mathbf{G}^F$ is of type $D_4$. We use the notation of Section 6.5. Since Assumption 6.26 holds for the character $\chi$ we know by Proposition 6.23 that $\psi := {}^*R_L^G(\chi)$ extends to $(G\mathcal{R}_{\mathcal{A}})_\psi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$ with $r \in J$. By assumption (ii), it follows that $\varphi(\psi)$ extends to $\mathrm{N}_{L\mathcal{R}_{\mathcal{A}}}(Q)_\psi$ for every Sylow $r$-subgroup $\mathcal{R}$ of $\mathcal{C}$ with $r \in J$. Applying Proposition 6.23 again yields that $\chi'$ extends to its inertia group $\mathrm{N}_{G\mathcal{B}}(Q)$. Thus, condition (ii) in Theorem 6.15 is also satisfied in this case.

From Lemma 6.24 and assumption (i) it follows that the first part of condition (iv) in Theorem 6.15 is satisfied. Moreover, Lemma 6.25 and assumption (i) imply that the second part of condition (iv) in Theorem 6.15 is satisfied. We have now verified all conditions of Theorem 6.15 except for condition (v). To prove this we will show the following:

**Lemma 6.28.** *There exist characters* $\tilde{\chi}_0 \in \mathrm{Irr}(\tilde{G}_\chi \mid \chi)$ *and* $\tilde{\chi}_0' \in \mathrm{Irr}(\tilde{M}_{\chi'} \mid \chi')$ *satisfying* $\mathrm{bl}(\tilde{\chi}_0')^{\tilde{G}_\chi} = \mathrm{bl}(\tilde{\chi}_0)$.

*Proof.* Let $\tilde{\chi}_0$ be a character of $J := \tilde{G}_\chi$ extending $\chi$. Let $J_L$ be the subgroup of $\tilde{L}/L$ corresponding to $J$ under the natural isomorphism $\tilde{L}/L \cong \tilde{G}/G$.

Recall that $C := G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}$ is a complex of $\mathcal{O}[(G \times L^{\mathrm{opp}})\Delta(\tilde{L})]$-modules. Moreover by [BR06, Proposition 1.1], we have a canonical isomorphism

$$\mathrm{Ind}_{(G\times L^{\mathrm{opp}})\Delta(\tilde{L})}^{\tilde{G}\times \tilde{L}^{\mathrm{opp}}} G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O}) e_s^L \cong G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}, \mathcal{O}) e_s^L$$

in $\mathrm{Ho}^b(\mathcal{O}[\tilde{G}\times \tilde{L}^{\mathrm{opp}}])$. By Lemma 2.32 and Theorem 2.37 the complex $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\tilde{\mathbf{G}}}, \mathcal{O}) e_s^L$ induces a splendid Rickard equivalence between $\mathcal{O}\tilde{G} e_s^G$ and $\mathcal{O}\tilde{L} e_s^L$. Thus, the complex $\tilde{C} := \mathrm{Ind}_{(G\times L^{\mathrm{opp}})\Delta(J_L)}^{J\times J_L^{\mathrm{opp}}} (G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})^{\mathrm{red}}) c$ induces a splendid Rickard equivalence between $\mathcal{O}Jb$ and $\mathcal{O}J_L c$, see Lemma 1.27. Denote $\tilde{b} := \mathrm{bl}(\tilde{\chi})$ and let $\tilde{c}$ be the block corresponding to $\tilde{b}$ under the Rickard equivalence induced by $\tilde{C}$. The cohomology of $C$ is concentrated in degree $d := \dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$ and $H^d(\tilde{C}) \cong \mathrm{Ind}_{(G\times L^{\mathrm{opp}})\Delta J_L}^{J\times J_L^{\mathrm{opp}}} H^d(C)$. By Theorem 1.24 the bimodule $H^d(\tilde{C})$ induces a Morita equivalence between $\mathcal{O}J_L c$ and $\mathcal{O}Jb$. We denote by

$$R : \mathrm{Irr}(J_L, c) \to \mathrm{Irr}(J, b)$$

the associated bijection between irreducible characters and its inverse by ${}^*R$.

The complex $\mathrm{Ind}_{(\mathrm{C}_J(Q)\times \mathrm{C}_{J_L}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{J_L}(Q))}^{\mathrm{N}_J(Q)\times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(\tilde{C}))$ induces a derived equivalence between the algebras $k\mathrm{N}_J(Q)B_Q$ and $k\mathrm{N}_{J_L}(Q)C_Q$, see Proposition 1.36. Denote

$$\tilde{C}_{\mathrm{loc}} := \mathrm{Ind}_{(\mathrm{C}_G(Q)\times \mathrm{C}_L(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{L}}(Q))}^{(\mathrm{N}_J(Q)\times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{L}}(Q))} (G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \mathcal{O}))C_Q.$$

By the proof of Lemma 1.46 it follows that

$$\tilde{C}_{\mathrm{loc}} \otimes_{\mathcal{O}} k \cong \mathrm{Ind}_{\mathrm{C}_J(Q)\times \mathrm{C}_{J_L}(Q)^{\mathrm{opp}}\Delta(\mathrm{N}_{J_L}(Q))}^{\mathrm{N}_J(Q)\times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}} (\mathrm{Br}_{\Delta Q}(\tilde{C}))C_Q$$

in $\mathrm{Ho}^b(k[\mathrm{N}_J(Q) \times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}])$.

The cohomology of $G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \mathcal{O})\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ is concentrated in degree $d_Q := \dim(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)})$ by Lemma 2.28. Moreover, the bimodule $H_c^{d_Q}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \mathcal{O})c_Q$ induces a Morita equivalence between $\mathcal{O}\,\mathrm{C}_L(Q)c_Q$ and $\mathcal{O}\,\mathrm{C}_G(Q)b_Q$. Now

Lemma 1.35 and Corollary 1.18 imply that $H^{d_Q}(\tilde{C}_{\mathrm{loc}})$ induces a Morita equivalence between $\mathcal{O}\mathrm{N}_{J_L}(Q)C_Q$ and $\mathcal{O}\mathrm{N}_J(Q)B_Q$. We denote the associated character bijection by $R_{\mathrm{loc}} : \mathrm{Irr}(\mathrm{N}_{J_L}(Q), C_Q) \to \mathrm{Irr}(\mathrm{N}_J(Q), B_Q)$ and its inverse by ${}^*R_{\mathrm{loc}} : \mathrm{Irr}(\mathrm{N}_J(Q), B_Q) \to \mathrm{Irr}(\mathrm{N}_{J_L}(Q), C_Q)$. Let $\tilde{\chi} = \mathrm{Ind}_J^{\tilde{G}}(\tilde{\chi}_0)$ and define

$$\tilde{\chi}' := R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)} \circ \tilde{\varphi} \circ {}^*R_{\tilde{L}}^{\tilde{G}}(\tilde{\chi}).$$

By construction, $\tilde{\chi}' \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q) \mid \chi')$. We let $\tilde{\chi}_0' \in \mathrm{Irr}(\tilde{M}_{\chi'})$ be the unique character covering $\chi'$ with $\mathrm{Ind}_{\tilde{M}_{\chi'}}^{\tilde{M}}(\tilde{\chi}_0') = \tilde{\chi}_0'$. Let $\theta := {}^*R_L^G(\chi)$ and $\tilde{\theta} := {}^*R_{\tilde{L}}^{\tilde{G}}(\chi)$. As in the proof of Theorem 4.28 we have

$$\mathrm{Ind}_{\mathrm{N}_J(Q)\times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}\Delta(\mathrm{N}_{\tilde{L}}(Q))}^{\mathrm{N}_{\tilde{G}}(Q)\times \mathrm{N}_{\tilde{L}}(Q)^{\mathrm{opp}}} H^{d_Q}(\tilde{C}_{\mathrm{loc}}) \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{N}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\tilde{\mathbf{G}}}(Q)}, \mathcal{O})\mathrm{Tr}_{\mathrm{N}_{\mathrm{J_L}}(\mathrm{Q})}^{\mathrm{N}_{\tilde{L}}(Q)}(C_Q).$$

Hence, by Remark 1.34 we obtain

$$\mathrm{Ind}_{\mathrm{N}_{J_L}(Q)}^{\mathrm{N}_{\tilde{L}}(Q)}({}^*R_{\mathrm{loc}}(\tilde{\chi}_0')) = {}^*R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\mathrm{Ind}_{\mathrm{N}_J(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\tilde{\chi}_0')) = {}^*R_{\mathrm{N}_{\tilde{L}}(Q)}^{\mathrm{N}_{\tilde{G}}(Q)}(\tilde{\chi}') = \tilde{\varphi}({}^*R_{\tilde{L}}^{\tilde{G}}(\tilde{\chi})) = \tilde{\theta}.$$

Thus, ${}^*R_{\mathrm{loc}}(\tilde{\chi}_0') \in \mathrm{Irr}(\mathrm{N}_{J_L}(Q) \mid \varphi(\theta))$ is the Clifford correspondent of $\tilde{\varphi}(\tilde{\theta})$. Consequently, we have

$$\mathrm{bl}({}^*R_{\mathrm{loc}}(\tilde{\chi_0'}))^{J_L} = \mathrm{bl}({}^*R(\tilde{\chi}_0))$$

by assumption (iii). In other words, $\mathrm{bl}({}^*R_{\mathrm{loc}}(\tilde{\chi_0'}))$ is the Harris–Knörr correspondent of $\mathrm{bl}({}^*R(\tilde{\chi}_0))$ in the sense of Corollary 1.44.

Moreover, the bimodule $H^{d_Q}(\tilde{C}_{\mathrm{loc}})\mathrm{bl}({}^*R_{\mathrm{loc}}(\tilde{\chi}_0'))$ induces a Morita equivalence between the blocks $\mathcal{O}\mathrm{N}_J(Q)\mathrm{bl}(\tilde{\chi}_0')$ and $\mathcal{O}\mathrm{N}_{J_L}(Q)\mathrm{bl}({}^*R_{\mathrm{loc}}(\tilde{\chi}_0'))$. On the other hand, as shown above, we have

$$\tilde{C}_{\mathrm{loc}} \otimes_{\mathcal{O}} k \cong \mathrm{Ind}_{\mathrm{C}_J(Q)\times \mathrm{C}_{J_L}(Q)^{\mathrm{opp}}\Delta(\mathrm{N}_{J_L}(Q))}^{\mathrm{N}_J(Q)\times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(\tilde{C}))$$

in $\mathrm{Ho}^b(k[\mathrm{N}_J(Q) \times \mathrm{N}_{J_L}(Q)^{\mathrm{opp}}])$. Since $\mathrm{bl}({}^*R_{\mathrm{loc}}(\tilde{\chi_0'}))$ is the Harris–Knörr correspondent of $\mathrm{bl}({}^*R(\tilde{\chi}_0))$, it follows by Remark 1.47 that $\mathrm{bl}(\tilde{\chi}_0')$ is the Harris–Knörr correspondent of $\mathrm{bl}(\tilde{\chi}_0)$. In other words we have $\mathrm{bl}(\tilde{\chi}_0')^J = \mathrm{bl}(\tilde{\chi}_0)$. □

We now complete the proof of Theorem 6.27. Lemma 6.28 together with Lemma 6.16 implies that condition (v) in Lemma 6.15 is satisfied. We let $Z := \mathrm{Z}(G) \cap \mathrm{Ker}(\chi)$. Theorem 6.15 applies and we obtain

$$((\tilde{G}\mathcal{B})_\chi/Z, G/Z, \overline{\chi}) \geq_b ((\tilde{M}\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}/Z, \mathrm{N}_G(Q), \overline{\chi'}).$$

By the Butterfly Theorem, see Theorem 6.11, it follows that $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(M, B_Q)$ is a strong iAM-bijection. Consequently, the block $b$ is iAM-good. □

## 6.7 Quasi-isolated blocks

We recall the notion of quasi-isolated blocks. Let $\mathbf{H}$ be a connected reductive group with Frobenius $F : \mathbf{H} \to \mathbf{H}$. Let $\mathbf{H}^*$ be its dual group with dual Frobenius $F^*$. Recall that an element $t \in \mathbf{H}^*$ is called *quasi-isolated* if $\mathrm{C}_{\mathbf{H}^*}(t)$ is not contained in a proper Levi subgroup of $\mathbf{H}^*$. We say that a block $b$ of $\mathbf{H}^F$ is *quasi-isolated* if it occurs in $\mathcal{O}\mathbf{H}^F e_t^{\mathbf{H}^F}$ for a quasi-isolated semisimple element $t \in (\mathbf{H}^*)^{F^*}$ of $\ell'$-order.

The following remark gives the generic structure of defect groups of blocks of groups Lie type:

**Remark 6.29.** Recall that for $\ell \geq 5$ (and $\ell \geq 7$ if $\mathbf{G}$ is of type $E_8$) we know by [CE99, Lemma 4.16] that every defect group $D$ of a block $b$ of $\mathbf{G}^F$ has a unique maximal abelian normal subgroup $Q$ such that the group extension

$$1 \to Q \to D \to D/Q \to 1$$

splits. Evidence suggests that working with $Q$ is more accessible than working with the defect group and the iAM-condition is usually checked by working with $\mathrm{N}_G(Q)$.

Following the terminology in [KM15, Section 3.4] we say that an $\ell$-group $D$ is *Cabanes* if it has a unique maximal abelian normal subgroup.

This motivates the following hypothesis:

**Hypothesis 6.30.** *Consider the class $\mathcal{H}_{\mathbf{G}}$ of tuples $(\mathbf{H}, F')$ consisting of a simple algebraic groups $\mathbf{H}$ of simply connected type over $\overline{\mathbb{F}_p}$ with Frobenius $F' : \mathbf{H} \to \mathbf{H}$ such that the Dynkin diagram of $\mathbf{H}$ is isomorphic to a subgraph of the Dynkin diagram of $\mathbf{G}$. Assume that for $(\mathbf{G}, F)$ one of the following holds:*

*(a) For every $(\mathbf{H}, F') \in \mathcal{H}_{\mathbf{G}}$ the group $\mathbf{H}^{F'}/\mathrm{Z}(\mathbf{H}^{F'})$ is an abstract simple group. Let $b$ be a quasi-isolated block of $\mathbf{H}^{F'}$ and assume that $b$ has a non-central defect group $D$. Then $D$ has a unique maximal normal abelian subgroup $Q$ such that $\mathrm{N}_{\mathbf{H}^{F'}}(Q) \lneq \mathbf{H}^{F'}$ and there exists an iAM-bijection $\Psi : \mathrm{Irr}_0(\mathbf{H}^{F'}, b) \to \mathrm{Irr}_0(\mathrm{N}_{\mathbf{H}^{F'}}(Q), B_Q)$.*

*(b) Let $(\mathbf{H}, F') \in \mathcal{H}_{\mathbf{G}}$ and $b$ a quasi-isolated block of $\mathbf{H}^{F'}$. If $b$ has a non-central defect group $D$ and $\mathbf{H}^{F'}/\mathrm{Z}(\mathbf{H}^{F'})$ is an abstract simple group then there exists an iAM-bijection $\Psi : \mathrm{Irr}_0(\mathbf{H}^{F'}, b) \to \mathrm{Irr}_0(\mathrm{N}_{\mathbf{H}^{F'}}(D), B_D)$.*

Let $\mathbf{G}$ be a simple algebraic group of simply connected type such that $\mathbf{G}^F/\mathrm{Z}(\mathbf{G})^F$ is simple and non-abelian. By the explicit description of automorphisms of $\mathbf{G}^F$ in Proposition 4.2, every automorphism $\mathrm{Aut}(\mathbf{G}^F)$ lifts to a bijective automorphism of $\mathbf{G}$ commuting with the action of $F$. However, in very small cases the group $\mathbf{G}^F / \mathrm{Z}(\mathbf{G})^F$ is solvable, see [MT11, Theorem 24.17] for a list of these exceptions.

In the proof of Proposition 6.33 we will use the following property of Cabanes groups in an essential way (see [KM15, Lemma 3.11] for a slightly more general statement):

**Lemma 6.31.** *Let $P := P_1 \times P_2$, where $P_1$ and $P_2$ are both Cabanes. Then $P$ is Cabanes with maximal normal abelian subgroup $A_1 \times A_2$ where $A_i$, $i = 1, 2$, is the maximal normal abelian subgroup of $P_i$.*

We keep the notation of Theorem 6.27. We aim to understand blocks of $[\mathbf{L}, \mathbf{L}]^F$ which are covered by $c$.

**Lemma 6.32.** *Suppose that we are in the situation of Theorem 6.27. Let $c_0$ be a block of $[\mathbf{L}, \mathbf{L}]^F$ covered by $c$. Then $c_0$ is a quasi-isolated block of $[\mathbf{L}, \mathbf{L}]^F$.*

*Proof.* The inclusion $\iota : [\mathbf{L}, \mathbf{L}] \hookrightarrow \mathbf{L}$ induces a dual morphism $\iota^* : \mathbf{L}^* \twoheadrightarrow [\mathbf{L}, \mathbf{L}]^*$. We let $\overline{s} := \iota^*(s)$ the image of $s$ under this map. Recall that $\mathbf{L}^*$ is the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)$. Hence the element $s$ is quasi-isolated in $\mathbf{L}^*$ and so $c$ is a quasi-isolated block of $\mathbf{L}^F$. By [Bon05, Proposition 2.3] it follows that $\overline{s}$ is quasi-isolated in $[\mathbf{L}, \mathbf{L}]^*$. Since $c$ is a block of $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$ it follows that $c_0$ is a block of $\mathcal{O}[\mathbf{L}, \mathbf{L}]^F e_{\overline{s}}^{[\mathbf{L},\mathbf{L}]^F}$. From this we conclude that $c_0$ is a quasi-isolated block of $[\mathbf{L}, \mathbf{L}]^F$. □

The proof of the following proposition is similar to the proof of [NS14, Corollary 6.3].

**Proposition 6.33.** *Let $c_0$ be a quasi-isolated block of $L_0 := [\mathbf{L}, \mathbf{L}]^F$ of non-central defect. If Hypothesis 6.30 holds for $(\mathbf{G}, F)$ then there exists a defect group $D_0$ of $c_0$ and a characteristic subgroup $Q_0$ of $D_0$ satisfying $\mathrm{N}_{L_0}(Q_0) \lneqq L_0$ and an iAM-bijection $\varphi_0 : \mathrm{Irr}_0(L_0, c_0) \to \mathrm{Irr}_0(\mathrm{N}_{L_0}(Q_0), (C_0)_{Q_0})$.*

*Proof.* Since $\mathbf{L}$ is Levi subgroup of a simple algebraic group $\mathbf{G}$ of simply connected type it follows that $[\mathbf{L}, \mathbf{L}]$ is semisimple of simply connected type, see [MT11, Proposition 12.4]. Thus, we have

$$[\mathbf{L}, \mathbf{L}] = \mathbf{H}_1 \times \cdots \times \mathbf{H}_r,$$

where the $\mathbf{H}_i$ are simple algebraic groups of simply connected type. We have a decomposition

$$[\mathbf{L}, \mathbf{L}]^* = \mathbf{H}_1^* \times \mathbf{H}_2^* \times \cdots \times \mathbf{H}_r^*$$

into adjoint simple groups.

The action of the Frobenius endomorphism $F$ induces a permutation $\pi$ on the set of simple components of $[\mathbf{L},\mathbf{L}]$. We let $\pi = \pi_1 \dots \pi_t$ be the decomposition of this permutation into disjoint cycles. For $i = 1, \dots, t$ choose $x_i \in \Pi_i$ in the support $\Pi_i$ of the permutation $\pi_i$ and let $n_i = |\Pi_i|$ be the length of $\pi_i$. For every $1 \leq i \leq t$ the inclusion map

$$\mathbf{H}_{x_i} \hookrightarrow \prod_{x \in \Pi_i} \mathbf{H}_x$$

induces isomorphisms between $\mathbf{H}_{x_i}^{F^{n_i}}$ and $(\prod_{x\in\Pi_i} \mathbf{H}_x)^F$. Consequently, we have

$$L_0 = [\mathbf{L},\mathbf{L}]^F \cong \mathbf{H}_{x_1}^{F^{n_1}} \times \cdots \times \mathbf{H}_{x_t}^{F^{n_t}}$$

and

$$[\mathbf{L}^*,\mathbf{L}^*]^F \cong (\mathbf{H}_{x_1}^*)^{F^{n_1}} \times \cdots \times (\mathbf{H}_{x_t}^*)^{F^{n_t}}$$

in the dual group. There exists a semisimple element $\overline{s} \in [\mathbf{L}^*,\mathbf{L}^*]^F$ of $\ell'$-order such that $c_0$ is a block of $\mathcal{O}[\mathbf{L},\mathbf{L}]^F e_{\underline{s}}^{[\mathbf{L},\mathbf{L}]^F}$.

Writing $\overline{s} = s_1 \dots s_t \in (\mathbf{H}_{x_1}^*)^{F^{n_1}} \times \cdots \times (\mathbf{H}_{x_t}^*)^{F^{n_t}}$ with $s_i \in (\mathbf{H}_{x_i}^*)^{F^{n_i}}$ we obtain a decomposition

$$e_{\overline{s}}^{[\mathbf{L},\mathbf{L}]^F} = e_{s_1}^{\mathbf{H}_{x_1}^F} \otimes \cdots \otimes e_{s_t}^{\mathbf{H}_{x_t}^F}.$$

In particular, the block $c_0$ can be written as $c_0 = c_{x_1} \otimes \cdots \otimes c_{x_t}$ where the $c_{x_i}$ are blocks of $\mathbf{H}_{x_i}^{F^{n_i}}$. Note that the blocks $c_{x_i}$ are quasi-isolated in $\mathbf{H}_{x_i}^{F^{n_i}}$ since $c_0$ is assumed to be quasi-isolated. In the following we denote $H_{x_i} := \mathbf{H}_{x_i}^{F^{n_i}}$. By possibly reordering the factors of $L_0$ we can assume that there exists some integer $v$ such that the factor $\mathbf{H}_{x_i}^{F^{n_i}}$ is quasi-simple if and only if $i \leq v$. Hence we can decompose $L_0$ as

$$L_0 := L_{0,\mathrm{simp}} \times L_{0,\mathrm{solv}},$$

such that $L_{0,\mathrm{simp}} = H_{x_1} \times \cdots \times H_{x_v}$ is a direct product of simple non-abelian finite groups and $L_{0,\mathrm{solv}}$ is a finite solvable group. This induces a decomposition $c_0 = c_{0,\mathrm{simp}} \otimes c_{0,\mathrm{solv}}$ of blocks.

By Hypothesis 6.30 there exist for each $i \leq v$ a characteristic subgroup $Q_i$ of a defect group $D_i$ of $c_i$ such that there exist an $\mathrm{N}_{\mathrm{Aut}(H_i)}(Q_i, C_i)$-equivariant iAM-bijection $\varphi_i : \mathrm{Irr}_0(H_{x_i}, c_{x_i}) \to \mathrm{Irr}_0(\mathrm{N}_{H_{x_i}}(Q_i), C_{x_i})$, where $C_{x_i}$ is the unique block of $\mathrm{Bl}(\mathrm{N}_{H_{x_i}}(Q_i) \mid D_i)$ with $C_{x_i}^{H_{x_i}} = c_{x_i}$. We let

$$\{x_1, \dots, x_v\} = A_1 \cup A_2 \cup \cdots \cup A_u,$$

be the partition such that $x_j, x_k \in A_i$ whenever $n_j = n_k$ and there exists a bijective morphism $\phi : \mathbf{H}_{x_j} \to \mathbf{H}_{x_k}$ commuting with the action of $F^{n_i}$ such that $\phi(c_{x_j}) = c_{x_k}$. For each $i$ we fix a representative $x_{i_j} \in A_i$. We denote $y_i := x_{ij}$ and $m_i := n_{i_j}$.

For a bijective morphism $\phi : [\mathbf{L}, \mathbf{L}] \to [\mathbf{L}, \mathbf{L}]$ commuting with $F$ it holds that $c_0$ is quasi-isolated if and only if $\phi(c_0)$ is quasi-isolated. Moreover, the conclusion of the proposition holds for $c_0$ if and only if it holds for $\phi(c_0)$. Hence, without loss of generality, we may conjugate the block $c_0$ by an element of $\mathrm{Aut}(L_{0,\mathrm{simp}})$ such that the block $c_{0,\mathrm{simp}}$ is of the form

$$c_{0,\mathrm{simp}} = \bigotimes_{i=1}^{u} c_{y_i}^{\otimes |A_i|},$$

where the $c_{y_i}$ are all distinct blocks. Therefore, the block stabilizer of $c_0$ satisfies

$$\mathrm{N}_{\mathrm{Aut}(L_0)}(c_{0,\mathrm{simp}}) \cong \prod_{i=1}^{u} \mathrm{N}_{\mathrm{Aut}(H_{y_i})}(c_{y_i}) \wr S_{|A_i|}.$$

It follows that $D_{0,\mathrm{simp}} := \prod_{i=1}^{u} D_{y_i}^{|A_i|}$ is a defect group of $c_{0,\mathrm{simp}}$. Moreover, $Q_{0,\mathrm{simp}} := \prod_{i=1}^{u} Q_{y_i}^{|A_i|}$ is a characteristic subgroup of $D_{0,\mathrm{simp}}$ by Lemma 6.31. We let $C_{0,\mathrm{simp}}$ be the unique block of $\mathrm{N}_{L_{0,\mathrm{simp}}}(Q_{0,\mathrm{simp}})$ with defect group $D_{0,\mathrm{simp}}$ satisfying $(C_{0,\mathrm{simp}})^{L_{0,\mathrm{simp}}} = c_{0,\mathrm{simp}}$. Define $H_{A_i} := H_{y_i}^{|A_i|}$. For every $i$ we have

$$\mathrm{N}_{H_{A_i}}(Q_{y_i}^{|A_i|}) = \mathrm{N}_{H_{A_{y_i}}}(Q_{y_i})^{|A_i|}.$$

Thus, for each $i = 1, \dots, u$ we obtain bijections

$$\varphi_{y_i}^{|A_i|} : \mathrm{Irr}_0(H_{A_i}, c_{y_i}^{\otimes |A_i|}) \to \mathrm{Irr}_0(\mathrm{N}_{H_{A_i}}(Q_{y_i}^{|A_i|}), C_{y_i}^{\otimes |A_i|}).$$

We claim that these bijections are iAM-bijections. Every character $\chi \in \mathrm{Irr}_0(H_{A_i}, c_{y_i}^{\otimes |A_i|})$ is $\mathrm{N}_{\mathrm{Aut}(H_{A_i})}(Q_{y_i}^{|A_i|}, c_{y_i}^{\otimes |A_i|})$-conjugate to a character $\prod_{i=1}^{u} \chi_i$ such that for every $i$ and $j$ we either have $\chi_i = \chi_j$ or $\chi_i$ and $\chi_j$ are not $\mathrm{Aut}(H_{A_i})$-conjugate. Since the relation $\geq_b$ is preserved by automorphisms we may assume that $\chi$ is of this form. Hence, $\chi = \prod_{l=1}^{t} \psi_k^{r_l}$, where the $\psi_l$ are all distinct characters and the $r_l$ are some integers. Therefore, we have

$$\mathrm{Aut}(H_{A_i})_\chi \cong \prod_{l=1}^{t} \mathrm{Aut}(H_{y_i})_{\psi_l} \wr S_{r_l}.$$

In other words, the stabilizer of $\chi$ is a direct product of wreath products. By [Spä18, Theorem 2.21] and [Spä18, Theorem 4.6] the relation $\geq_b$ is compatible with wreath products and by [Spä18, Theorem 2.18] and [Spä18, Theorem 4.6] it is compatible with direct products. Hence we can conclude that the bijection $\varphi_{y_i}^{|A_i|}$ is an iAM-bijection.

Using this it follows directy by [Spä18, Theorem 2.18] and [Spä18, Theorem 4.6] that the bijection

$$\varphi_{0,\mathrm{simp}} := \prod_{i=1}^{u} \varphi_{x_{i_j}}^{|A_i|} : \mathrm{Irr}_0(L_{0,\mathrm{simp}}, c_{0,\mathrm{simp}}) \to \mathrm{Irr}_0(\mathrm{N}_{L_{0,\mathrm{simp}}}(Q_{0,\mathrm{simp}}), C_{0,\mathrm{simp}})$$

is an iAM-bijection.

Let $D_{0,\mathrm{solv}}$ be a defect group of $c_{0,\mathrm{solv}}$ and $C_{0,\mathrm{solv}}$ be the Brauer correspondent of $c_{0,\mathrm{solv}}$ in $\mathrm{N}_{L_{0,\mathrm{solv}}}(D_0)$. By [NS14, Theorem 7.1] there exists a strong iAM-bijection

$$\varphi_{0,\mathrm{solv}} : \mathrm{Irr}_0(L_{0,\mathrm{solv}}, c_{0,\mathrm{solv}}) \to \mathrm{Irr}_0(\mathrm{N}_{L_0}(D_{0,\mathrm{solv}}), C_{0,\mathrm{solv}}).$$

We note $D_0 = D_{0,\mathrm{simp}} \times D_{0,\mathrm{solv}}$ is a defect group of the block $c_0$ and $Q_0 := Q_{0,\mathrm{simp}} \times D_{0,\mathrm{solv}}$ is a characteristic subgroup of $D_0$. Since the image under an automorphism of a solvable (resp. quasi-simple) finite group is solvable (resp. quasi-simple), we obtain

$$\mathrm{Aut}(L_0) \cong \mathrm{Aut}(L_{0,\mathrm{simp}}) \times \mathrm{Aut}(L_{0,\mathrm{solv}}).$$

Hence, by [Spä18, Theorem 2.18] and [Spä18, Theorem 4.6] we obtain an iAM-bijection

$$\varphi_0 := \varphi_{0,\mathrm{simp}} \times \varphi_{0,\mathrm{solv}} : \mathrm{Irr}_0(L_0, c_0) \to \mathrm{Irr}_0(\mathrm{N}_{L_0}(Q_0), C_0).$$

It finally remains to show that $Q_0$ is non-central in $L_0$. By assumption the block $c_0$ of $L_0$ has non-central defect group $D_0 = D_{0,\mathrm{simp}} \times D_{0,\mathrm{solv}}$. Hence, it follows that either $D_{0,\mathrm{simp}}$ is non-central in $L_{0,\mathrm{simp}}$ or $D_{0,\mathrm{solv}}$ is non-central in $L_{0,\mathrm{solv}}$. In the latter case it follows immediately that $Q_0$ is non-central in $L_0$. In the former case, we observe that there exists some $i$ with $1 \leq i \leq v$ such that $c_{x_i}$ has non-central defect group. In particular, $\mathrm{N}_{H_{x_i}}(Q_i) \subsetneq H_{x_i}$. From this we conclude that $Q_0$ is non-central in $L_0$. □

## 6.8 Normal subgroups and character triple bijections

In this section we will recall two general statements from the theory of character triples which we will use in the next sections. The notation will therefore be unrelated to the notation of the previous sections.

We need a variant of [NS14, Proposition 4.7(b)]:

**Proposition 6.34.** *Let $X$ be a finite group. Suppose that $N \lhd X$ and $H \leq X$ such that $X = NH$. Let $e$ be a block of $N$ with defect group $D_0$. Assume that $M := N \cap H$ satisfies $H = M\mathrm{N}_X(D_0)$ and let $E \in \mathrm{Bl}(M \mid D_0)$ such that $E^H = e$. Suppose there exists an $\mathrm{N}_X(D_0, E)$-equivariant bijection*

$$\varphi : \mathrm{Irr}_0(N, e) \to \mathrm{Irr}_0(M, E)$$

*such that $(X_\theta, N, \theta) \geq_b (H_\theta, M, \varphi(\theta))$. Furthermore let $J \lhd X$ such that $N \leq J$. Let $c$ be a block of $J$ covering $f$ and $D$ a defect group of $c$ satisfying $D \cap N = D_0$ and let $C \in \mathrm{Bl}(J \cap H \mid D)$ with $C^J = c$. Then there exists an $\mathrm{N}_H(D, C)$-equivariant bijection*

$$\pi_J : \mathrm{Irr}_0(J, c) \to \mathrm{Irr}_0(J \cap H, C)$$

*such that $(X_\tau, J, \tau) \geq_b (H_\tau, J \cap H, \pi_J(\tau))$ for all $\tau \in \mathrm{Irr}_0(J, c)$.*

*Proof.* This is proved as in [NS14, Proposition 4.7(b)]. We note that the assumptions in in [NS14, Proposition 4.7(b)] are stronger. However, one can with our weaker assumption and with the same proof show that the statement of the proposition holds. □

We use the following statement, which is a consequence of the Dade–Nagao–Glauberman correspondence. Recall that if $N \lhd G$ such that $G/N$ is an $\ell$-group then every $\ell$-block of $N$ is covered by a unique block of $G$, see [Nav98, Corollary 9.6].

**Lemma 6.35.** *Let $X$ be a finite group, $M \lhd X$ and $N \lhd M$ such that $M/N$ is an $\ell$-group. Let $D_0 \lhd N$ be an $\ell$-subgroup with $D_0 \leq \mathrm{Z}(M)$. Suppose that $c_0$ is an $M$-invariant block of $N$. Let $e$ be the unique block of $M$ covering $c_0$ and let with Brauer correspondent $E_D$ in $\mathrm{N}_M(D)$. Then there exists an $\mathrm{N}_X(D, E_D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(M, e) \to \mathrm{Irr}_0(\mathrm{N}_M(D), E_D)$$

*with $(X_\tau, M, \tau) \geq_b (\mathrm{N}_X(D)_\tau, \mathrm{N}_M(D), \Pi_D(\tau))$ for every $\tau \in \mathrm{Irr}_0(M, e)$.*

*Proof.* This is a direct consequence of [NS14, Corollary 5.14]. □

## 6.9 Application of character triples

We keep the notation of Section 6.7. Furthermore, we fix a block $c_0$ of $L_0 = [\mathbf{L}, \mathbf{L}]^F$ covered by the block $c$ of $L$ with defect group $D_0$. By [Nav98, Theorem

9.26] we can assume that the defect group $D$ of $c$ satisfies $D_0 = D \cap L_0$. Our aim in this section is to obtain an iAM-bijection for the block $c$.

If the defect group $D_0$ of $c_0$ is non-central in $L_0$ then Proposition 6.33 yields an iAM-bijection for the block $c_0$. Let us now consider the case when the defect group $D_0$ of $c_0$ is central in $L_0$.

**Lemma 6.36.** *Suppose that $D_0 \leq \mathrm{Z}(L_0)$. Let $e$ be the unique block of $L_1 := L_0D$ covering $c_0$. Let $E_D$ be the Brauer correspondent of $e$ in $\mathrm{N}_{L_1}(D)$. Then there exists an iAM-bijection*

$$\varphi_0 : \mathrm{Irr}_0(L_1, e) \to \mathrm{Irr}_0(\mathrm{N}_{L_1}(D), E_D).$$

*Proof.* We show that we can apply Lemma 6.35 to the case $M := L_1$, $N := L_0$ and $X := L_1 \rtimes \mathrm{Aut}(L_1)$. Note that $c_0$ is indeed $L_1 = L_0D$-invariant. Since $\mathbf{L} = [\mathbf{L}, \mathbf{L}]\mathrm{Z}^\circ(\mathbf{L})$ we have $\mathrm{Z}([\mathbf{L}, \mathbf{L}])^F \leq \mathrm{Z}(\mathbf{L})^F$. Consequently,

$$\mathrm{Z}([\mathbf{L}, \mathbf{L}]^F) = \mathrm{Z}([\mathbf{L}, \mathbf{L}])^F \leq \mathrm{Z}(\mathbf{L})^F = \mathrm{Z}(\mathbf{L}^F)$$

by [Bon06, Remark 6.2]. This implies $\mathrm{Z}(L_0) \leq \mathrm{Z}(L_1)$ and therefore $D_0 \leq \mathrm{Z}(L_1)$. Thus Lemma 6.35 applies and the statement follows from this. □

To simplify the following calculations we introduce another notation.

**Notation 6.37.**

- Assume that $D_0$ is central in $L_0$. Then as before we fix a defect group $D$ of $c$ satisfying $D \cap L_0 = D_0$. We define $L_1 := L_0D$ and we let $e$ be the unique block of $L_1$ covering $c_0$. In addition, we set $Q := D$.
- If $D_0$ is not central in $L_0$ then we set $L_1 := L_0$, $e := c_0$ and $Q := D_0$.

We note that in both cases we have $\mathrm{N}_L(Q) \lneq L$. In the first case this follows from the assumption that $c$ has non-central defect group and in the second case this follows by the construction in Proposition 6.33. Moreover, in both cases $c$ is a block of $L$ covering $e$. This is because $c$ covers the block $c_0$ and $e$ is the unique block of $L_1$ covering $c_0$.

**Lemma 6.38.** *If Hypothesis 6.30 holds for $(\mathbf{G}, F)$ then there exists an $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q)$-equivariant bijection $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q)$ such that*

$$((\tilde{L}\mathcal{A})_\chi, L, \chi) \geq_b (\mathrm{N}_{\tilde{L}\mathcal{A}}(Q))_{\varphi(\chi)}, \mathrm{N}_L(Q), \varphi(\chi)).$$

*for every character $\chi \in \mathrm{Irr}_0(L, c)$.*

*Proof.* Denote by $E_Q \in \mathrm{Bl}(\mathrm{N}_{L_1}(Q) \mid D)$ the unique block which satisfies $(E_Q)^{L_1} = e$. By Proposition 6.33 and Lemma 6.36 there exists an iAM-bijection

$$\varphi_0 : \mathrm{Irr}_0(L_1, e) \to \mathrm{Irr}_0(\mathrm{N}_{L_1}(Q), E_Q).$$

We claim that $(\tilde{L}\mathcal{A})_\chi$ acts on $L_1$. If the defect group $D_0$ of $c_0$ is non-central in $L$ then $L_1 = L_0$ is a characteristic subgroup of $L$, so the claim is clear. Assume therefore that the defect group $D_0$ is central in $L_0$. Then the group $(\tilde{L}\mathcal{A})_\chi$ acts on $L$ and stabilizes the block $c$. Thus, $(\tilde{L}\mathcal{A})_\chi$ stabilizes the defect group $D$ up to $L$-conjugation. In particular, $L_1 = L_0 D$ is $(\tilde{L}\mathcal{A})_\chi$-stable.

Hence, we can apply the Butterfly Theorem, see Theorem 6.11, and we conclude that the bijection $\varphi_0 : \mathrm{Irr}_0(L_1, e) \to \mathrm{Irr}_0(\mathrm{N}_{L_1}(Q), E_Q)$ satisfies

$$((\tilde{L}\mathcal{A})_\chi, L_1, \chi) \geq_b (\mathrm{N}_{\tilde{L}\mathcal{A}}(Q)_{\varphi_0(\chi)}, \mathrm{N}_{L_1}(Q), \varphi_0(\chi)).$$

for every character $\chi \in \mathrm{Irr}_0(L_1, e)$.

Now we apply Proposition 6.34 in the case $H = \mathrm{N}_{\tilde{L}\mathcal{A}}(Q)$, $N = L_1$ and $J = L$. We obtain an $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q)$-equivariant bijection $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q)$ such that

$$((\tilde{L}\mathcal{A})_\chi, L, \chi) \geq_b (\mathrm{N}_{\tilde{L}\mathcal{A}}(Q)_{\varphi(\chi)}, \mathrm{N}_L(Q), \varphi(\chi))$$

holds for every character $\chi \in \mathrm{Irr}_0(L, c)$. Finally, note that

$$\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q) = \mathrm{N}_{\tilde{L}\mathcal{A}}(D, C_D)\mathrm{N}_L(Q),$$

which proves that $\varphi$ is $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q)$-equivariant. □

**Lemma 6.39.** *Suppose that Hypothesis 6.30 holds for* $(\mathbf{G}, F)$. *Then there exists a bijection* $\tilde{\varphi} : \mathrm{Irr}(\tilde{L} \mid \mathrm{Irr}_0(c)) \to \mathrm{Irr}(\mathrm{N}_{\tilde{L}}(Q) \mid \mathrm{Irr}_0(C_Q))$ *such that* $\tilde{\varphi}$ *together with the bijection* $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q)$ *constructed in the proof of Lemma 6.38 satisfy assumptions (i)-(iii) of Theorem 6.27.*

*Proof.* Choose a transversal $\mathcal{T}$ of the characters in $\mathrm{Irr}_0(L, c)$ under $\mathrm{N}_{\tilde{L}}(c)$-conjugation. By Lemma 6.38 we obtain an $\mathrm{N}_{\tilde{L}\mathcal{A}}(Q, C_Q)$-equivariant bijection $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q)$ such that

$$((\tilde{L}\mathcal{A})_\chi, L, \chi) \geq_b (\mathrm{N}_{\tilde{L}\mathcal{A}}(Q)_{\varphi(\chi)}, \mathrm{N}_L(Q), \varphi(\chi)),$$

for every character $\chi \in \mathrm{Irr}_0(L, c)$. Hence, for every character $\chi \in \mathcal{T}$ we obtain by Theorem 6.3 a bijection

$$\sigma^{(\chi)}_{\tilde{L}_\chi} : \mathrm{Irr}(\tilde{L}_\chi \mid \chi) \to \mathrm{Irr}(\mathrm{N}_{\tilde{L}}(Q)_\chi \mid \varphi(\chi)).$$

By Clifford correspondence we then obtain a bijection

$$\tilde{\varphi}_\chi : \mathrm{Irr}(\tilde{L} \mid \chi) \to \mathrm{Irr}(\tilde{M}_L \mid \varphi(\chi)).$$

The disjoint union of the bijections $\tilde{\varphi}_\chi$, $\chi \in \mathcal{T}$, induces a bijective map

$$\tilde{\varphi} : \mathrm{Irr}(\tilde{L} \mid \mathrm{Irr}_0(c)) \to \mathrm{Irr}(\tilde{M}_L \mid \mathrm{Irr}_0(C_D)).$$

By Lemma 6.4, the bijection $\tilde{\varphi}$ is $\mathrm{Irr}(\tilde{M}_L/M_L) \rtimes \hat{M}_L$-equivariant. Together with Lemma 6.5, it follows that the bijections $\varphi$ and $\tilde{\varphi}$ satisfy assumption (i) of Theorem 6.27. Moreover, Lemma 6.4 show that assumption (ii) in Theorem 6.27 is satisfied.

Let $\chi \in \mathrm{Irr}_0(L, c)$ and $\tilde{\chi} \in \mathrm{Irr}(\tilde{L} \mid \chi)$. Let $\chi_0 \in \mathrm{Irr}(\tilde{L}_\chi \mid \chi)$ be the Clifford correspondent of $\tilde{\chi} \in \mathrm{Irr}(\tilde{L})$. By Definition 6.6(ii) we have

$$\mathrm{bl}(\chi_0) = \mathrm{bl}(\sigma_{\tilde{L}_\chi}(\chi_0))^{\tilde{L}_\chi}.$$

By construction of the map $\tilde{\varphi}$, the character $\sigma_{\tilde{L}_\chi}(\chi_0) \in \mathrm{Irr}(M_{\tilde{L}} \mid \varphi(\chi))$ is the Clifford correspondent of $\tilde{\varphi}(\tilde{\chi})$. This shows that assumption (iii) in Theorem 6.27 is satisfied. □

## 6.10 Jordan decomposition for the Alperin–McKay conjecture

We can now prove our main theorem.

**Theorem 6.40.** *Let* $\mathbf{G}$ *be a simple algebraic group of simply connected type with Frobenius* $F : \mathbf{G} \to \mathbf{G}$*. Suppose that* $S := \mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F)$ *is simple and* $\mathbf{G}^F$ *is its universal covering group. Let* $b$ *be a block of* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *for a semisimple element* $s \in (\mathbf{G}^*)^{F^*}$ *of* $\ell'$*-order. If Assumption 6.26 holds for* $e_s^{\mathbf{G}^F}$ *and Hypothesis 6.30 holds for the group* $(\mathbf{G}, F)$ *then every* $\ell$*-block* $b$ *of* $\mathbf{G}^F$ *is AM-good for* $\ell$*.*

*Proof.* As in Theorem 6.27 let $c \in \mathrm{Z}(\mathcal{O}Le_s^L)$ be the block idempotent corresponding to $b$ under the Morita equivalence between $\mathcal{O}Le_s^L$ and $\mathcal{O}Ge_s^G$ given by $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})e_s^L$. By Lemma 6.39 there exists a bijection $\tilde{\varphi} : \mathrm{Irr}(\tilde{L} \mid \mathrm{Irr}_0(c)) \to \mathrm{Irr}(\mathrm{N}_{\tilde{L}}(Q) \mid \mathrm{Irr}_0(C_Q))$ such that $\tilde{\varphi}$ together with the bijection $\varphi : \mathrm{Irr}_0(L, c) \to \mathrm{Irr}_0(\mathrm{N}_L(Q), C_Q)$ constructed in the proof of Lemma 6.38 satisfy assumptions (i)-(iii) of Theorem 6.27. Hence, by Theorem 6.27 the block $b$ is therefore AM-good for $\ell$. □

**Remark 6.41.** Assumption 6.26 has been proved for all simple simply connected groups $\mathbf{G}$ not of type $D$, see [CS19, Theorem B], and conjectured to hold as well in this type. Therefore, Theorem F from the introduction is a consequence of Theorem 6.40.

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